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SOME REVERSE DYNAMIC INEQUALITIES ON TIME SCALES

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Abstract

In this paper, we prove some new reverse dynamic inequalities of Renaud- and Bennett-type on time scales. The results are established using the time scales Fubini theorem, the reverse Hölder inequality and a time scales chain rule.

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1. Introduction

The celebrated Hardy inequality for series (see [5], [7, Theorem 326]) asserts that, for $p > 1$, and $a_n \geq 0$ and $A_n = \sum_{i=1}^{n} a_i$ for $n \geq 1$,

$$\sum_{n=1}^{\infty} n^{-p} A_n^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p. \tag{1.1}$$

Hardy (see [6], [7, Theorem 327]) established an integral analogue of (1.1) which states that, for $p > 1$ and $f \geq 0$ over $(0, \infty)$,

$$\int_0^{\infty} \left(\frac{F(x)}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{\infty} f^p(x)\, dx, \tag{1.2}$$

where $F(x) = \int_0^x f(t)\, dt$. The constant $(p/(p-1))^p$ in (1.1) and (1.2) is the best possible.

Copson [4] (see also [7, Theorem 331]) proved some new variants for the discrete Hardy inequality (1.1). One of his inequalities is

$$\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} a_k\right)^p \leq p^p \sum_{n=1}^{\infty} (n a_n)^p, \tag{1.3}$$

for $p > 1$ and $\{a_n\}$ a sequence with nonnegative terms.

Lyons [8] proved a type of reverse inequality for the discrete inequality (1.1) in the special case $p = 2$. Namely, if $\{a_n\}$ is a monotone nonincreasing sequence of nonnegative numbers, then

$$\sum_{n=0}^{\infty}\Big(\frac{1}{n+1}\sum_{k=0}^{n} a_k\Big)^2 \geq \frac{\pi^2}{6}\sum_{n=0}^{\infty} a_n^2.$$

Renaud [11] generalised the result of Lyons and established a reverse of the two Hardy inequalities (1.1) and (1.2) for an arbitrary exponent $p > 1$:

$$\sum_{n=1}^{\infty} n^{-p} A_n^p \geq \zeta(p)\sum_{n=1}^{\infty} a_n^p, \tag{1.4}$$

where $\zeta(p)$ is the Riemann zeta function and $\{a_n\}$ is a monotone nonincreasing sequence with nonnegative terms, and

$$\int_0^{\infty}\Big(\frac{F(x)}{x}\Big)^p dx \geq \frac{p}{p-1}\int_0^{\infty} f^p(x)\,dx, \tag{1.5}$$

where $f(x)$ is a nonnegative monotone nonincreasing function (see also [9, 13]). In the same paper, Renaud proved a reverse of the discrete inequality of Copson (1.3):

$$\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty} a_k\Big)^p \geq \sum_{n=1}^{\infty}(na_n)^p, \tag{1.6}$$

where $p > 1$ and $\{a_n\}$ is a monotone nonincreasing sequence with nonnegative terms. Renaud also gave an integral analogue of (1.6) by showing that

$$\int_0^{\infty}\Big(\int_x^{\infty} f(t)\,dt\Big)^p dx \geq \int_0^{\infty}(xf(x))^p\,dx, \tag{1.7}$$

where $p > 1$ and $f(x)$ is a nonnegative monotone nonincreasing function. Bennett [1, Theorem 2] proved for $p \geq 1$, that

$$\Big(\sum_{j=1}^{m}\Big(\sum_{k=1}^{n} a_{jk}x_k\Big)^p\Big)^{1/p} \geq \lambda\Big(\sum_{j=1}^{m}|x_j|^p\Big)^{1/p}, \tag{1.8}$$

where

$$\lambda^p = \min_{1\leq r\leq n}\frac{1}{r}\sum_{j=1}^{m}\Big(\sum_{k=1}^{r} a_{jk}\Big)^p,$$

and Bennett [1, Theorem 7] also established the integral analogue of (1.8), namely

$$\Big(\int_0^{a}\Big(\int_0^{t} K(x,y)g(y)\,dy\Big)^p\Big)^{1/p} dx \geq \lambda\Big(\int_0^{a}|g(x)|^p\,dx\Big)^{1/p}, \tag{1.9}$$

where

$$\lambda^p = \inf_{0<t<a}\int_0^{a}\frac{1}{t}\Big(\int_0^{t} K(x,y)\,dy\Big)^p dx.$$

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In 2005, Řehák [10] gave a time scales version of the classical Hardy inequality (1.2) in the form

$$\int_a^{\infty}\Big(\frac{1}{\sigma(t)-a}\int_a^{\sigma(t)} f(s)\Delta s\Big)^p \Delta t \le \Big(\frac{p}{p-1}\Big)^p \int_a^{\infty}(f(t))^p\Delta t,$$

where $p > 1$ and f is a nonnegative function.

In this paper we establish new reverse dynamic inequalities and discuss inequalities (1.5), (1.8) and (1.9) as special cases. The main results will be proved using the time scales Fubini theorem and a time scales chain rule. The paper is organised as follows. In Section 2, we present the basic concepts on the calculus on time scales. In Section 3, we prove the time scales version of the reverse Hardy inequalities (1.5) and (1.7) due to Renaud. Finally, in Section 4 we prove the time scales version of the reverse inequality (1.9) with general kernel due to Bennett.

2. Preliminaries and basic lemmas

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ respectively, where $\sup\emptyset = \inf\mathbb{T}$. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t > \inf\mathbb{T}$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous) provided g is continuous at right-dense points, and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. The set of all rd-continuous functions is denoted by $C_{\mathrm{rd}}(\mathbb{T})$.

The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t \ge 0$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. We assume that $\sup\mathbb{T} = \infty$, and define the time scale interval $[a, b]_{\mathbb{T}}$ by $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$.

Define $f^{\Delta}(t)$ to be the number (if it exists) with the property that given any $\epsilon > 0$ there is a neighbourhood U of t with

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \le \epsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

In this case, we say $f^{\Delta}(t)$ is the (delta) derivative of f at t and that f is (delta) differentiable at t. The product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$ and g^{σ} is the composition function $g \circ \sigma$) of two differentiable functions f and g are given by

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma},$$
$$\Big(\frac{f}{g}\Big)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}.$$

We can define the (delta) integral as follows. If $G^{\Delta}(t) = g(t)$, then the Cauchy (delta) integral of g is defined by $\int_a^t g(s)\Delta s := G(t) - G(a)$. It is known (see [3]) that if $g \in C_{\mathrm{rd}}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists for $t_0 \in \mathbb{T}$, and

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satisfies $G^{\Delta}(t) = g(t)$ for $t \in \mathbb{T}$. An improper (delta) integral can be defined by $\int_a^\infty f(t)\Delta t = \lim_{b\to\infty} \int_a^b f(t)\Delta t$.

The time scales chain rule (see [3, Theorem 1.87]) is given by

$$(g \circ \delta)^{\Delta}(t) = g'(\delta(d))\delta^{\Delta}(t) \quad \text{for } d \in [t, \sigma(t)], \tag{2.1}$$

where it is assumed that $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $\delta : \mathbb{T} \to \mathbb{R}$ is delta differentiable. A simple consequence of Keller's chain rule [3, Theorem 1.90] is

$$(u^{\gamma}(t))^{\Delta} = \gamma \int_0^1 [hu^{\sigma}(t) + (1-h)u(t)]^{\gamma-1}\, dh u^{\Delta}(t) \tag{2.2}$$

and the integration by parts formula on time scales, for $a, b \in \mathbb{T}$, is given by

$$\int_a^b u(t)v^{\Delta}(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t. \tag{2.3}$$

The reverse Hölder inequality on time scales (see [12, Lemma 1]) is given by

$$\Big(\frac{m}{M}\Big)^{-1/\gamma\nu} \int_0^\infty |f(t)g(t)|\Delta t \geq \Big[\int_0^\infty |f(t)|^{\gamma}\Delta t\Big]^{1/\gamma}\Big[\int_0^\infty |g(t)|^{\nu}\Delta t\Big]^{1/\nu}, \tag{2.4}$$

with f, $g \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ satisfying $0 < m \leq f(t)^{\gamma}/g(t)^{\nu} \leq M < \infty$, for $t \in [0, \infty)_{\mathbb{T}}$, and $1/\gamma + 1/\nu = 1$.

We next state the time scales Fubini theorem due to Bibi *et al.* [2, Theorem 1.1].

LEMMA 2.1. *Let* (X, M, μ_{Δ}) *and* (Y, L, ν_{Δ}) *be two finite-dimensional time scales measure spaces. If* $f : X \times Y \to \mathbb{R}$ *is a* Δ*-integrable function and if we define the functions*

$$\varphi(y) = \int_X f(x, y)\, d\mu_{\Delta}(x) \quad \textit{for } y \in Y,$$

and

$$\psi(x) = \int_Y f(x, y)\, d\nu_{\Delta}(y) \quad \textit{for } x \in X,$$

then φ *is* Δ*-integrable on* Y *and* ψ *is* Δ*-integrable on* X *and*

$$\int_X d\mu_{\Delta}(x) \int_Y f(x, y)\, d\nu_{\Delta}(y) = \int_Y d\nu_{\Delta}(y) \int_X f(x, y)\, d\mu_{\Delta}(x).$$

Throughout the paper, we will assume (without mentioning it) that the functions in the statements of the theorems are nonnegative and rd-continuous and the integrals exist (finite and convergent).

3. Renaud-type inequalities

We state and prove our first main result which is the time scales version of (1.5).

Theorem 3.1. *Let $\mathbb{T}$ be a time scale with $0 \in \mathbb{T}$ and let f be a decreasing function. If $p > 1$, then*

$$\int_0^\infty \Big(\frac{F(x)}{x}\Big)^p \Delta x \geq \frac{p}{p-1} \int_0^\infty f^p(x)\Delta x, \tag{3.1}$$

where

$$F(x) = \int_0^x f(t)\Delta t.$$

Proof. For $x \geq 0$, since f is a decreasing function,

$$F(x) = \int_0^x f(t)\Delta t \geq x f(x),$$

so

$$p f(x) F^{p-1}(x) \geq p x^{p-1} f^p(x),$$

and as a result

$$\int_0^x p f(t) F^{p-1}(t)\Delta t \geq p \int_0^x t^{p-1} f^p(t)\Delta t. \tag{3.2}$$

By the time scales chain rule (2.2), since $F^\Delta(t) = f(t) > 0$,

$$\begin{aligned}(F^p(t))^\Delta &= p \int_0^1 [hF^\sigma(t) + (1-h)F(t)]^{p-1}\, dh\, F^\Delta(t) \\ &\geq p \int_0^1 [hF(t) + (1-h)F(t)]^{p-1}\, dh\, f(t) = p f(t) F^{p-1}.\end{aligned}$$

Thus

$$F^p(x) = \int_0^x (F^p(t))^\Delta \Delta t \geq \int_0^x p f(t) F^{p-1}(t)\Delta t.$$

Using this in (3.2),

$$\int_0^\infty \Big(\frac{F(x)}{x}\Big)^p \Delta x \geq \int_0^\infty \frac{p}{x^p}\Big(\int_0^x t^{p-1} f^p(t)\Delta t\Big)\Delta x. \tag{3.3}$$

Applying Fubini's theorem on the right-hand side of (3.3) yields

$$\int_0^\infty \Big(\frac{F(x)}{x}\Big)^p \Delta x \geq \int_0^\infty \Big(\int_t^\infty \frac{p t^{p-1} f^p(t)}{x^p}\Delta x\Big)\Delta t. \tag{3.4}$$

Now, by the time scales chain rule (2.2),

$$\begin{aligned}(x^{1-p})^\Delta &= (1-p)\int_0^1 [hx^\sigma + (1-h)x]^{-p}\, dh\, x^\Delta \\ &= (1-p)\int_0^1 \frac{dh}{[hx^\sigma + (1-h)x]^p} \\ &\geq (1-p)\int_0^1 \frac{dh}{[hx + (1-h)x]^p} = (1-p)x^{-p}.\end{aligned}$$

Substituting this in (3.4) and observing that $p > 1$,

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p \Delta x \geq \int_0^\infty \left(pt^{p-1} f^p(t) \frac{x^{1-p}}{1-p}\bigg|_t^\infty\right)\Delta t = \frac{p}{p-1}\int_0^\infty f^p(t)\Delta t,$$

which is the required inequality (3.1). This completes the proof. □

REMARK 3.2. If $\mathbb{T} = \mathbb{N}$, then the inequality in Theorem 3.1 reduces to the discrete result

$$\sum_{n=1}^\infty n^{-p} A_n^p \geq \frac{p}{p-1}\sum_{n=1}^\infty a_n^p,$$

which is different from the discrete inequality (1.4) due to Renaud.

REMARK 3.3. If $\mathbb{T} = \mathbb{R}$, then the inequality in Theorem 3.1 reduces to the continuous result (1.5) due to Renaud.

REMARK 3.4. We note that Theorem 3.1 not only extends the discrete inequality (1.4) and its continuous counterpart (1.5) to an arbitrary time scale $\mathbb{T}$, but it also unifies the constant in these two results.

Next, we extend the reverse inequality (1.7) to an arbitrary time scale $\mathbb{T}$.

THEOREM 3.5. *Let $\mathbb{T}$ be a time scale with $0 \in \mathbb{T}$ and let f be a decreasing function with $\int_0^\infty f(t)\Delta t < \infty$. Set $F(x) = \int_x^\infty f(t)\Delta t$. If $p > 1$, then*

$$\int_0^\infty \left(\int_x^\infty f(t)\Delta t\right)^p \Delta x \geq (kp)^p \int_0^\infty x^p f^p(x)\Delta x, \tag{3.5}$$

where

$$k \equiv \left(\frac{m}{M}\right)^{(p-1)/p^2} \quad \text{and} \quad 0 < m \leq \frac{x^p f^p(x)}{F^p(\sigma(x))} \leq M < \infty \quad \text{for } x \in [0,\infty)_{\mathbb{T}}.$$

PROOF. By integration by parts with $v^\sigma(x) = F^p(\sigma(x))$ and $u^\Delta(x) = 1$,

$$\int_0^\infty F^p(\sigma(x))\Delta x = xF^p(x)|_0^\infty + \int_0^\infty x(-F^p(x))^\Delta \Delta x.$$

Since $\lim_{x\to 0^+} xF^p(x) = 0$ (by assumption $\int_0^\infty f(t)\Delta t < \infty$) and $xF^p(x) \geq 0$ for $x > 0$,

$$\int_0^\infty F^p(\sigma(x))\Delta x \geq \int_0^\infty x(-F^p(x))^\Delta \Delta x. \tag{3.6}$$

Using the time scales chain rule (2.1) yields

$$-(F^p(x))^\Delta = -pF^{p-1}(d)(F^\Delta(x)),$$

where $d \in [x, \sigma(x)]$. Since $F^\Delta(x) = -f(x) \leq 0$ and $d \leq \sigma(x)$,

$$-(F^p(x))^\Delta = \frac{pf(x)}{F^{1-p}(d)} \geq pF^{p-1}(\sigma(x))f(x).$$

Substituting this in (3.6) gives

$$\int_0^\infty F^p(\sigma(x))\Delta x \geq p \int_0^\infty xF^{p-1}(\sigma(x))f(x)\Delta x. \tag{3.7}$$

Applying the reverse Hölder inequality (2.4) to the term $\int_0^\infty xF^{p-1}(\sigma(x))f(x)\Delta x$ with indices $\gamma = p > 1$ and $\nu = p/(p-1)$,

$$\begin{aligned}
&\int_0^\infty xF^{p-1}(\sigma(x))f(x)\Delta x \\
&\geq \Big(\frac{m}{M}\Big)^{(p-1)/p^2}\Big\{\int_0^\infty [xf(x)]^p\Delta x\Big\}^{1/p}\Big\{\int_0^\infty [F^{p-1}(\sigma(x))]^{p/(p-1)}\Delta x\Big\}^{(p-1)/p} \\
&= k\Big(\int_0^\infty f^p(x)x^p\Delta x\Big)^{1/p}\Big(\int_0^\infty F^p(\sigma(x))\Delta x\Big)^{(p-1)/p}.
\end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7) and raising both sides to the pth power,

$$\Big(\int_0^\infty F^p(\sigma(x))\Delta x\Big)^p \geq (kp)^p\Big(\int_0^\infty f^p(x)x^p\Delta x\Big)\Big(\int_0^\infty F^p(\sigma(x))\Delta x\Big)^{p-1},$$

which is the required inequality (3.5). This completes the proof. □

REMARK 3.6. In the proof of Theorem 3.5, the condition $\int_0^\infty f(t)\Delta t < \infty$ could be replaced with $\lim_{x\to 0^+} x\big(\int_x^\infty f(t)\Delta t\big)^p = 0$. With this condition, (3.6) still holds.

4. Bennett-type inequalities

In this section, we state and prove the time scales version of Bennett's inequalities with general kernels. For simplicity, we write

$$G(a) = \int_0^a K(x,y)g(y)\Delta y \quad \text{and} \quad F(y) = \int_0^y K(x,t)\Delta t,$$

where $K(x,y) \geq 0$ for $0 < x,\ y < a$.

THEOREM 4.1. *Let $\mathbb{T}$ be a time scale with $0, a \in \mathbb{T}$ and let g be a decreasing function. If $p \geq 1$, then*

$$\Big(\int_0^a \Big(\int_0^t K(x,y)g(y)\Delta y\Big)^p\Big)^{1/p}\Delta x \geq \lambda\Big(\int_0^a |g(x)|^p\Delta x\Big)^{1/p}, \tag{4.1}$$

where

$$\lambda^p = \inf_{0<t<a}\int_0^a \frac{1}{t}\Big(\int_0^t K(x,y)\Delta y\Big)^p\Delta x. \tag{4.2}$$

PROOF. For $x, y > 0$, since g is a decreasing function,

$$G(y) = \int_0^y K(x,t)g(t)\Delta t \geq g(y)\int_0^y K(x,t)\Delta t,$$

so

$$\Big(\int_0^y K(x,t)g(t)\Delta t\Big)^{p-1}K(x,y)g(y)\ge g^p(y)\Big(\int_0^y K(x,t)\Delta t\Big)^{p-1}K(x,y),$$

and as a result

$$\begin{aligned}\int_0^a\Big(\int_0^y K(x,t)g(t)\Delta t\Big)^{p-1}&K(x,y)g(y)\Delta y\\ &\ge\int_0^a g^p(y)\Big(\int_0^y K(x,t)\Delta t\Big)^{p-1}K(x,y)\Delta y.\end{aligned}\tag{4.3}$$

Using the time scales chain rule (2.2), since $G^\Delta(y)=K(x,y)g(y)\ge 0$,

$$\begin{aligned}(G^p(y))^\Delta&=p\int_0^1[hG^\sigma(y)+(1-h)G(y)]^{p-1}dh\,G^\Delta(y)\\ &\ge p\int_0^1[hG(y)+(1-h)G(y)]^{p-1}dh\,G^\Delta(y)=pG^{p-1}(y)G^\Delta(y).\end{aligned}$$

Thus

$$G^p(a)=\int_0^a(G^p(y))^\Delta\Delta y\ge\int_0^a pG^{p-1}(y)G^\Delta(y)\Delta y$$

and, using this in (4.3),

$$\Big(\int_0^a K(x,y)g(y)\Delta y\Big)^p\ge p\int_0^a g^p(y)\Big(\int_0^y K(x,t)\Delta t\Big)^{p-1}K(x,y)\Delta y.\tag{4.4}$$

Now using the time scales chain rule (2.2), since $F^\Delta(y)=K(x,y)\ge 0$,

$$\begin{aligned}(F^p(y))^\Delta&=p\int_0^1[hF^\sigma(y)+(1-h)F(y)]^{p-1}dh\,F^\Delta(y)\\ &\ge p\int_0^1[hF(y)+(1-h)F(y)]^{p-1}dh\,F^\Delta(y)=pF^{p-1}(y)F^\Delta(y),\end{aligned}$$

and so

$$F^p(a)=\int_0^a(F^p(y))^\Delta\Delta y\ge\int_0^a pF^{p-1}(y)F^\Delta(y)\Delta y.$$

Integrating by parts with $u=g^p(y)$ and $v^\Delta=F^{p-1}F^\Delta$ gives

$$\begin{aligned}\int_0^a g^pF^{p-1}F^\Delta\Delta y&=\frac{1}{p}g^p(y)F^p(y)|_0^a-\frac{1}{p}\int_0^a(g^p(y))^\Delta F^p(\sigma(y))\Delta y\\ &=\frac{1}{p}\Big(g^p(a)F^p(a)-\int_0^a(g^p)^\Delta F^p(\sigma(y))\Delta y\Big),\end{aligned}$$

and substituting this in (4.4) and integrating by parts again gives

$$\begin{aligned}\int_0^a\Big(\int_0^a &K(x,y)g(y)\Delta y\Big)^p\Delta x\\ &\ge g^p(a)\int_0^a\Big(\int_0^a K(x,t)\Delta t\Big)^p\Delta x-\int_0^a\int_0^a(g^p(y))^\Delta F^p(\sigma(y))\Delta y\Delta x.\end{aligned}$$

Applying Fubini's theorem to the last term on the right-hand side,

$$\int_0^a \Big(\int_0^a K(x,y)g(y)\Delta y\Big)^p \Delta x$$
$$\geq g^p(a)\int_0^a \Big(\int_0^a K(x,t)\Delta t\Big)^p \Delta x - \int_0^a (g^p(y))^\Delta \int_0^a \Big(\int_0^{\sigma(y)} K(x,t)\Delta t\Big)^p \Delta x \Delta y.$$

Now, with λ as in (4.2),

$$\int_0^a \Big(\int_0^a K(x,y)g(y)\Delta y\Big)^p \Delta x \geq ag^p(a)\lambda^p - \lambda^p \int_0^a \sigma(y)(g^p(y))^\Delta \Delta y.$$

Using the integration by parts formula (2.3) with $v^\sigma = \sigma(y)$ and $u^\Delta = (g^p(y))^\Delta$,

$$\int_0^a \Big(\int_0^a K(x,y)g(y)\Delta y\Big)^p \Delta x \geq ag^p(a)\lambda^p - \lambda^p\Big(yg^p(y)|_0^a - \int_0^a g^p(y)\Delta y\Big)$$
$$= \lambda^p \int_0^a g^p(y)\Delta y,$$

and so we obtain inequality (4.1). This completes the proof. □

REMARK 4.2. As special cases of Theorem 4.1, we have the following results.

(1) If $\mathbb{T} = \mathbb{N}$, then the result in Theorem 4.1 reduces to the discrete result (1.8) due to Bennett.

(2) If $\mathbb{T} = \mathbb{R}$, then the result in Theorem 4.1 reduces to the continuous result (1.9) due to Bennett.

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