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ADDITION THEOREM OF ABEL TYPE FOR HYPER-LOGARITHMS

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Several kinds of generalizations of classical Abel Theorem in algebraic curves are known, for example see [12] and [13]. It seems to the author these are all regarded as local relations among rational differential forms. In this article we shall try to generalize *Abel Theorem* for integrals of rational forms in some specific cases where these can be described in terms of *hyper-logarithms* (for the definition see [3] and [4], Theorem 2). Trigonometric functions have been generalized to higher dimensional cases by L. Schläfli who has obtained a very important variational formula related to the volume of a spherical simplex [16]. In particular, the volume $V(\Delta)$ of a 3-dimensional double rectangular tetrahedron Δ with the dihedral angles α, β, γ, $\pi/2$, $\pi/2$, $\pi/2$ can be expressed in terms of the *di-logarithm* as follows: For

$$e^{-2i\mu} = \frac{\cos\alpha\cos\gamma - \sqrt{\sin^2\alpha\sin^2\gamma - \cos^2\beta}}{\cos\alpha\cos\gamma + \sqrt{\sin^2\alpha\sin^2\gamma - \cos^2\beta}},$$

$$\begin{aligned}\frac{1}{4}V(\Delta) = &\frac{1}{2}\left\{\psi(e^{-2i(\mu-\alpha)}) + \psi(e^{-2i(\mu+\alpha)})\right\}\\ &-\frac{1}{2}\left\{\psi(-e^{-2i(\mu+\beta)}) + \psi(-e^{-2i(\mu-\beta)})\right\}\\ &+\frac{1}{2}\left\{\psi(e^{-2i(\mu-\gamma)}) + \psi(e^{-2i(\mu+\gamma)})\right\}\\ &-\psi(-e^{-2i\mu}) - \left(\frac{\pi}{2}-\alpha\right)^2 + \beta^2 - \left(\frac{\pi}{2}-\gamma\right)^2,\end{aligned} \tag{0.1}$$

where $\psi(x)$ denotes the di-logarithm

$$\psi(x) = -\int_0^x \frac{\log(1-x)}{x}dx = -\int_0^x d\log x \cdot d\log(1-x)$$

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(see [9], p. 13). On the other hand it is known that $\psi(x)$ can be characterized by the functional equation [17]:

$$\psi\Big(\frac{xy}{(1-x)(1-y)}\Big) = \psi\Big(\frac{x}{1-x}\Big) + \psi\Big(\frac{y}{1-y}\Big) - \psi(x) - \psi(y) \\ - \log(1-x)(1-y)\,. \tag{0.2}$$

This equality comes from the *co-algebra property* of iterated integrals of 1-forms $\omega_1, \omega_2, \cdots, \omega_m$ along two paths γ, γ' connecting each other:

$$\int_{\gamma\cdot\gamma'} \omega_1 \cdots \omega_m = \sum_{p=0}^{m} \int_\gamma \omega_1 \cdots \omega_p \cdot \int_{\gamma'} \omega_{p+1} \cdots \omega_m\,. \tag{0.3}$$

This also corresponds to *the additive property of the volume* itself (see the Remark in § 1). To generalize it, we consider the integral of a differential form ω over a chain X:

$$W(X) = \int_X \omega\,, \tag{0.4}$$

ω being fixed and regarded as a functional of X, $W(X)$ has the "*additive property*":

$$W(X \cup Y) + W(X \cap Y) = W(X) + W(Y)\,, \tag{0.5}$$

which is a so-called "*content-mass*" function studied in detail by H. Hadwiger [14]. As is well-known, *the additive and invariant properties of* $W(X)$ also correspond to *the cocycle condition* for the cohomology of Lie groups (see for example, [10], [19]). In this article we shall show *these properties characterize hyper-logarithms* in the following two cases:

i) *Schläfli function, the volume of a spherical simplex, which has been discussed in* [2] and

ii) *hyper-logarithms associated with the configuration of hyperplanes in the real projective space* RP^n,

(see Theorem 1_n and Theorem 4). This will be done in the framework of *iterated integrals of differential* 1-*forms due to K. T. Chen* [7] and [8]. The cases a) and c) of Theorem 2 in [4] immediately follows from the above formulae (see [4], p. 356). The conformal case b) in [4] will be discussed in a forth-coming paper.

Actually Theorem 1_n can be regarded as an analytic and spherical version of the Hadwiger functional theorem [14], 221–222.

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§1. Addition formula of Schläfli function (orthogonal case)

Let $S_1, S_2, \cdots, S_{n+1}$ be arbitrary hyperplanes through the origin: $f_j(=\sum_{\nu=1}^{n+1} l_{j\nu}x_\nu) = 0$ in general position in R^{n+1}. We consider the integral

$$V(\Delta) = \int_\Delta \sum_{i=1}^{n+1} (-1)^{i-1} x_i dx_1 \wedge \cdots \langle i \rangle \cdots \wedge dx_{n+1}$$

over the spherical simplex Δ defined by $f_j \geqq 0$ in the unit sphere S^n. We denote by $\Delta_{i_1 i_2 \cdots i_p}$ the facettes defined by $\Delta \cap S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_p}$. Let A be the symmetric matrix $((a_{ij}))_{1 \leqq i, j \leqq n+1}$ with $a_{ij} = -\cos\langle i, j\rangle$ and $a_{ii} = 1$. Then $V(\Delta)$ is the *hyper-logarithm* with respect to A expressed by

$$(1.1) \qquad V(\Delta) = \sum_{I_0 \subset I_1 \subset \cdots \subset I_{[(n+1)/2]}} \sum_{\sigma=0}^{[(n+1)/2]} \int_E^A d\left\langle \begin{matrix} I_0 \\ I_1 \end{matrix} \right\rangle d\left\langle \begin{matrix} I_1 \\ I_2 \end{matrix} \right\rangle \cdots d\left\langle \begin{matrix} I_{\sigma-1} \\ I_\sigma \end{matrix} \right\rangle \frac{|S^{n-2\sigma}|}{2^{n+1-2\sigma}}$$

where $\left\langle \begin{matrix} I_p \\ I_{p+1} \end{matrix} \right\rangle$, $I_p = (i_1, \cdots, i_p)$ and $I_{p+1} = (i_1, \cdots, i_p, i_{p+1}, i_{p+2})$ represent the *dihedral angles* between $\Delta_{i_1 \cdots i_p} \cap S_{i_{p+1}}$ and $\Delta_{i_1 \cdots i_p} \cap S_{i_{p+2}}$ subtended by the $(n-p)$ simplex $\Delta_{i_1 i_2 \cdots i_p}$, and $|S^n|$ denotes the volume of the unit sphere. This is the simple consequence of the following formula due to L. Schläfli:

$$(1.2) \qquad dV(\Delta) = \sum_{1 \leqq i < j \leqq n+1} V_{ij}(\Delta) d\langle i, j\rangle$$

where $V_{ij}(\Delta)$ denotes $V(\Delta_{ij})$, [2] and [20].

Let $S_1, S_2, \cdots, S_{n+1}, S_{n+2}$ be arbitrary hyperplanes through the origin in general position in R^{n+1}, and Δ'_i, $1 \leqq i \leqq n+2$, be the n-simplices in S_i bounded by $S_1 \cup \cdots \cup S_{i-1} \cup S_{i+1} \cup \cdots \cup S_{n+1}$. Then the *content-mass property* of $V(\Delta)$ implies the following identity relation [6], [14]:

$$(1.3) \qquad \sum_{i=1}^{n+1} (-1)^{i-1} V(\Delta'_i) = 0\,.$$

$V(\Delta)$ is a locally analytic function of A whose singularities are in the loci $X_{i_1 \cdots i_p}$ defined by $\det A(i_1, \cdots, i_p) = 0$, where $\det A(i_1, \cdots, i_p)$, $1 \leqq i_1 < \cdots < i_p \leqq n+1$, $1 \leqq p \leqq n+1$ denote the subdeterminant with the i_1 th, $\cdots$, i_p th lines and columns of A.

Let A' be a symmetric matrix of order $(n+2)$, $((a'_{i,j}))_{1 \leqq i, j \leqq n+2}$, with $a_{ii} = 1$. We denote by $\tilde{A}'_i$, $1 \leqq i \leqq n+2$, the sub-determinant matrix obtained by deleting the ith line and column from A'. Then (1.3) can be stated as

PROPOSITION 1.2. *The strong additive relation*

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$$\sum_{i=1}^{n+2}(-1)^{i-1}V(A_i') \equiv 0 \ (\det A')^{n/2} \tag{1.4}$$

holds, where T_i *denotes* $\mathrm{Diag}\,[\underbrace{-1, \cdots, -1}_{i-1}, \underbrace{1, \cdots, 1}_{n+2-i}]$ *and* $A_i' = T_i\tilde{A}_i'T_i$. *In particular, the weak additive relation*:

$$\det A' = 0\,, \tag{R1. n}$$

implies

$$\sum_{i=1}^{n+2}(-1)^{i-1}V(A_i') = 0\,. \tag{1.5}$$

We are now going to prove the converse of the preceding Proposition. In fact we have

THEOREM 1_n. *Let* $F(A)$ *be a continuous and locally analytic function of* A *in the domain* $X = \{A \in R^{n(n+1)/2} | A \geqq 0\}$ *satisfying the following condition* (H1. n): *i)* $F(A) = 0$ *if* $\det A = 0$ *and* $a_{i,j} < 0$, $1 \leqq i, j \leqq n+1$,

ii) *symmetry property, namely* $F(\sigma A) = F(A)$ *where* σ *denotes an arbitrary permutation of the indices* $1, 2, \cdots, n+1$ *such that* $(\sigma^{-1}A)_{i,j} = A_{\sigma(i)\sigma(j)}$.

iii) *the strong additive property* (R2. n), *namely*

$$\sum_{i=1}^{n+2}(-1)^{i-1}F(A_i') = 0 \mod (\det A')^{n/2}\,. \tag{1.4*}$$

Then $F(A)$ *is a constant multiple of* $V(A)$ *in* X.

It is unknown to the author if the weak additivity implies the strong one or not. This kind of theorem has been implicitly investigated by W. Maier and A. Effenberger [18].

Proof of the Theorem. In view of Gauss-Bonnet theorem [2] the even case is reduced to the odd one. So we have only to prove it in the odd case. We shall prove the Theorem by induction with respect to n. When $n = 1$, the Theorem is nothing but the well-known addition theorem of *arccos x*.

1*st step.* LEMMA 1.1. *Assume* $n \geqq 3$. *Suppose that the Theorem* 1_k *has been proved for* $k < n$. *Then the differential* $dF(A)$ *can be expressed as*

$$\begin{aligned} dF(A) = \sum_{i<j} V_{ij}(A)\cdot d\langle i,j\rangle \cdot \\ \varphi(\langle i,j\rangle, \langle i,1\rangle, \langle i,2\rangle, \cdots \hat{j} \cdots, \langle i,n+1\rangle; \\ \langle j,1\rangle, \cdots \hat{i} \cdots, \langle j,n+1\rangle) \end{aligned} \tag{1.6}$$

where φ *denotes a suitable locally analytic function on* X.

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Proof. The function $F_{ij} = \partial F(A)/\partial\langle i, j\rangle$ represents a locally analytic function of the simplex Δ_{ij} with the dihedral angles $\left\langle {ij \atop ijkl} \right\rangle$ and satisfies the hypothesis (H1. $n-2$). In fact (1.5) implies

$$\sum_{\substack{k\neq i,j\\ i<j}} F_{ij}(A'_\kappa)(-1)^{k-1} d\langle i, j\rangle \equiv 0 \bmod (\det A')^{(n-1)/2}, \tag{1.7}$$

in other words,

$$\sum_{k\neq i,j} F_{ij}(A'_k)(-1)^{k-1} \equiv 0 \bmod (\det A')^{(n-1)/2}. \tag{1.8}$$

By induction hypothesis, as a function of $\left\langle {ij \atop ijkl} \right\rangle$, i, j fixed, F_{ij} is equal to a constant multiple of $V_{ij}(A)$, so that there exists a function of the $2n-1$ variables $\langle i, j\rangle$, $\langle i, \lambda\rangle$, $\langle j, \lambda\rangle$ for $\lambda \in \{1, 2, \cdots, i, \cdots, n+1\} - \{i, j\}$ such that

$$F_{ij}(A) = V_{ij}(A)\cdot \varphi_{ij}(\langle i, j\rangle, \langle i, \lambda\rangle, \langle j, \lambda\rangle). \tag{1.9}$$

Because of the symmetry

$$F_{\sigma(i)\sigma(j)}(\sigma^{-1}A) = F_{ij}(A), \qquad V_{\sigma(i)\sigma(j)}(\sigma^{-1}A) = V_{ij}(A), \tag{1.10}$$

we have

$$\varphi_{\sigma(i)\sigma(j)}(\sigma\langle i, j\rangle;\ \sigma\langle i, \lambda\rangle, \sigma\langle j, \lambda\rangle) = \varphi_{ij}(\langle i, j\rangle;\ \langle i, \lambda\rangle, \langle j, \lambda\rangle) \tag{1.11}$$

namely

$$\varphi_{\sigma(i)\sigma(j)} = \varphi_{ij}, \tag{1.12}$$

which we shall denote by φ. The Lemma has been proved.

LEMMA 1.2. *When $n \geqq 5$, φ_{ij} is constant and does not depend on i, j.*

Proof. Let i, j, k, l arbitrary different indices in $\{1, 2, \cdots, n+1\}$. Then

$$\partial F_{ij}/\partial\langle k, l\rangle = \partial F_{kl}/\partial\langle i, j\rangle, \tag{1.13}$$

which implies

$$\varphi_{ij}(\langle i, j\rangle;\ \langle i, \lambda\rangle, \langle j, \lambda\rangle) = \varphi_{kl}(\langle k, l\rangle;\ \langle k, \lambda\rangle, \langle l, \lambda\rangle) \tag{1.14}$$

in view of the equality

$$\partial V_{ij}/\partial\langle k, l\rangle = \partial V_{kl}/\partial\langle i, j\rangle. \tag{1.15}$$

Consequently both sides in (1.14) depend only on $\langle i, k\rangle$, $\langle i, l\rangle$, $\langle j, k\rangle$, $\langle j, l\rangle$. Let $i_1, i_2, \cdots, i_6$ be different indices in $\{1, 2, \cdots, n+1\}$. Then (1.14) shows

$$(1.16)\qquad \varphi(\langle i_1, i_5\rangle, \langle i_1, i_6\rangle; \langle i_2, i_5\rangle, \langle i_2, i_6\rangle) = \varphi(\langle i_3, i_5\rangle, \langle i_3, i_6\rangle; \langle i_4, i_5\rangle, \langle i_4, i_6\rangle)$$

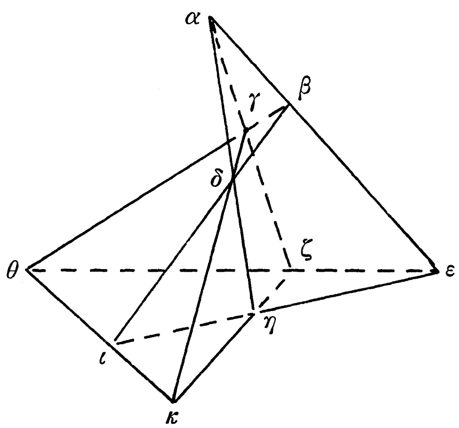
which implies φ is constant.

COROLLARY. *When $n = 4$, for arbitrary different indices i, j, k, l we have the relations*:

$$(1.17)\qquad \begin{aligned} &\text{i)}\ \ \varphi(\langle i, k\rangle, \langle i, l\rangle; \langle j, k\rangle, \langle j, l\rangle) = \varphi(\langle i, l\rangle, \langle i, k\rangle; \langle j, l\rangle, \langle j, k\rangle) \\ &\text{ii)}\ \ \varphi(\langle i, k\rangle, \langle j, k\rangle; \langle i, l\rangle, \langle j, l\rangle) = \varphi(\langle i, k\rangle, \langle i, l\rangle; \langle j, k\rangle, \langle j, l\rangle) \end{aligned}$$

LEMMA 1.3. *Theorem 1_3 holds.*

Proof. We put $\varphi(\alpha, \beta, \gamma, \pi/2) = \varphi_1(\alpha, \beta, \gamma)$, $\varphi(\alpha, \beta, \pi/2, \pi/2) = \varphi_2(\alpha, \beta)$, $\varphi(\alpha, \pi/2, \pi/2, \pi/2) = \varphi_3(\alpha)$ and $\varphi(\pi/2, \pi/2, \pi/2, \pi/2) = \varphi_4$. Consider a configuration of hyperplanes in S^3 as in the figure which has 10 vertices $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \iota, \kappa$, such that we have the 5 simplices $\varDelta_1' = [2, 3, 4, 5]$, $\varDelta_2' = [1, 3, 4, 5]$, $\varDelta_3' = [1, 2, 4, 5]$, $\varDelta_4' = [1, 2, 3, 5]$ and $\varDelta_5' = [1, 2, 3, 4]$ and that $\{\varepsilon, \zeta, \eta, \theta, \iota, \kappa\} \subset S_1$, $\{\beta, \gamma, \delta, \theta, \iota, \kappa\} \subset S_2$, $\{\alpha, \gamma, \delta, \zeta, \eta\} \in S_3$, $\{\alpha, \beta, \delta, \varepsilon, \eta\} \in S_4$, $\{\alpha, \beta, \gamma, \varepsilon, \zeta\} \subset S_5$. The system of angles subtended by each $\varDelta_i'$ are as follows:



$\varDelta_1'$: $\langle 2, 3\rangle$, $\langle 2, 4\rangle$, $\langle 2, 5\rangle$, $\langle 3, 4\rangle$, $\langle 3, 5\rangle$, $\langle 4, 5\rangle$
$\varDelta_2'$: $\langle 3, 4\rangle$, $\langle 4, 5\rangle$, $\langle 3, 5\rangle$, $\pi - \langle 3, 1\rangle$, $\pi - \langle 4, 1\rangle$, $\pi - \langle 5, 1\rangle$
$\varDelta_3'$: $\langle 1, 2\rangle$, $\pi - \langle 1, 4\rangle$, $\pi - \langle 1, 5\rangle$, $\pi - \langle 2, 4\rangle$, $\pi - \langle 2, 5\rangle$, $\langle 4, 5\rangle$,
$\varDelta_4'$: $\langle 1, 2\rangle$, $\langle 1, 3\rangle$, $\pi - \langle 1, 5\rangle$, $\langle 2, 3\rangle$, $\pi - \langle 2, 5\rangle$, $\pi - \langle 3, 5\rangle$,
$\varDelta_5'$: $\langle 1, 2\rangle$, $\langle 1, 3\rangle$, $\langle 1, 4\rangle$, $\langle 2, 3\rangle$, $\langle 2, 4\rangle$, $\langle 3, 4\rangle$.

We have only to prove that (1.6) *determines φ in a unique way except for a constant multiple.*

1*st step*. We assume $\langle 1,5\rangle=\langle 2,5\rangle=\langle 3,5\rangle=\langle 4,5\rangle=\pi/2$ so that A' has the following expression:

$$A'=\begin{vmatrix} 1, & a_{12}, & a_{13}, & a_{14}, & 0 \\ a_{21}, & 1, & a_{23}, & a_{24}, & 0 \\ a_{31}, & a_{32}, & 1, & a_{34}, & 0 \\ a_{41}, & a_{42}, & a_{43}, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1 \end{vmatrix} \tag{1.18}$$

(R1. 3) shows

$$0=(1-a_{13}^2-a_{14}^2)(1-a_{23}^2-a_{24}^2)-(a_{12}-a_{13}\cdot a_{23}-a_{14}\cdot a_{24})^2\,. \tag{1.19}$$

Owing to (1.8) we have

$$\begin{aligned} &\{V_{12}(\Delta_3')\varphi_2(\langle 1,4\rangle,\langle 2,4\rangle)-V_{12}(\Delta_4')\varphi_2(\langle 1,3\rangle,\langle 2,3\rangle)\\ &\qquad\qquad +V_{12}(\Delta_5')\varphi(\langle 1,3\rangle,\langle 1,4\rangle;\langle 2,3\rangle,\langle 2,4\rangle)\}d\langle 1,2\rangle\\ &\quad+\{V_{13}(\Delta_2')\varphi_3(\langle 1,4\rangle)-V_{13}(\Delta_4')\cdot\varphi_2(\langle 1,2\rangle,\langle 3,2\rangle)\\ &\qquad\qquad +V_{13}(\Delta_5')\varphi_1(\langle 1,2\rangle,\langle 1,4\rangle,\langle 3,2\rangle)\}d\langle 1,3\rangle\\ &\quad+\{V_{14}(\Delta_2')\varphi_3(\langle 1,3\rangle)-V_{14}(\Delta_3')\varphi_2(\langle 1,2\rangle,\langle 4,2\rangle)\\ &\qquad\qquad +V_{14}(\Delta_5')\varphi_1(\langle 1,2\rangle,\langle 1,3\rangle,\langle 4,2\rangle)\}d\langle 1,4\rangle\\ &\quad+\{V_{23}(\Delta_1')\varphi_3(\langle 2,4\rangle)-V_{23}(\Delta_4')\varphi_2(\langle 2,1\rangle,\langle 3,1\rangle)\\ &\qquad\qquad +V_{23}(\Delta_5')\varphi_1(\langle 2,1\rangle,\langle 2,4\rangle,\langle 3,1\rangle)\}d\langle 2,3\rangle\\ &\quad+\{V_{24}(\Delta_1')\varphi_3(\langle 2,3\rangle)-V_{24}(\Delta_3')\varphi_2(\langle 2,1\rangle,\langle 4,1\rangle)\\ &\qquad\qquad +V_{24}(\Delta_5')\varphi_1(\langle 2,1\rangle,\langle 2,3\rangle,\langle 4,1\rangle)\}d\langle 2,4\rangle\\ &\quad+\{V_{34}(\Delta_1')\varphi_2(\langle 3,2\rangle,\langle 4,2\rangle)-V_{34}(\Delta_2')\varphi_2(\langle 3,1\rangle,\langle 4,1\rangle)\\ &\qquad\qquad +V_{34}(\Delta_5')\cdot\varphi(\langle 3,1\rangle,\langle 3,2\rangle;\langle 4,1\rangle,\langle 4,2\rangle)\}\\ &\quad\cdot d\langle 3,4\rangle=0\,. \end{aligned} \tag{1.20}$$

In view of (1.19) we can take as independent variables $\langle 1,3\rangle$, $\langle 1,4\rangle$, $\langle 2,3\rangle$, $\langle 2,4\rangle$ and $\langle 3,4\rangle$, so that we have

$$\begin{aligned} &V_{12}(\Delta_5')\partial\langle 1,2\rangle/\partial\langle 1,3\rangle\cdot\varphi(\langle 1,3\rangle,\langle 1,4\rangle;\langle 2,3\rangle,\langle 2,4\rangle)\\ &\quad=\{-V_{12}(\Delta_3')\varphi_2(\langle 1,4\rangle,\langle 2,4\rangle)\\ &\qquad\qquad +V_{12}(\Delta_4')\varphi_2(\langle 1,3\rangle,\langle 2,3\rangle)\}\partial\langle 1,2\rangle/\partial\langle 1,3\rangle\\ &\quad\quad-\{V_{13}(\Delta_2')\varphi_3(\langle 1,4\rangle)-V_{13}(\Delta_4')\varphi_2(\langle 1,2\rangle,\langle 3,2\rangle)\\ &\qquad\qquad +V_{13}(\Delta_5')\varphi_3(\langle 1,2\rangle,\langle 1,4\rangle,\langle 3,2\rangle)\}\,. \end{aligned} \tag{1.21}$$

Because $V_{12}(\Delta_3')\cdot\partial\langle 1,2\rangle/\partial\langle 1,3\rangle\neq 0$, the local analytic function φ can be expressed by means of the functions $V_{ij}(A)$, φ_1, φ_2, φ_3 and φ_4.

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2nd step. We assume further $\langle 1,4\rangle=\pi/2$, namely $a_{14}=0$. Then (1.19) becomes

$$0=(1-a_{13}^2)(1-a_{23}^2-a_{24}^2)-(a_{12}-a_{13}\cdot a_{23})^2\,. \tag{1.22}$$

We can take $\langle 1,2\rangle$, $\langle 1,3\rangle$, $\langle 2,3\rangle$, and $\langle 3,4\rangle$ as independent variables. By (1.21) again, φ_1 can be expressed by means of $V_{ij}(A)$ and $\varphi_2, \varphi_3, \varphi_4$.

3rd step. Let A have the form

$$A=\begin{bmatrix} 1 & a_{12} & a_{13} & 0 \\ a_{21} & 1 & a_{23} & 0 \\ a_{31} & a_{32} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

then (1.6) becomes

$$\begin{aligned} dF(A) &= \frac{\pi}{2}\cdot\varphi_2(\langle 1,3\rangle,\langle 2,3\rangle)d\langle 1,2\rangle \\ &\quad+\frac{\pi}{2}\cdot\varphi_2(\langle 1,2\rangle,\langle 3,2\rangle)d\langle 1,3\rangle \\ &\quad+\frac{\pi}{2}\cdot\varphi_2(\langle 2,1\rangle,\langle 3,1\rangle)\mathrm{d}\langle 2,3\rangle \end{aligned} \tag{1.23}$$

because $V_{12}(A)=V_{13}(A)=V_{23}(A)=\pi/2$. The integrability condition shows

$$\frac{\partial\varphi_2(\langle 1,3\rangle,\langle 2,3\rangle)}{\partial\langle 1,3\rangle}=\frac{\partial\varphi_2(\langle 1,2\rangle,\langle 3,2\rangle)}{\partial\langle 1,2\rangle}\,. \tag{1.24}$$

As a consequence the left hand side is independent of $\langle 1,3\rangle$. φ_2 being symmetric, $\varphi_2(\langle 1,3\rangle,\langle 2,3\rangle)$ can be described as

$$\varphi_2(\langle 1,3\rangle,\langle 2,3\rangle)=c_0(\langle 1,3\rangle+\langle 2,3\rangle)+c_1 \tag{1.25}$$

where $\varphi_4=c_0\pi+c_1$ and $\varphi_3(\langle 1,3\rangle)=(c_0(\pi/2)+c_1)+c_0\langle 1,3\rangle$.

4th step. Let A' have

$$A'=\begin{bmatrix} 1 & a_{12} & 0 & a_{14} & 0 \\ a_{21} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ a_{41} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.26}$$

with the condition (R1. 3) $1-a_{12}^2-a_{14}^2=0$. By considering the solid angle subtended by Δ_5' at ι (see the figure), we have

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$$\langle 1, 2\rangle + \langle 1, 4\rangle = \langle 2, 4\rangle = \frac{\pi}{2}. \tag{1.27}$$

(1.20) becomes

$$\begin{aligned} &\{V_{12}(\varDelta_3')\varphi_3(\langle 1, 4\rangle) - V_{12}(\varDelta_4')\varphi_4 + V_{12}(\varDelta_5')\varphi_3(\langle 1, 4\rangle)\}d\langle 1, 2\rangle \\ &\quad + \{V_{14}(\varDelta_2')\varphi_4 - V_{14}(\varDelta_3')\varphi_3(\langle 1, 2\rangle) + V_{14}(\varDelta_5')\varphi_3(\langle 1, 2\rangle)\}d\langle 1, 4\rangle = 0 \end{aligned} \tag{1.28}$$

where $V_{12}(\varDelta_3') = V_{12}(\varDelta_4') = V_{12}(\varDelta_5') = \pi/2$, and $-V_{14}(\varDelta_2') = -V_{14}(\varDelta_3') = V_{14}(\varDelta_5') = \pi/2$, namely

$$d\langle 1, 2\rangle\{2\varphi_3(\langle 1, 4\rangle) - \varphi_4\} + d\langle 1, 4\rangle\{2\varphi_3(\langle 1, 2\rangle) - \varphi_4\} = 0\,. \tag{1.29}$$

From (1.27) we have $d\langle 1, 2\rangle(\varphi_3(\langle 1, 4\rangle) - \varphi_3(\langle 1, 2\rangle)) = 0$, which implies c_0 vanishes. Theorem has been completely proved.

Remark. Let A' be the following Jacobi matrix

$$A' = \begin{pmatrix} 1 & -\cos\alpha & & & \\ -\cos\alpha & 1 & -\cos\beta & & \\ & -\cos\beta & 1 & -\cos\gamma & \\ & & -\cos\gamma & 1 & -\cos\delta \\ & & & -\cos\delta & 1 \end{pmatrix}. \tag{1.30}$$

Then the sub-matrices A_1', A_2', A_3', A_4' and A_5' define 5 double rectangular tetrahedra $\varDelta_1$, $\varDelta_2$, $\varDelta_3$, $\varDelta_4$ and $\varDelta_5$ respectively. Each volume can be described as follows (see [9] p. 13):

$$\begin{aligned} \frac{1}{4}V(\varDelta_1) = &\frac{1}{2}\{\psi(e^{-2i(\lambda-\beta)}) + \psi(e^{-2i(\lambda+\beta)})\} \\ &-\frac{1}{2}\{\psi(-e^{-2i(\lambda+\gamma)}) + \psi(-e^{-2i(\lambda-\gamma)})\} \\ &+\frac{1}{2}\{\psi(e^{-2i(\lambda-\delta)}) + \psi(e^{-2i(\lambda+\delta)})\} \\ &-\psi(-e^{-2i\lambda}) - \left(\frac{\pi}{2} - \beta\right)^2 + \gamma^2 - \left(\frac{\pi}{2} - \delta\right)^2, \end{aligned} \tag{1.31}$$

$$V(\varDelta_2) = \frac{\pi}{4}\left(\gamma + \delta - \frac{\pi}{2}\right), \tag{1.32}$$

$$V(\varDelta_3) = \frac{\alpha\delta}{2}, \tag{1.33}$$

$$V(\varDelta_4) = \frac{\pi}{4}\left(\alpha + \beta - \frac{\pi}{2}\right), \tag{1.34}$$

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and $(1/4)V(\Delta_5)$ is the same as (0.1), where $e^{-2i\lambda}$ is defined to be

$$\frac{\cos\beta\cdot\cos\delta-\sqrt{\sin^2\beta\cdot\sin^2\delta-\cos^2\gamma}}{\cos\beta\cdot\cos\delta-\sqrt{\sin^2\beta\cdot\sin^2\delta-\cos^2\gamma}}\,.$$

We assume now that the determinant of A' vanishes, namely

$$\begin{aligned}(\mathrm{R1.\,4})\qquad 0=1&-\cos^2\alpha-\cos^2\beta-\cos^2\gamma-\cos^2\delta\\&+\cos^2\alpha\cdot\cos^2\gamma+\cos^2\beta\cdot\cos^2\delta+\cos^2\alpha\cdot\cos^2\delta\,.\end{aligned}$$

Then, according to (1.5), the following equality holds:

$$\begin{aligned}(1.35)\qquad V(\Delta_1)+V(\Delta_5)&=V(\Delta_2)-V(\Delta_3)+V(\Delta_4)\\&=\frac{\pi}{4}(\alpha+\beta+\gamma+\delta-\pi)-\frac{\alpha\delta}{2}\,.\end{aligned}$$

We assume further that $\beta+\gamma=\pi/2$. Then (R1. 4) is equivalent to the following:

$$(\mathrm{R1.\,4})'\qquad \cos^2\delta=\frac{-\cos^2\alpha\cdot\cos^2\beta}{1-\cos^2\alpha-\cos^2\beta}\,.$$

Therefore $e^{-2i\lambda}$, $e^{-2i\mu}$, $e^{-2i\gamma}$ and $e^{-2i\delta}$ all become rational functions of $e^{2i\alpha}$, $e^{2i\beta}$ as follows:

$$(1.36)\qquad e^{-2i\lambda}=-e^{-2i\mu}=-e^{-2i\gamma}=e^{2i\beta}\,,$$

and

$$(1.37)\qquad e^{2i\delta}=\frac{e^{2i\alpha}+e^{2i\beta}}{1+e^{2i\alpha}\cdot e^{2i\beta}}\,.$$

Then (1.36) becomes

$$\begin{aligned}(1.38)\qquad &\frac{1}{2}\{\psi(e^{2i\beta}\cdot e^{2i\delta})+\psi(e^{2i\beta}\cdot e^{-2i\delta})+\psi(-e^{2i\alpha})+\psi(-e^{-2i\alpha})\}\\&\qquad-\psi(-e^{2i\beta})-\psi(e^{2i\beta})-\left(\frac{\pi}{2}-\delta\right)^2-\left(\frac{\pi}{2}-\alpha\right)^2\\&=\frac{\pi}{4}\left(\alpha+\delta-\frac{\pi}{2}\right)-\frac{\alpha\delta}{2}\,.\end{aligned}$$

This is equivalent to the identity (0.2).

§ 2. Hyper-logarithms in projective case

Let M be the complement $C^n-S^+\cup S^-$ where S^+ and S^- denote the union of hyperplanes $S_j: f_j = 0$, $1 \leqq j \leqq m$, and $-n \leqq j \leqq 0$ respectively which are in general position. We denote $\Omega^{\cdot}(*S^+)$ the space of rational forms whose poles are located in S^+. The following is well-known (see [4]).

LEMMA 2.1. *An arbitrary n-form $\Omega^n(*S)$ is rationally homologous to a linear combination of logarithmic forms*

$$\varphi_{i_1 i_2 \cdots i_n} = df_{i_1}/f_{i_1} \wedge \cdots \wedge df_{i_n}/f_{i_n}, \qquad 1 \leqq i_1 < \cdots < i_n \leqq m . \tag{2.1}$$

From now on we shall assume $f_{-n}, f_{-(n-1)}, \cdots, f_0$ are all real such that the region

$$f_{-n} \geqq 0, \qquad f_{-n+1} \geqq 0, \cdots, f_0 \geqq 0 \tag{2.2}$$

defines a n-simplex Δ with its facettes $\Delta_{j_1 j_2 \cdots j_p} = S_{j_1} \cap \cdots \cap S_{j_p} \cap \Delta$, $-n \leqq j_1 < j_2 < \cdots < j_p \leqq 0$, $0 \leqq p \leqq n$. For each sequence of indices $I = \{i_1, \cdots, i_n\}$ consider the integral

$$\hat{\varphi}_{i_1 i_2 \cdots i_n} = \int_\Delta \varphi_{i_1 i_2 \cdots i_n} . \tag{2.3}$$

We denote by $[\alpha_1, \alpha_2, \cdots, \alpha_{n+1}]$ and $[\infty, \beta_1, \cdots, \beta_n]$ the determiniants

$$\begin{vmatrix} a_{\alpha_1 0}, & a_{\alpha_1 1}, & \cdots, & a_{\alpha_1 n} \\ a_{\alpha_2 0}, & a_{\alpha_2 1}, & \cdots, & a_{\alpha_2 n} \\ & \cdots\cdots & & \\ a_{\alpha_{n+1} 0}, & a_{\alpha_{n+1} 1}, & \cdots, & a_{\alpha_{n+1} n} \end{vmatrix}, \qquad \begin{vmatrix} a_{\beta_1 1}, & \cdots, & a_{\beta_1 n} \\ a_{\beta_2 1}, & \cdots, & a_{\beta_2 n} \\ & \cdots\cdots & \\ a_{\beta_n 1}, & \cdots, & a_{\beta_n n} \end{vmatrix}$$

respectively for $f_j = \sum_{\nu=1}^n a_{j\nu} x_\nu + a_{j0}$, with the conditions $a_{j0}^2 + \sum_{\nu=1}^n a_{j\nu}^2 = 1$. Then $\hat{\varphi}(i_1, \cdots, i_n)$ is analytic function on *the configuration space* X of S, parametrized by Plücker coordinates $[\alpha_1, \alpha_2, \cdots, \alpha_{n+1}]$ and $[\infty, \beta_1, \cdots, \beta_n]$. We have the following formula analogous to Schläfli's:

LEMMA 2.2.

$$\begin{aligned} d\hat{\varphi}(i_1, \cdots, i_n) = \sum_{-n \leqq j \leqq 0} \sum_{\sigma=1}^{n} (-1)^\sigma \int_{\Delta_j} df_{i_1}/f_{i_1} \wedge \cdots \langle \sigma \rangle \cdots \wedge df_{i_n}/f_{i_n} \\ \cdot d \log([j, i_1, \cdots, i_n]/[\infty, j, i_1, \cdots \langle \sigma \rangle \cdots, i_n]) . \end{aligned} \tag{2.4}$$

Proof. The integral

$$\hat{\varphi}_\lambda(i_1, \cdots, i_n) = \int_\Delta f_{-n}^{\lambda_0} f_{-n+1}^{\lambda_1} \cdots f_{-1}^{\lambda_{n-1}} f_0^{\lambda_n} \cdot \varphi(i_1, \cdots, i_n) \tag{2.5}$$

satisfies the following Gauss-Manin connection:

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$$d\hat{\varphi}_\lambda(i_1, \cdots, i_n) = \sum_{i_0 \notin \{i_1, \cdots, i_n\}} \sum_{\sigma=0}^{n} (-1)^\sigma \lambda_{i_0} \hat{\varphi}_\lambda(i_0, i_1, \cdots \langle\sigma\rangle \cdots, i_n) \cdot d\log([i_0, i_1, \cdots, i_n]/[\infty, i_0, i_1, \cdots \langle\sigma\rangle \cdots, i_n]) \tag{2.6}$$

(see [1] p. 60). On the other hand for any $(n-1)$-form $\psi \in \Omega^{n-1}(*S)$ we have

$$\lim_{\lambda_{i_0} \to 0} \lambda_{i_0} \int_\Delta (df_{i_0}/f_{i_0}) \wedge \psi = \int_{\Delta_{i_0}} \psi \tag{2.7}$$

if $i_0 \in \{-n, \cdots, -1, 0\}$, equal to zero otherwise. (2.6) and (2.7) imply the Lemma.

For $-n \leqq j_0 < j_1 < \cdots < j_{p-1} \leqq 0$, $1 \leqq i_1 < i_2 < \cdots < i_{n-p} \leqq n$, put

$$\hat{\varphi}(i_1, \cdots, i_{n-p}; j_0, j_1, \cdots, j_{p-1}) = \int_{\Delta_{j_0 j_1 \cdots j_{p-1}}} \varphi(i_1, \cdots, i_{n-p}) \tag{2.8}$$

where $\varphi(i_1, \cdots, i_{n-p})$ denotes the form $df_{i_1}/f_{i_1} \wedge \cdots \wedge df_{i_{n-p}}/f_{i_{n-p}}$. These are symmetric or skew-symmetric with respect to $i_1, \cdots, i_{n-p}$ or $j_0, j_1, \cdots, j_{p-1}$ respectively. Applying Lemma 2.2 repeatedly we have

LEMMA 2.3.

$$\begin{aligned} &d\hat{\varphi}(i_1, \cdots, i_{n-p}; j_0, j_1, \cdots, j_{p-1}) \\ &= \int_0^\xi \sum_{j \notin \{-n, -n+1, \cdots, 0\} - \{j_0, \cdots, j_{p-1}\}} \sum_{\sigma=1}^{n-p} (-1)^\sigma d\log [j, j_0, \cdots, j_{p-1}, i_1, \cdots, i_{n-p}]/ [\infty, j, j_0, \cdots, j_{p-1}, i_1, \cdots \langle\sigma\rangle \cdots, i_{n-p}] \\ &\quad \cdot \hat{\varphi}(i_1, \cdots \langle i_\sigma \rangle \cdots i_{n-p}, j, j_0, \cdots, j_{p-1}), \end{aligned} \tag{2.9}$$

where ξ denotes an arbitrary point of X and 0 *the point of X such that Δ shrinks to a point where* $\det[-n, -n+1, \cdots, 1, 0] = 0$.

Combining Lemmas 2.2 and 2.3 we have the following expression by means of *hyper-logarithms*:

PROPOSITION 2.

$$\begin{aligned} \hat{\varphi}(i_1, \cdots, i_n) = &\sum_{\substack{(j_0, j_1, \cdots, j_{n-1}) \\ (\sigma_1, \sigma_2 \cdots, \sigma_n)}} (-1)^{n(n+1)/2} \cdot \operatorname{sgn}(\sigma_1, \cdots, \sigma_n) \\ &\cdot \int_0^\xi d\log \frac{[j_0, i_1, \cdots, i_n]}{[\infty, j_0, i_1, \cdots \langle i_{\sigma_n} \rangle \cdots, i_n]} \\ &\cdot d\log \frac{[j_0, j_1, i_1 \cdots \langle i_{\sigma_n} \rangle \cdots, i_n]}{[\infty, j_0, j_1, i_1, \cdots \langle i_{\sigma_{n-1}} \rangle \cdots \langle i_{\sigma_n} \rangle \cdots, i_n]} \cdots \\ &\cdot d\log \frac{[j_0, j_1, \cdots, j_{n-1}, i_1, \cdots, \langle i_{\sigma_1} \rangle \cdots \langle i_{\sigma_2} \rangle \cdots \langle i_{\sigma_{n-1}} \rangle \cdots, i_n]}{[\infty, j_0, \cdots, j_{n-1}]} \end{aligned} \tag{2.10}$$

where $(j_0, j_1, \cdots, j_{n-1})$ *or* $(\sigma_1, \cdots, \sigma_n)$ *run over all the different sequences of indices in* $\{-n, -n+1, \cdots, -1, 0\}$ *or* $\{1, 2, \cdots, n\}$ *respectively. The right hand side represents iterated integrals in the sense of K. T. Chen* (see [3] and also [7]).

Proposition 2 shows immediately the following which can be regarded as a generalization of Abel Theorem, in terms of hyper-logarithms instead of logarithms:

THEOREM 2. *For any* $\omega \in \Omega^n(*S)$, *as a function on* X, *the integral*

$$\int_{\Delta} \omega \tag{2.11}$$

can be described as a sum of

$$(\textit{rational functions}) \times (\textit{hyper-logarithms at most n-th order}), \tag{2.12}$$

whose singularities are all located in the union Y *of the subsets defined by*

$$[\alpha_1, \alpha_2, \cdots, \alpha_{n+1}] = 0 \text{ and } [\infty, \beta_1, \cdots, \beta_n] = 0, \quad -n \leqq \alpha_1 < \cdots < \alpha_{n+1} \leqq m$$

and $-n \leqq \beta_1 < \cdots < \beta_n \leqq m$ *respectively.*

Proof. Owing to Lemma 2.1 and Stokes formula, (2.11) turns out to be equal to a linear combination of $\hat{\varphi}(i_1, \cdots, i_n)$ and integrals of rational forms *over lower dimensional simplices.* Repeating this procedure for the latter step by step, we arrive at the Theorem.

Consider *the de Rham algebra* $\Omega(X, \log\langle Y\rangle)$ *generated by* $d\log[i_0, i_1, \cdots, i_n]$ *for* $\{i_0, i_1, \cdots, i_n\} \subset K = \{\infty, -n, -n+1, \cdots, -1, 0, 1, \cdots, m\}$ so that $\Omega^1(X, \log\langle Y\rangle)$ is spanned by $d\log[i_0, i_1, \cdots, i_n]$. All the exterior products $d\log[i_0, i_1, \cdots, i_n] \wedge d\log[j_0, j_1, \cdots, j_n]$ are not linearly independent, but the following relations hold: *For any choice of indices* α, β, γ, δ *and* $i_1, \cdots, i_{n-1}$ *in* K *put*

$$\begin{aligned}
&\theta(\alpha, \beta, \gamma, \delta;\ i_1, \cdots, i_{n-1}) \\
&\quad = d\log[\alpha, \beta, i_1, \cdots, i_{n-1}] \wedge d\log[\alpha, \gamma, i_1, \cdots, i_{n-1}] \\
&\qquad + d\log[\alpha, \gamma, i_1, \cdots, i_{n-1}] \wedge d\log[\alpha, \delta, i_1, \cdots, i_{n-1}] \\
&\qquad + d\log[\alpha, \delta, i_1, \cdots, i_{n-1}] \wedge d\log[\alpha, \beta, i_1, \cdots, i_{n-1}]\,,
\end{aligned} \tag{2.13}$$

then

LEMMA 2.4.

$$\begin{aligned}
&\theta(\alpha, \beta, \gamma, \delta; i_1, \cdots, i_{n-1}) - \theta(\beta, \gamma, \delta, \alpha; i_1, \cdots, i_{n-1}) \\
&\quad + \theta(\gamma, \delta, \alpha, \beta; i_1, \cdots, i_{n-1}) - \theta(\delta, \alpha, \beta, \gamma; i_1, \cdots, i_{n-1}) = 0\,.
\end{aligned} \tag{2.14}$$

Proof. Logarithmic forms of the above type are determined by their residues. We have only to prove all the residues of the left hand side vanish. This can be easily done.

According to Chen's formula about the differentiation of iterated integrals [7] Prop. 4.1.2, (2.14) shows that *the iterated integrals* (2, 10) *depends only on homotopy classes of paths from* 0 *to* ξ.

Remark. (2.4) gives the fundamental relations in $\Omega^2(X, \log\langle Y\rangle)$ with respect to the generators $d\log[i_1, i_2, \cdots, i_{n+1}]$. In view of the general principle of $K(\Pi, 1)$ spaces, one may ask the following questions: *Is* $\Omega^{\cdot}(X, \log\langle Y\rangle)$ *isomorphic to the singular cohomology of* $X - Y$?

Is the space $X - Y$ $K(\Pi, 1)$? *Does* (2.14) *give a fundamental system of relations of* $\Omega^{\cdot}(X, \log\langle Y\rangle)$ *with respect to* $d\log[i_1, i_2, \cdots, i_{n+1}]$? When $n = 1$, it is well-known that these are true. For the more extensive treatment see [15].

Let $\mathfrak{g}$ be the holonomy Lie algebra over $\boldsymbol{C}$ generated by the symbols $u_{i_1, i_2 \cdots i_{n+1}}$ which have the *fundamental relations* (*integrability condition for the connection form* $\sum_{\{i_1, i_2, \cdots, i_{n+1}\} \subset K} u_{i_1, i_2, \cdots, i_{n+1}} d\log[i_1, i_2, \cdots, i_{n+1}]$):

$$0 = \sum u_{i_1 i_2 \cdots i_{n+1}} d\log[i_1, i_2, \cdots, i_{n+1}] \wedge \sum u_{i_1 i_2 \cdots i_{n+1}} d\log[i_1, i_2, \cdots, i_{n+1}] \tag{2.15}$$

namely for different indices $\alpha, \beta, \gamma, \delta, i_1, \cdots, i_{n-1} \subset K$

$$\begin{aligned} 0 &= \Big[\sum_{\lambda \in K} u_{\lambda\alpha i_1 \cdots i_{n-1}}, u_{\beta\alpha i_1 \cdots i_{n-1}}\Big], \\ 0 &= [u_{\alpha\beta i_1 \cdots i_{n-1}} + u_{\beta\gamma i_1 \cdots i_{n-1}} + u_{\gamma\alpha i_1 \cdots i_{n-1}}, u_{\alpha\beta i_1 \cdots i_{n-1}}] \\ 0 &= [u_{\alpha\beta i_1 \cdots i_{n-1}}, u_{\gamma\delta i_1 \cdots i_{n-1}}]\,. \end{aligned} \tag{2.16}$$

Then exactly in the same way as in [3] we can conclude

THEOREM 3. *The space of hyper-logarithms is isomorphic to the dual of the envelopping algebra* $\mathscr{E}(\mathfrak{g})$ *of* $\mathfrak{g}$.

§ 3. Addition formula for hyper-logarithms in projective case

The group $G = GL(n+1, R)$ acts on R^n as projective transformations in such a way that logarithmic forms of the type (2.1) are transformed into themselves. Therefore $\varphi(i_1, \cdots, i_n)$ has the following properties:

(H2.1) *It is invariant by* G.

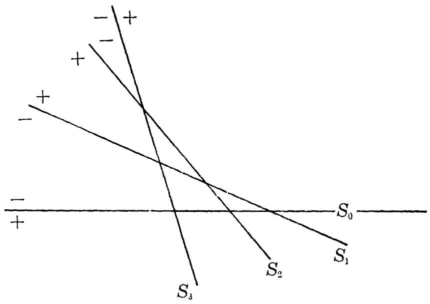
(H2.2) *It is symmetric or skew-symmetric with respect to* $S_{-n}, S_{-n+1}, \cdots, S_0$ *or* $i_1, \cdots, i_n$ *respectively.*

Now consider the $(n+2)$ hyperplanes $S_{-n-1}, S_{-n}, \cdots, S_0$, with S_j: $f_j = 0$, such that the region $-f_0 \geqq 0, \cdots, -f_{-\nu+1} \geqq 0, f_{-\nu-1} \geqq 0, \cdots, f_{-n-1} \geqq 0$ defines each simplex Δ'_ν with suitable orientations, and

(3.1) $$\sum_{\nu=0}^{n+2} (-1)^\nu \cdot \Delta'_\nu = 0 .$$

Let $\hat{\varphi}'_\nu(i_1, \cdots, i_n)$ be

(3.2) $$\int_{\Delta'} \varphi(i_1, \cdots, i_n) .$$



This is a locally analytic function of the configuration $\{S_{-n-1}, \cdots, \langle\nu\rangle, \cdots, S_0; 1, 2, \cdots, m\}$ such that the obvious identity relation holds:

(H2.3) $$\sum_{\nu=0}^{n+2} (-1) \cdot \hat{\varphi}'_\nu(i_1, \cdots, i_n) = 0 .$$

Now we are going to prove the converse is also true:

THEOREM 4. *Let $F(\xi) = F(i_1, \cdots, i_n)$ be a locally analytic function on X with singularities in Y, satisfying* (H2.1)-(H2.3) *such that it vanishes at $\xi = 0$. Then F is equal to a constant multiple of $\hat{\varphi}(i_1, \cdots, i_n)$. In other words the function $\hat{\varphi}(i_1, \cdots, i_n)$ is characterized by* (H2.1)-(H2.3).

Proof. We shall prove the Theorem by induction with respect to n. By the invariance property (H2.1) we may assume $f_{i_1} = x_1, \cdots, f_{i_n} = x_n$ so that $[j, i_1, i_2 \cdots, i_n] = a_{j0}$ and $(-1)^{\nu-1}$ $[\infty, j, i_1, \cdots, \langle i_\nu\rangle, \cdots, i_n] = a_{j\nu}$. F being a function of the configuration S, depends only on the ratios $a_{j0}/a_{j\nu}$. We may assume $a_{j0} = 1$. When $n = 0$, the Theorem is trivial in view of the definition of the logarithm. Assume $n > 1$. We consider the variation of F which can be expressed as follows:

$$dF = \sum_{j=-n}^{0} \sum_{\nu=1}^{n} F_{j\nu} d \log a_{j\nu}, \tag{3.3}$$

where $F_{j\nu}$ is uniquely determined. $a_{j\nu}$, $1 \leqq \nu \leqq n$ being fixed, $F_{j\nu}$ satisfies the assumption of the Theorem for $n-1$. Therefore by induction hypothesis $F_{j\nu}$ is a constant multiple of $\phi(i_1, \cdots, \langle i_\nu \rangle, \cdots, i_n; j)$, namely there exists a suitable locally analytic function $u_{j\nu}(a_{j1}, \cdots, a_{jn})$ such that $F_{j\nu}$ can be expressed as

$$F_{j\nu} = u_{j\nu}(a_{j1}, \cdots, a_{jn}) \cdot \phi(i_1, \cdots, \langle i_\nu \rangle, \cdots, i_n; j) \cdot (-1)^\nu. \tag{3.4}$$

By the integrability condition we have for any two (j, μ) and (k, ν), $j \neq k$, the following:

$$\partial F_{j\mu} / \partial a_{k\nu} = \partial F_{k\mu} / \partial a_{j\mu} \tag{3.5}$$

In the same way, according to Lemma 2.3

$$\begin{aligned} &\partial\{(-1)^\mu \phi'(i_1, \cdots, \langle i_\mu \rangle, \cdots, i_n; j)\} / \partial a_{k\nu} \\ &\quad = \partial\{(-1)^\nu \phi'(i_1, \cdots, \langle i_\nu \rangle, \cdots, i_n; k)\} / \partial a_{j\mu} \neq 0. \end{aligned} \tag{3.6}$$

Therefore we have

$$u_{j\mu}(a_{j1}, \cdots, a_{jn}) = u_{k\nu}(a_{k1}, \cdots, a_{kn}) \tag{3.7}$$

which implies that $u_{j\mu}$ is equal to a constant independent of j and μ. The Theorem has been completely proved.

Added in proof. In [21] is defined the di-logarighm form on the configuration space. It seems still uncertain if it is related to the hyper-logarithms discussed here.

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