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CONTINUITY OF UNIVERSALLY MEASURABLE HOMOMORPHISMS

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Abstract

Answering a longstanding problem originating in Christensen's seminal work on Haar null sets [*Math. Scand.* **28** (1971), 124–128; *Israel J. Math.* **13** (1972), 255–260; *Topology and Borel Structure. Descriptive Topology and Set Theory with Applications to Functional Analysis and Measure Theory*, North-Holland Mathematics Studies, 10 (Notas de Matematica, No. 51). (North-Holland Publishing Co., Amsterdam–London; American Elsevier Publishing Co., Inc., New York, 1974), iii+133 pp], we show that a universally measurable homomorphism between Polish groups is automatically continuous. Using our general analysis of continuity of group homomorphisms, this result is used to calibrate the strength of the existence of a discontinuous homomorphism between Polish groups. In particular, it is shown that, modulo ZF+DC, the existence of a discontinuous homomorphism between Polish groups implies that the Hamming graph on $\{0, 1\}^{\mathbb{N}}$ has finite chromatic number.

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1. Continuity of homomorphisms

The question of whether a measurable homomorphism between topological groups is continuous has a long and illustrious history. For example, in the very first issue of Fundamenta Matematicae, no less than three papers by Banach, Sierpiński and Steinhaus are dedicated to the question of continuity of Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{R}$ satisfying Cauchy's functional equation

$$f(x + y) = f(x) + f(y).$$

Banach [1] and Sierpiński [19] each show that such f must be continuous, which is also established by Fréchet [8], while Steinhaus [22] expands on the methods of Sierpinski [18, 19] to show that, if $A \subseteq \mathbb{R}$ is a Lebesgue measurable set of positive measure, then $A - A$ contains 0 in its interior. Steinhaus' result is subsequently generalized to arbitrary locally compact groups by Weil (see [25, page 50]), that is, if A is a Haar measurable set of positive Haar measure in a locally compact group, then AA^{-1} is an identity neighbourhood. In turn, this implies by a simple argument that every Haar measurable homomorphism between locally compact Polish groups is continuous.

Of course, as shown by Weil [25], in groups that are not locally compact there is no notion of translation invariant σ-finite measure and, in particular, no notion of Haar measurable set. Instead, in a Polish group G, one may consider the *universally measurable* sets, that is, sets A that are measurable with respect to every Borel probability measure μ on G. One particular reason for their interest is the construction by Mokobodzki (see [12, 13]) and Christensen [4] of medial limits under CH. We recall that a *medial limit* is a finitely additive translation invariant probability measure μ on $\mathbb{N}$, which is universally measurable as a function $\mu\colon \mathcal{P}(\mathbb{N}) \to [0, 1]$. Alternatively, via integration, medial limits induce translation invariant positive linear functionals $\mathfrak{m}\colon \ell^\infty \to \mathbb{R}$ satisfying a universal measurability condition so that $\liminf_n x_n \leqslant \mathfrak{m}(x) \leqslant \limsup_n x_n$ for all $x = (x_n) \in \ell^\infty$. While the assumption of CH is weakened to Martin's Axiom by Normann [14], the existence of medial limits is independent of ZFC itself as shown by Larson [10]. (A more thorough discussion of the existence of medial limits can be found at https://math.stackexchange.com/questions/54554/medial-limit-of-mokobodzki-case-of-banach-limit.)

In connection with this, Christensen [2] studies the question of whether every universally measurable homomorphism between Polish groups is continuous. He shows the following Steinhaus type principle (see [2, Theorem 5]), which turns out to be central to our study.

THEOREM 1. *Suppose $G = \bigcup_{i=1}^\infty A_i$ is a covering of a Polish group G by universally measurable sets A_i and U is an identity neighbourhood. Then there are a finite set $F \subseteq U$ and some i so that*

$$\bigcup_{g \in F} g A_i A_i^{-1} g^{-1}$$

is an identity neighbourhood.

From this he immediately deduces that every universally measurable homomorphism $G \xrightarrow{\pi} H$ between Polish groups is continuous provided H

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is *SIN*, that is, admits a bi-invariant compatible metric. In particular, this applies if either G or H is abelian and also provides an alternative proof of A. Douady's result (published by Schwartz [17]) that every universally measurable linear operator between Banach spaces is continuous. However, the general problem has remained open thus far.

PROBLEM 2. *Is every universally measurable homomorphism $G \xrightarrow{\pi} H$ between Polish groups continuous?*

Partially motivated by this and by applications to differentiability of Lipschitz mappings, Christensen [3, 4] and other authors have developed a theory of Haar null sets and related notions of smallness in Polish groups (see [6] for a recent survey). One of the principal aims of this theory is to find robust notions of smallness satisfying a variant of Steinhaus' Theorem. For example, in [21], Solecki studies *left Haar null sets* and isolates a class of Polish groups G said to be *amenable at* 1 for which every universally measurable homomorphism $G \xrightarrow{\pi} H$ into an arbitrary Polish group H is continuous. In another direction, in [16] we show that Problem 2 has a positive answer when H is locally compact or non-Archimedean (see also [7, 23] for strengthenings in the abelian case). The main result of the present paper solves the general case of Problem 2.

THEOREM 3. *Let $G \xrightarrow{\pi} H$ be a universally measurable homomorphism from a Polish group G to a separable topological group H. Then π is continuous.*

Somewhat surprisingly, the proof proceeds by showing that the conclusion of Theorem 1 is already enough for the general solution and thus entirely circumvents any further considerations of universal measurability.

LEMMA 4. *Let $G \xrightarrow{\pi} H$ be a homomorphism from a Polish group G to a separable topological group H. Assume also that, for all identity neighbourhoods $U \subseteq G$ and $V \subseteq H$, there is a finite set $F \subseteq U$ so that*

$$\bigcup_{f \in F} f \cdot \pi^{-1}(V) \cdot f^{-1}$$

is an identity neighbourhood in G. Then π is continuous.

For this reason, our proof also allows us to address a different but related question of logic, namely the strength of the existence of a discontinuous homomorphism between Polish groups. Therefore, the discussion that follows is relative to ZF+DC, that is, Zermelo–Fraenkel–Skolem set theory without the

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full axiom of choice, but only with the principle of dependent choice. This latter principle is sufficient to establish the Baire category theorem and treat basic concepts of analysis.

Various results of the literature indicate that some amount of AC is needed to construct discontinuous homomorphisms between Polish groups. For example, Larson and Zapletal [11] show that, if there is a discontinuous additive homomorphism between two separable Banach spaces, then there is a *Vitali set*, that is, a set $T \subseteq \mathbb{R}$ intersecting every translate of $\mathbb{Q}$ in a single point. However, without a linear structure on the groups, little is known.

In the following, for $k \geqslant 2$, by k^∞ we denote the profinite group $\prod_{n=1}^{\infty} \mathbb{Z}/k\mathbb{Z}$. The *Hamming graph* on k^∞ is then the graph with vertex set k^∞ and so that two elements $\alpha, \beta \in k^\infty$ form an edge if they differ in exactly one coordinate $n \in \mathbb{N}$. Also, by

$$\chi(k)$$

we denote the *chromatic number* of the Hamming graph on k^∞, that is the smallest cardinality κ so that there is a graph colouring $c\colon k^\infty \to \kappa$, that is, so that neighbouring vertices get different colours under c. Since the Hamming graph on k^∞ has cliques of size k, we always have $\chi(k) \geqslant k$. Conversely, as we shall show later, if there is a Vitali set, then the Hamming graph has chromatic number $\chi(k) = k$ for all $k \geqslant 2$. Also, if $\chi(k) = k$ for some k, then $\chi(k^n) = k^n$ for all $n \geqslant 1$. Similarly, if just some $\chi(k)$ is finite, then all the chromatic numbers $\chi(k)$ are finite.

Anticipating our general analysis of homomorphisms, if $G \xrightarrow{\pi} H$ is a homomorphism between Polish groups, we define a closed subgroup of H by

$$N = \bigcap_V \overline{\pi[V]},$$

where V ranges over identity neighbourhoods in G. Then N gauges the discontinuity of π. Indeed, assuming that $\pi[G]$ is dense in H, then N is normal in H and the induced homomorphism

$$G \xrightarrow{\tilde{\pi}} H/N$$

has closed graph and thus is continuous.

THEOREM 5. *In every model of ZF+DC, one of the following conditions hold.*

(1) *Every homomorphism between Polish groups is continuous,*

(2) *the chromatic number $\chi(k)$ is finite for all $k \geqslant 2$ and, if $G \xrightarrow{\pi} H$ is a homomorphism between Polish groups, then N is compact and connected,*

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(3) *for infinitely many* $k \geqslant 2$, *we have* $\chi(k) = k$ *and, if* $G \xrightarrow{\pi} H$ *is a homomorphism between Polish groups, then* N *is compact,*

(4) *there is a Vitali set.*

In the above theorem, we see that the conclusions about continuity of homomorphisms weaken as we go from (1) to (4), while, on the other hand, the graph theoretical conclusions strengthen. For example, if (2) holds and H is a Polish group without compact connected subgroups other than $\{1\}$, then every homomorphism from a Polish group into H must have $N = \{1\}$ and thus is continuous. Similarly, if (3) holds, then every linear operator between two Banach spaces is continuous.

Note also that by a result of L. Pontryagin every compact connected group N is *pro-Lie*, that is, for every identity neighbourhood $V \subseteq N$, there is a compact normal subgroup $K \subseteq V$ so that N/K is a finite-dimensional Lie group. In particular, this applies to the subgroup N in condition (2).

We should mention that the intermediate option (3) cannot be avoided. Indeed, di Prisco and Todorčević [15] have under the assumption of large cardinals constructed a model of ZF+DC in which there is no Vitali set (or equivalently, no transversal for E_0), but nevertheless containing a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Viewing $\mathcal{U}$ as an index 2 subgroup of $\prod_n \mathbb{Z}/2\mathbb{Z}$, this gives a discontinuous homomorphism from $\prod_n \mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$.

2. Continuity of homomorphisms

In the following, consider a homomorphism

$$G \xrightarrow{\pi} H$$

between Hausdorff topological groups G and H. Associated to this, we define a closed subgroup of H by

$$N = \bigcap_U \overline{\pi[U]} = \Big\{h \in H \mid h = \lim_\alpha \pi(g_\alpha) \text{ for some net } g_\alpha \xrightarrow[\alpha]{} 1\Big\},$$

where U varies over all identity neighbourhoods in G. Indeed, note that, if V and U are identity neighbourhoods in G so that $VV^{-1} \subseteq U$, then also

$$\overline{\pi[V]} \cdot \overline{\pi[V]}^{-1} \subseteq \overline{\pi[V]} \cdot \overline{\pi[V^{-1}]} \subseteq \overline{\pi[VV^{-1}]} \subseteq \overline{\pi[U]}.$$

So $NN^{-1} \subseteq N$ and N is a subgroup.

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LEMMA 6. *Suppose that the image of G is dense in H. Then N is the smallest closed normal subgroup of H so that the induced map*

$$G \xrightarrow{\tilde{\pi}} H/N$$

has closed graph $\mathcal{G}\tilde{\pi}$.

Proof. First, to verify that N is normal in H, since $\pi[G]$ is dense in H and N is closed, it suffices to show that $\pi(f)N\pi(f)^{-1} \subseteq N$ for all $f \in G$. So fix $h \in N$, $f \in G$ and let U be any identity neighbourhood in G. Then

$$h \in \overline{\pi[f^{-1}Uf]} = \overline{\pi(f)^{-1}\pi[U]\pi(f)} = \pi(f)^{-1} \cdot \overline{\pi[U]} \cdot \pi(f),$$

that is, $\pi(f)h\pi(f)^{-1} \in \overline{\pi[U]}$. So $\pi(f)N\pi(f)^{-1} \subseteq N$ as required.

Consider now the quotient group H/N equipped with the quotient topology, making it a Hausdorff topological group.

Note that the graph $\mathcal{G}\tilde{\pi}$ is a subgroup of $G \times H/N$ and hence so is its closure $\overline{\mathcal{G}\tilde{\pi}}$. Therefore, if $(g, hN) \in \overline{\mathcal{G}\tilde{\pi}} \setminus \mathcal{G}\tilde{\pi}$, then also $(1, \pi(g)^{-1}hN) \in \overline{\mathcal{G}\tilde{\pi}} \setminus \mathcal{G}\tilde{\pi}$. Thus, to see that $\tilde{\pi}$ has closed graph, it suffices to show that $(1, fN) \notin \overline{\mathcal{G}\tilde{\pi}}$ whenever $f \notin N$.

So fix $f \notin N$. Then there is an open identity neighbourhood $U \subseteq G$ so that $f \notin \overline{\pi[U]}$ and thus also an open neighbourhood V of f with $V \cap \overline{\pi[U]} = \emptyset$. Since U is open, we have $\pi[U]N \subseteq \overline{\pi[U]}$. Indeed, given $u \in U$, let W be an identity neighbourhood in G so that $uW \subseteq U$. Then

$$\pi(u)N \subseteq \pi(u)\overline{\pi[W]} = \overline{\pi[uW]} \subseteq \overline{\pi[U]}$$

as claimed. This thus implies that $VN \cap \pi[U]N = \emptyset$ and hence that $\underline{U \times VN/N}$ is a neighbourhood of $(1, fN)$ disjoint from $\mathcal{G}\tilde{\pi}$. So $(1, fN) \notin \overline{\mathcal{G}\tilde{\pi}}$ as required.

To see that N is the smallest closest normal subgroup K of H so that the induced map $G \xrightarrow{\hat{\pi}} H/K$ has closed graph, observe that, if K is any closed normal subgroup of H and $h \in N$, then there is a net g_α in G so that $g_\alpha \to 1$ and $\pi(g_\alpha) \to h$, whereby also $\pi(g_\alpha)K \to hK$, that is, $(1, hK)$ lies in the closure of the graph of $G \xrightarrow{\hat{\pi}} H/K$. Thus, if $G \xrightarrow{\hat{\pi}} H/K$ has closed graph, we see that $h \in K$ for every $h \in N$. □

In order to establish Lemma 4 and thus ultimately Theorem 3, the following lemma can be avoided. Instead, it is used to prove Lemma 9 that slightly strengthens Lemma 4 and that is relevant when determining the class of measures with respect to which measurability implies continuity.

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LEMMA 7. *Let $G \xrightarrow{\pi} H$ be an arbitrary homomorphism between Polish groups and suppose that $X \subseteq G$ is comeagre and $W \subseteq G$ is open. Then*

$$\overline{\pi[W]} = \overline{\pi[W \cap X]}.$$

Proof. Suppose first that U and V are identity neighbourhoods in G and H, respectively, that $Y \subseteq G$ is comeagre and let $g_n \in G$ be chosen so that $\{\pi(g_n)\}_n$ is dense in $\overline{\pi[G]}$. Then $G = \bigcup_n g_n \pi^{-1}(V)$ and hence some $g_n \pi^{-1}(V)$ is nonmeagre in U. Let $f \in \pi^{-1}(V)$ be such that $g_n f \in U$. Then

$$f^{-1}\pi^{-1}(V) = (g_n f)^{-1} \cdot g_n \pi^{-1}(V)$$

is nonmeagre in $(g_n f)^{-1} \cdot U \subseteq U^{-1}U$ and so $\pi^{-1}(V^{-1}V)$ is nonmeagre in $U^{-1}U$. It thus follows that $\pi^{-1}(V^{-1}V) \cap U^{-1}U \cap Y \neq \emptyset$.

As U and V were arbitrary, this shows that $\pi^{-1}(V) \cap U \cap Y \neq \emptyset$ for all comeagre sets $Y \subseteq G$ and identity neighbourhoods U and V in G and H, respectively.

Suppose now that $h \in \overline{\pi[W]}$ and pick $w_n \in W$ so that $\pi(w_n) \to h$. Let $Y = \bigcap_n X w_n^{-1}$, which is still comeagre in G. Let also U_n be identity neighbourhoods in G so that $U_n w_n \subseteq W$ and let $\{V_n\}$ be a neighbourhood basis at the identity in H.

By the above, we have $\pi^{-1}(V_n) \cap U_n \cap Y \neq \emptyset$, so, for each n, find g_n in this intersection. Then $g_n w_n \in U_n w_n \subseteq W$, while, as $g_n \in \pi^{-1}(V_n)$, we have $\pi(g_n) \to 1$, and finally, as $g_n \in Y \subseteq X w_n^{-1}$, we get $g_n w_n \in X$. Thus $g_n w_n \in W \cap X$, but

$$\lim_n \pi(g_n w_n) = \lim_n \pi(g_n) \cdot \lim_n \pi(w_n) = h.$$

So $h \in \overline{\pi[W \cap X]}$. □

LEMMA 8. *Let N be a Hausdorff topological group with the property that, for every identity neighbourhood V, there is a finite set F so that $N = \bigcup_{f \in F} f V f^{-1}$. Then $N = \{1\}$.*

Proof. We first claim that, for every identity neighbourhood U, there is a finite set E so that $N = EU$. For suppose not. Pick another identity neighbourhood $V \subseteq U$ so that $VV^{-1} \subset U$ and find a set F of minimal cardinality for which there is a finite set E with

$$N = EVF.$$

Fix such a set E and pick any $f \in F$. Observe then that, by our assumption, $N \neq EVV^{-1} \subseteq EU$. So take $g \notin EVV^{-1} = EVf \cdot (Vf)^{-1}$, whereby $gVf \cap EVf = \emptyset$

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and thus $gVf \subseteq EVF \setminus EVf \subseteq EV \cdot (F \setminus \{f\})$. It follows that

$$EVf \subseteq Eg^{-1} \cdot gVf \subseteq Eg^{-1} \cdot EV \cdot (F \setminus \{f\})$$

and thus that

$$N = EVF = (Eg^{-1}E \cup E)V \cdot (F \setminus \{f\}),$$

which contradicts the minimality of F and hence establishing our claim.

We now show that, for any symmetric identity neighbourhood U, one has $N = U^3$. Since N is Hausdorff, this shows that $N = \{1\}$. Thus, to see that $N = U^3$, let first E be a finite set so that $N = EU$ and let $V = \bigcap_{g \in E} gUg^{-1}$. Let also F be a finite set so that $N = \bigcup_{f \in F} fVf^{-1}$. For $f \in F$, write $f^{-1} = gu$ for some $g \in E$ and $u \in U$ and observe that

$$fVf^{-1} = u^{-1}g^{-1}Vgu \subseteq u^{-1}Uu \subseteq U^3.$$

So $N = \bigcup_{f \in F} fVf^{-1} = U^3$. □

We now finally arrive at the central lemma of the paper, which is a slightly strengthened version of Lemma 4.

LEMMA 9. *Let $G \xrightarrow{\pi} H$ be a homomorphism from a Polish group G to a separable topological group H. Suppose that, for all identity neighbourhoods $U \subseteq G$ and $V \subseteq H$, there is a finite set $E \subseteq U$ so that*

$$\bigcup_{g \in E} g \cdot \pi^{-1}(V) \cdot g^{-1}$$

is comeagre in an open identity neighbourhood. Then π is continuous.

Proof. Let M be the closed normal subgroup of H consisting of all elements that cannot be separated from the identity by an open set. Then the quotient group H/M is Hausdorff. Moreover, since H/M is separable, any nonempty open set covers H/M by countably many translates and thus, by a result of Guran [9], H/M is embeddable into a direct product $\prod_{i \in I} K_i$ of separable metrisable groups. Taking completions in the two-sided uniformity, we may assume that the K_i are Polish.

By taking compositions with the quotient map $H \to H/M$, the embedding $H/M \to \prod_{i \in I} K_i$ and the factor projections $\prod_{i \in I} K_i \to K_j$, we thus have homomorphisms $G \xrightarrow{\pi_j} K_j$, $j \in I$, and to see that $G \xrightarrow{\pi} H$ is continuous it suffices to show that each π_j is continuous. Since the maps π_j still satisfy the condition that, for all identity neighbourhoods $U \subseteq G$ and $V \subseteq K_j$, there is a

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finite set $E \subseteq G$ for which $\bigcup_{g\in E} g \cdot \pi_j^{-1}(V) \cdot g^{-1}$ is comeagre in an open identity neighbourhood, we have thus reduced to problem to the case when H is a Polish group.

So assume without loss of generality that H itself is Polish, that $\pi[G]$ is dense in H and let N and $\tilde{\pi}$ be as before. Since $G \xrightarrow{\tilde{\pi}} H/N$ has closed graph and both G and H/N are Polish, $\tilde{\pi}$ is continuous. Therefore, to see that π is continuous, it suffices to show that $N = \{1\}$ or by Lemma 8 that, for every identity neighbourhood V in H, there is a finite set $F \subseteq N$ so that $N \subseteq \bigcup_{f\in F} fVf^{-1}$.

So let V be any open identity neighbourhood in H and let W be an open identity neighbourhood so that $\overline{WWW^{-1}} \subseteq V$. As $\tilde{\pi}$ is continuous, $\pi^{-1}(NW)$ is an identity neighbourhood in G. Therefore, by assumption, there is a finite set $E \subseteq \pi^{-1}(NW)$ so that

$$B = \bigcup_{g\in E} g\pi^{-1}(W)g^{-1}$$

is comeagre in an open identity neighbourhood U in G. In particular, by Lemma 7,

$$N \subseteq \overline{\pi[U]} \subseteq \overline{\pi[B]}.$$

Now, let $F \subseteq N$ be a finite set so that $\pi[E] \subseteq FW$. Then

$$\pi[B] \subseteq \bigcup_{g\in E} \pi(g)W\pi(g)^{-1} \subseteq \bigcup_{f\in F} fWWW^{-1}f^{-1},$$

and so

$$N \subseteq \overline{\pi[B]} \subseteq \bigcup_{f\in F} f\overline{WWW^{-1}}f^{-1} \subseteq \bigcup_{f\in F} fVf^{-1}$$

as required. □

We now turn to the proof of our principal result, Theorem 3. To avoid any ambiguity, recall that, given a Borel probability measure μ on a Polish space X, $\mathrm{Meas}_\mu(X)$ is the σ-algebra of all μ-measurable sets $A \subseteq X$, that is, sets so that, for some Borel set $B \subseteq X$, $A \triangle B$ is μ-null. The σ-algebra

$$\Sigma = \bigcap_{\mu} \mathrm{Meas}_\mu(X),$$

where μ varies over all Borel probability measures on X, is the algebra of *universally measurable sets*. Of course, this is an extremely impredicative definition and little is known about how to generate the absolutely measurable sets by a more explicit procedure. To some extent, absolute measurability is

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therefore a placeholder for the requirement that all arguments about these sets should principally be measure-theoretic and not involve definability (for example, projective sets) or Baire category.

A map $X \xrightarrow{\phi} Y$ from a Polish space X to a topological space Y is *universally measurable* if $\phi^{-1}(V)$ is universally measurable for every open set $V \subseteq Y$.

THEOREM 10. *Let $G \xrightarrow{\pi} H$ be a universally measurable homomorphism from a Polish group G to a separable topological group H. Then π is continuous.*

Proof. We verify the hypothesis of Lemma 9. For this, suppose V is an identity neighbourhood in H and pick an open identity neighbourhood W so that $WW^{-1} \subseteq V$. Then $A = \pi^{-1}(W)$ is universally measurable and, as H is separable, covers G by countably many right translates. By Theorem 1, this means that, for any identity neighbourhood U in G there are $g_1, \ldots, g_n \in U$ so that

$$g_1 AA^{-1} g_1^{-1} \cup \cdots \cup g_n AA^{-1} g_n^{-1}$$

and hence also

$$g_1 \pi^{-1}(V) g_1^{-1} \cup \cdots \cup g_n \pi^{-1}(V) g_n^{-1}$$

is an identity neighbourhood. This verifies the conditions of Lemma 9 and thus proves the theorem. □

It is debatable whether Theorem 3 is a positive or a negative result. On the one hand, it is certainly a regularity theorem for universally measurable sets, but, on the other hand, it shows that there is no homomorphism analogue of the useful medial limits of Mokobodzki.

It is of course natural to wonder whether measurability with respect to all measures is really required for the proof of Theorem 10. For example, if the domain group G is locally compact, the Steinhaus–Weil theorem says that any homomorphism π, which is measurable with respect to just the Haar measure on G, is necessarily continuous. So can the requirement of universal measurability be relaxed to demanding that π be measurable with respect to some single judiciously chosen σ-finite Borel measure μ on G?

In general, the answer is no, as the following example shows, but it is still interesting to identify specific σ-finite Borel measures on Polish groups with this property. Observe that, as any σ-finite Borel measure is equivalent to a Borel probability measure, we may consider these instead.

EXAMPLE 11. Let μ be a Borel probability measure on a separable infinite-dimensional real Banach space X. Then there is a μ-measurable discontinuous

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functional $\phi\colon X \to \mathbb{R}$. To see this, let $\{x_n\}$ be a countable dense sequence in X and observe that by tightness of μ, we may find symmetric compact sets $K_n \subseteq X$ so that $\mu(X \setminus K_n) < 1/n$ and $0, x_n \in K_n$. Without loss of generality, assume that $K_1 \subseteq K_2 \subseteq \cdots$. Observe then that $C_n = \overline{\mathrm{conv}}(K_n)$ is a compact convex set and hence that

$$V_n = \mathrm{span}(C_n) = \bigcup_{m\geqslant 1} m \cdot C_n$$

is a K_σ linear subspace. As V_n are increasing and contain x_n, it therefore follows that $W = \bigcup_n V_n$ is a dense K_σ linear subspace of X. Moreover, $\mu(W) = 1$. As X is not itself K_σ, W is a proper subspace of X and therefore contained in some hyperplane Z of X. Then, if ϕ is a nonzero linear functional on X that vanishes on Z, we see that, for open sets $U \subseteq \mathbb{R}$, the preimage $\phi^{-1}(U)$ either has full or zero measure depending on whether $0 \in U$ or not. So ϕ is μ-measurable, but discontinuous.

On the other hand, Stroock [23] shows that, if $T : X \to Y$ is a linear operator between real Banach spaces that is measurable with respect to every centred Gaussian measure on X, then T is bounded. Thus, in the case of Banach spaces, we have a geometrically defined class of measures that suffices for continuity.

Recall that, when X is a Polish space, the space of Borel probability measures $P(X)$ on X is Polish when equipped with the initial topology given by the maps

$$\mu \mapsto \int_X f\, d\mu,$$

where f ranges over continuous bounded real-valued functions on X.

PROBLEM 12. *Suppose $G \xrightarrow{\pi} H$ is a* residually measurable *homomorphism between Polish groups, that is, so that π is measurable with respect to a comeagre set of probability measures μ on G. Is π continuous?*

A second issue arising from our proof is that the statement of Lemma 9 is an entirely algebraic-topological criterion for continuity of homomorphisms and it is far from clear what rôle the completeness of G plays in it. Of course, ultimately, the proof makes a heavy recourse to the closed graph theorem, but can this be avoided?

PROBLEM 13. *Is Lemma 9 valid for all separable metrisable topological groups G and H?*

For the purpose of our analysis in Section 3, we now return to studying the subgroup N associated to a homomorphism $G \xrightarrow{\pi} H$.

LEMMA 14. *The following are equivalent for a homomorphism $G \xrightarrow{\pi} H$ between Polish groups.*

(1) $N = \bigcap_U \overline{\pi[U]}$ *is compact,*

(2) *for all identity neighbourhoods $U \subseteq G$ and $V \subseteq H$, there is a finite set $E \subseteq U$ so that $E \cdot \pi^{-1}(V)$ is an identity neighbourhood in G,*

(3) *for all identity neighbourhoods $U \subseteq G$ and $V \subseteq H$, there is a finite set $E \subseteq U$ so that $E \cdot \pi^{-1}(V) \cdot E$ is an identity neighbourhood in G.*

Proof. Without loss of generality, we assume $\pi[G]$ is dense and thus that N is normal in H.

(1)⇒(2): suppose N is compact and that identity neighbourhoods $U \subseteq G$ and $V \subseteq H$ are given. Fix some symmetric open identity neighbourhood $W \subseteq H$ so that $W^3 \subseteq V$. As N is compact, pick a finite set $F \subseteq N$ so that $N \subseteq FW$. Then, as $F \subseteq N \subseteq \overline{\pi[U]} \subseteq \pi[U]W$, we can find a finite set $E \subseteq U$ with $F \subseteq \pi[E]W$ and thus

$$NW \subseteq FW^2 \subseteq \pi[E]W^3 \subseteq \pi[E]V.$$

Since $G \xrightarrow{\tilde{\pi}} H/N$ is continuous, it follows that $\pi^{-1}(NW)$ is open in G and thus also that $\pi^{-1}(\pi[E]V) = E\pi^{-1}(V)$ is an identity neighbourhood.

(3)⇒(1): assume (3). To see that N is compact, by a result independently due to Solecki [20] and Uspenskiĭ [24], it is enough to show that, for every identity neighbourhood V in H there is a finite set $F \subseteq N$ so that $N \subseteq FVF$. So let V be given and find a symmetric open identity neighbourhood W so that $\overline{W^3} \subseteq V$. As $G \xrightarrow{\tilde{\pi}} H/N$ is continuous, the set $\pi^{-1}(NW)$ is open in G. There is therefore a finite set $E \subseteq \pi^{-1}(NW)$ so that $U = E\pi^{-1}(W)E$ is an identity neighbourhood in G. As $E \subseteq \pi^{-1}(NW) = \pi^{-1}(WN)$, we have $\pi[E] \subseteq FW \cap WF$ for some finite set $F \subseteq N$. Thus, by definition of N,

$$N \subseteq \overline{\pi[U]} \subseteq \overline{\pi[E]W\pi[E]} \subseteq \overline{FWWWF} \subseteq F\overline{WWW}F \subseteq FVF,$$

so N is compact. □

3. A quadrichotomy for homomorphisms

We will now use our analysis of discontinuous homomorphisms to address the amount of choice needed to produce these. For this, we must of course work in some suitable weak background theory, which we here take to be ZF+DC. This is appropriate, as dependent choice, DC, suffices to establish the basic concepts of analysis that do not directly involve choice, for example, the Baire category theorem.

3.1. Vitali sets and chromatic numbers of Hamming graphs. Recall that by k^∞ we denote the infinite direct product $\prod_{n=1}^{\infty} \mathbb{Z}/k\mathbb{Z}$ and that the *Hamming graph* on k^∞ is the graph with vertex set k^∞ so that two vertices $\alpha, \beta \in k^\infty$ form an edge if and only if α and β differ in exactly one coordinate $n \in \mathbb{N}$. A *graph colouring* is just a function $c\colon k^\infty \to X$ into some set X so that $c(\alpha) \neq c(\beta)$ whenever $\{\alpha, \beta\}$ is an edge. Then the *chromatic number*

$$\chi(k)$$

is the smallest cardinality κ so that there is a graph colouring $c\colon k^\infty \to X$ into a set X of cardinality κ.

Now, suppose $c\colon k^\infty \to X$ is a graph colouring and $m = k^n$ for some $n \geqslant 1$. Fix a bijection $\phi\colon \mathbb{Z}/m\mathbb{Z} \to \prod_{i=1}^{n} \mathbb{Z}/k\mathbb{Z}$ and let $(\cdot)_i = \mathrm{proj}_i \circ \phi$ for $i = 1, \ldots, n$. We then define $c_i\colon m^\infty \to X$ for $i = 1, \ldots, n$ by letting

$$c_i(\alpha) = c((\alpha_1)_i, (\alpha_2)_i, (\alpha_3)_i, \ldots)$$

and let $C\colon m^\infty \to X^n$ be given by $C(\alpha) = (c_1(\alpha), \ldots, c_n(\alpha))$. Then C is also a graph colouring, which shows that

$$\chi(k^n) \leqslant \chi(k)^n$$

for all $n \geqslant 1$. In particular, this shows that, if $\chi(k) = k$ for some k, then actually $\chi(k) = k$ for infinitely many k.

The equivalence relation on k^∞ of belonging to the same connected component of the Hamming graph will be denoted by $E_0(k)$. We observe that $\alpha, \beta \in k^\infty$ are $E_0(k)$-equivalent if they differ in only finitely many coordinates. Also, a *transversal* for $E_0(k)$ is a set $T \subseteq k^\infty$ intersecting every equivalence class in exactly one point. In case T is such a transversal, one easily sees that $\chi(k) = k$. Indeed, a colouring $c\colon k^\infty \to \mathbb{Z}/k\mathbb{Z}$ is then defined by letting

$$c(\alpha) = \sum_{i=1}^{\infty} (\alpha - \hat{\alpha})_i \qquad \mod k,$$

where $\hat{\alpha} \in T$ is the unique representative of the equivalence class of α in T.

As mentioned earlier, a *Vitali set* is a set $T \subseteq \mathbb{R}$ intersecting every translate of $\mathbb{Q}$ in a single point. For each $k \geqslant 2$, since both $E_0(k)$ and the equivalence relation

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on $\mathbb{R}$ of belonging to the same translate of $\mathbb{Q}$ are hyperfinite Borel equivalence relations, they are Borel bireducible and thus, in every model of ZF+DC, there is a Vitali set if and only if $E_0(k)$ admits a transversal. This means that we now have a sequence of implications holding under ZF+DC.

$$\begin{aligned}\text{There is a Vitali set} &\Rightarrow \chi(k) = k \text{ for all } k\\ &\Rightarrow \chi(k) = k \text{ for some or, equivalently, infinitely many } k\\ &\Rightarrow \chi(k) < \infty \text{ for all } k.\end{aligned}$$

Let us also observe the well-known fact that there can be no Baire or Haar measurable colouring $c\colon 2^\infty \to \mathbb{N}$. Indeed, given such a map c, there is some colour $n \in \mathbb{N}$ so that $c^{-1}(n)$ is nonmeagre, respectively, nonnull. Hence by Pettis' Lemma, respectively, the Steinhaus–Weil Theorem, $c^{-1}(n) - c^{-1}(n)$ contains some element γ with a single non-zero coordinate. Writing $\gamma = \alpha - \beta$ where $c(\alpha) = c(\beta)$, we see that α and β are neighbouring vertices in the Hamming graph and so c fails to be a graph colouring.

3.2. The quadrichotomy. For the next lemma, a subset A of a group G is said to be *right σ-syndetic* provided it covers G by countably many right translates, that is, $G = \bigcup_{n=1}^\infty Af_n$ for some $f_n \in G$.

LEMMA 15 (ZF+DC). *Suppose there is no Vitali set. Then, for every right σ-syndetic subset A of a Polish group G and identity neighbourhood $U \subseteq G$, there is finite set $E \subseteq U$ so that*

$$EAA^{-1}E$$

has nonempty interior.

Proof. Write $G = \bigcup_{n=1}^\infty Af_n$ for some $f_n \in G$ and suppose that $EAA^{-1}E$ has empty interior for every finite set $E \subseteq U$. Without loss of generality, we assume U is symmetric. Then, by induction, we can find $g_1, g_2, \ldots \in U$ so that

(1) $g_{i_1} \cdots g_{i_n} \in U$ for all $i_1 < \cdots < i_n$,

(2) for all $i_1 < \cdots < i_n < k$ and $j_1 < \cdots < j_m < k$ we have

$$g_k \notin g_{i_n}^{-1} \cdots g_{i_1}^{-1} \cdot AA^{-1} \cdot g_{j_1} \cdots g_{j_m},$$

(3) for all $i_1 < i_2 < \cdots$ the infinite product

$$g_{i_1} g_{i_2} g_{i_3} \cdots$$

converges in G,

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(4) the map $\phi\colon 2^\infty \to G$ defined by

$$\phi(\alpha) = g_1^{\alpha(1)} g_2^{\alpha(2)} \cdots,$$

where $g^0 = 1$, is a continuous injection and thus a homeomorphism with its image.

Assume that α and β are $E_0(2)$-equivalent but distinct, say $\alpha(k) = 1$, $\beta(k) = 0$ and $\alpha(n) = \beta(n)$ for all $n > k$. Then we can write

$$\phi(\alpha) = g_{i_1} \cdots g_{i_n} g_k h, \quad \phi(\beta) = g_{j_1} \cdots g_{j_m} h$$

for some $i_1 < \cdots < i_n < k$, $j_1 < \cdots < j_m < k$ and $h \in G$. Thus

$$\phi(\alpha)\phi(\beta)^{-1} = g_{i_1} \cdots g_{i_n} \cdot g_k \cdot g_{j_m}^{-1} \cdots g_{j_1}^{-1} \notin AA^{-1}.$$

In particular, this shows that each set $B_m = \phi^{-1}(Af_m)$ can only intersect an $E_0(2)$-equivalence class in a single point and hence is a partial $E_0(2)$-transversal.

Now, since $2^\infty = \bigcup_m B_m$, this means that a transversal $T \subseteq 2^\infty$ for $E_0(2)$ can be defined by

$$T = \bigcup_m (B_m \setminus [B_1 \cup \cdots \cup B_{m-1}]_{E_0}).$$

Thus, if the conclusion of the lemma fails, there is a transversal for $E_0(2)$ and hence also a Vitali set. $\square$

LEMMA 16 (ZF+DC). *Suppose $G \xrightarrow{\pi} H$ is a homomorphism between Polish groups so that $N = \bigcap_U \overline{\pi[U]}$ is compact. Assume that $V, W \subseteq H$ are symmetric open identity neighbourhoods so that*

(1) *$hVh^{-1} = V$ for all $h \in N$,*

(2) *there is a set $F \subseteq N$ with $|F| = p$ and $N \subseteq VF$,*

(3) *there is a set $E \subseteq N$ with $|E| = k$ and $VW^2h_1 \cap VW^2h_2 = \emptyset$ for all distinct $h_1, h_2 \in E$.*

Then $\chi(k) \leqslant p$.

Proof. Since N is compact and VF is open, by shrinking W, we may assume that $NW \subseteq VF$. Let then $U \subseteq G$ be a symmetric open identity neighbourhood so that $\overline{U} \subseteq \pi^{-1}(NW)$, which is possible since the induced map $G \to \overline{\pi[G]}/N$ is continuous.

Now, for every identity neighbourhood O in G, $N \subseteq W\pi[O]$, so we may inductively choose sets $E_1, E_2, \ldots \subseteq G$ so that

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(1) $E_1 \cdots E_n \subseteq U$ for all n,

(2) $|E_i| = k$,

(3) $VW\pi(g) \cap VW\pi(f) = \emptyset$ for all distinct $g, f \in E_n$,

(4) the map $\phi\colon \prod_{i=1}^{\infty} E_i \to \overline{U}$ given by

$$\phi(g_1, g_2, g_3, \ldots) = g_1 g_2 g_3 \ldots$$

is well defined, injective and continuous with respect to the product topology on $\prod_{i=1}^{\infty} E_i$.

Now, suppose $\alpha, \beta \in \prod_{i=1}^{\infty} E_i$ differ in a single coordinate n. Then we can write $\phi(\alpha) = ugx$ and $\phi(\beta) = ufx$ for some $u \in U \subseteq \pi^{-1}(NW)$, $x \in G$ and distinct $g, f \in E_n$, whereby $\pi(\phi(\alpha)) = hw\pi(g)\pi(x)$ and $\pi(\phi(\beta)) = hw\pi(f)\pi(x)$ for some $h \in N$ and $w \in W$. Thus

$$\begin{aligned} V\pi(\phi(\alpha)) \cap V\pi(\phi(\beta)) &= Vhw\pi(g)\pi(x) \cap Vhw\pi(h)\pi(x) \\ &\subseteq hVW\pi(g)\pi(x) \cap hVW\pi(h)\pi(x) \\ &= h\big[VW\pi(g) \cap VW\pi(f)\big]\pi(x) \\ &= \emptyset \end{aligned}$$

and so $\pi(\phi(\alpha))$ and $\pi(\phi(\beta))$ cannot belong to the same right translate Vz of V by any $z \in H$. Since $\mathrm{im}(\phi) \subseteq \overline{U} \subseteq \pi^{-1}(VF)$, it thus follows that the sets

$$A_z = (\pi\phi)^{-1}(Vz)$$

for $z \in F$ cover $\prod_{i=1}^{\infty} E_i$ by sets that are discrete in the Hamming graph on the product $\prod_{i=1}^{\infty} E_i$, which we may identify with k^{∞}. So $\chi(k) \leqslant |F| = p$. □

THEOREM 17 (ZF+DC). *One of the following conditions hold.*

(1) *Every homomorphism between Polish groups is continuous,*

(2) *the chromatic number $\chi(k)$ is finite for all $k \geqslant 2$ and, if $G \xrightarrow{\pi} H$ is a homomorphism between Polish groups, then N is compact and connected,*

(3) *for infinitely many $k \geqslant 2$, we have $\chi(k) = k$ and, if $G \xrightarrow{\pi} H$ is a homomorphism between Polish groups, then N is compact,*

(4) *there is a Vitali set.*

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Proof. In all of the proof, we suppose that (4) fails, that is, there is no Vitali set. Then, by Lemma 15, for every right σ-syndetic set $A \subseteq G$ and identity neighbourhood $U \subseteq G$, there is a finite set $E \subseteq U$ so that $EAA^{-1}E$ is an identity neighbourhood in G. So suppose $G \xrightarrow{\pi} H$ is a homomorphism between Polish groups so that $\pi[G]$ is dense in H and $U \subseteq G$, $V \subseteq H$ identity neighbourhoods. Pick a symmetric open identity neighbourhood W so that $W^2 \subseteq V$. Then $A = \pi^{-1}(W)$ is right σ-syndetic, so, for some finite $E \subseteq U$,

$$E \cdot \pi^{-1}(W) \cdot \pi^{-1}(W)^{-1} \cdot E \;\subseteq\; E \cdot \pi^{-1}(V) \cdot E$$

are identity neighbourhoods in G. By Lemma 14, this shows that N is compact.

Now, assume there is some $G \xrightarrow{\pi} H$ so that N is compact but not connected. Then, by a theorem of van Dantzig [5], N has a clopen proper normal subgroup $M \trianglelefteq N$. Since N is compact, the index $k = [N : M] > 1$ is finite and so the cosets of M in N,

$$M_1, \ldots, M_k,$$

are pairwise disjoint compact sets in H. We may thus choose a symmetric open identity neighbourhood $W \subseteq H$ so that also

$$W^3 M_1, \ldots, W^3 M_k$$

are pairwise disjoint. Moreover, as N is compact, we may suppose that $hWh^{-1} = W$ for all $h \in N$. Then $V = WM$ is symmetric and open in H and $VW^2h = WMW^2h = W^3Mh$ for all $h \in N$. It thus follows that $VW^2h_1 \cap VW^2h_2 = \emptyset$ whenever $h_1, h_2 \in N$ belong to distinct M-cosets. Letting $F \subseteq N$ be a set of coset representative for M in N, we see that $N \subseteq VF = WMF = WN$. Letting also $E = F$, Lemma 16 implies that $\chi(k) = k$ and thus that $\chi(k^m) = k^m$ for all $m \geqslant 1$.

Now, assume instead there is a discontinuous homomorphism $G \xrightarrow{\pi} H$ between Polish groups. Then N is compact but $N \neq \{1\}$. We may thus find some symmetric open identity neighbourhood $V \subseteq H$ so that $hVh^{-1} = V$ for all $h \in N$ and so that $V^3h_1 \cap V^3h_2 = \emptyset$ for some elements $h_1, h_2 \in N$. Let then $E = \{h_1, h_2\}$, $W = V$ and let $F \subseteq N$ be any finite set so that $N \subseteq VF$. By Lemma 16, it follows that $\chi(2) \leqslant |F|$ and hence that $\chi(k) < \infty$ for all $k \geqslant 2$. □

Corollary 18 (ZF+DC). *Suppose that $\chi(k) > k$ for all $k \geqslant 2$ and let H be a countable index subgroup of a Polish group G. Then H is open in G.*

Proof. Consider the group $\mathrm{Sym}(G/H)$ of all permutations of the left-coset space G/H of H. Equipped with the permutation group topology, that is, where pointwise stabilizers are declared open, $\mathrm{Sym}(G/H)$ is a Polish group. Since the only compact connected subgroup $N \leqslant \mathrm{Sym}(G/H)$ is $N = \{1\}$, by Theorem 17, the homomorphism

$$G \xrightarrow{\pi} \mathrm{Sym}(G/H),$$

is continuous and hence H is open in G. □

3.3. Γ-measurability. It is not hard to verify that the proof of our quadrichotomy relativizes to Γ-measurability, where Γ is an adequate pointclass in Polish spaces in the sense below. That is, in Theorem 17, we may replace abstract homomorphisms by Γ-measurable homomorphisms, transversals by Γ-measurable transversals and colourings by Γ-measurable colourings. The proofs are exactly the same, except that one must track the Γ-measurability of all sets involved in the constructions.

DEFINITION 19. A pointclass Γ in Polish spaces is said to be *adequate* if, for every Polish space X, $\Gamma(X)$ is a σ-algebra containing the Borel sets and, whenever $\phi\colon X \to Y$ is a homeomorphism between Polish spaces, then $A \subseteq X$ is Γ-measurable if and only if $\phi[A]$ is.

For example, Γ could be the pointclass of universally measurable sets or simply the pointclass of all subsets of Polish spaces.

One particular application that seems most interesting when involving Γ-measurability is the following corollary.

COROLLARY 20 (ZF+DC). *Let Γ be an adequate point class in Polish spaces so that, for all $k \geqslant 2$, the Hamming graph on k^∞ has Γ-chromatic number $> k$. Then, for every symmetric, right σ-syndetic, Γ-measurable subset $A \ni 1$ of a Polish group G, there is a power A^n with nonempty interior.*

Proof. Let $H = \langle A\rangle$ be the subgroup generated by A, which as A is right σ-syndetic must have countable index in G. Write $G = \bigcup_n Af_n$ for some $f_1, f_2, \ldots \in G$ and observe that, if $Af_n \cap H \neq \emptyset$, then $f_n \in H$. It follows that $H = \bigcup_{f_n \in H} Af_n$ and hence H is itself Γ-measurable and thus open by Corollary 18.

Again, by Lemma 15, we find a finite set $E \subseteq H$, so that EA^2E has nonempty interior. As $A \ni 1$ is a symmetric generating set for H, we have $E \subseteq A^k$ for some k and thus A^{2k+2} has nonempty interior. □

In this connection, we should mention one problem that Theorem 3 does not address, but which is certainly of high interest in the study of universally measurable sets.

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PROBLEM 21. *Is there a number $n \geqslant 1$ so that* $\mathrm{int}(A^n) \neq \emptyset$ *whenever $A \ni 1$ is a universally measurable, symmetric, right σ-syndetic subset of a Polish group?*

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