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EXISTENCE THEOREMS FOR SOME WEAK ABSTRACT VARIABLE DOMAIN HYPERBOLIC PROBLEMS

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1. Introduction. In this paper we prove some existence theorems for some weak problems with variable domains arising from hyperbolic equations of the type

$$u_{tt} + Au = f \tag{1.1}$$

where $A = \{A(t)\}$ is, for example, a family of elliptic differential operators in space variables $x = (x_1, \ldots, x_n)$. Thus let H be a separable Hilbert space and let $V(t) \subset H$ be a family of Hilbert spaces dense in H with continuous injections $i(t)\colon V(t) \to H$ $(0 \leqq t \leqq T < \infty)$. Let $V'(t)$ be the antidual of $V(t)$ (i.e. the space of continuous conjugate linear maps $V(t) \to \mathbf{C}$) and using standard identifications one writes $V(t) \subset H \subset V'(t)$. If $R(t)\colon H \to V(t)$ (into) is determined by $((R(t)x, y))_t = (x, y)$ for $x \in H$ and $y \in V(t)$, where $((\ ,\ ))_t$ (resp. $(\ ,\ )$) denotes the scalar product in $V(t)$ (resp. H), then $R^{-1}(t)$ is a positive selfadjoint operator in H and $S(t) = R^{-\frac{1}{2}}(t)$ maps its domain $V(t) = D(S(t))$ one to one onto H with $((x, y))_t = (Sx, Sy)$ (see [**9**] for details). We will always make hypotheses which insure that $||S^{-1}(t)|| \leqq c_1$ with $S^{-1}(\cdot)h$ measurable in H for $h \in H$ (i.e. $(S^{-1}(\cdot)h, k)$ is to be measurable for $k \in H$). The use of such standard operators has been quite successful in describing the variation of the $V(t)$ in weak abstract evolution problems (see [**9**; **10**; **11**; **12**; **15**; **16**; **17**; **20**; **21**; **34**; **44**] and see also [**22**; **23**] for related work in some strong problems; this was reported on in part in [**13**; **24**]). For the intrinsic meaning of the $S(t)$ see [**9**, **14**]. Let now $W = L^2(V(t)) = \{u \in L^2(H)$ on $[0, T]|\ u(t) \in V(t)$ a.e.; $Su \in L^2(H)$ on $[0, T]\}$ with scalar product

$$((u, v))_W = \int_0^T ((u, v))_t dt = \int_0^T (Su, Sv)dt$$

(Su means $S(\cdot)u(\cdot)$, for example, and we suppress the t argument in the integrals for simplicity). Note that W depends on $((\ ,\ ))_t$ and hence on $S(t)$. Under the hypotheses indicated for $S^{-1}(\cdot)$ it follows that $W \subset L^2(H)$ (cf. [**9**]). Let $a(t, \cdot, \cdot)$ be a continuous sesquilinear form on $V(t) \times V(t)$ with $|a(t, x, y)| \leqq c_2||x||_t||y||_t$ for $x, y \in V(t)$. One writes

$$a(t, x, y) = ((\mathfrak{A}(t)x, y))$$

and we assume that $\mathfrak{A}(\cdot)$ is a measurable family so that $\mathfrak{A}u \in W$ when

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$u \in W$ (cf. [**9**]); we will call $a(t, \cdot, \cdot)$ a measurable family of continuous sesquilinear forms. Then a natural weak problem related to (1.1) is to find $u \in W$ with $u' \in L^2(H)$ and $u(0) = 0$ ($'$ in $\mathfrak{D}'(H) = H$ valued distributions on $(0, T)$; note that u will be continuous (cf. [**9**]) and we need not work on $(-\infty, T)$) such that

$$(1.2) \qquad -\int_0^T (u', v')dt + \int_0^T (B(t)u', v)dt + \int_0^T a(t, u, v)dt = \int_0^T (f, v)dt + (u_1, v(0))$$

for all $v \in W$ such that $v' \in L^2(H)$ and $v(T) = 0$ (cf. [**32**]); the problem with $u(0) \neq 0$ is somewhat more complicated as indicated in [**32**] and we will not discuss it. Here $f \in L^2(H)$ and $u_1 \in H$ are given while one assumes that $B(t) \in \mathfrak{L}(H)$ with $||B(t)|| \leqq c_3$ and $B(\cdot)h$ measurable in H for $h \in H$ ($\mathfrak{L}(H)$ is the space of bounded operators $H \to H$). Under the assumptions listed, everything in (1.2) makes sense (note that $B(\cdot)u'(\cdot)$ is measurable since $(B(t)u', h) = (u', B^*(t)h)$ for $h \in H$, while $B^*(\cdot)h$ is measurable since for $k \in H$, $(B(t)k, h) = (k, B^*(t)h)$). We remark that Lions only considers $V(t) = V$ in [**32**] and variable domain hyperbolic problems in general present serious difficulties not encountered in the variable domain parabolic situation (this is perhaps related to characteristics, retrograde light cones, etc. in the hyperbolic case); some variable domain hyperbolic results appear in [**8**; **28**; **31**; **33**; **40**]† and other recent work on abstract hyperbolic problems can be found in [**1**; **2**; **4**; **6**; **7**; **25**; **26**; **27**; **29**; **30**; **32**; **33**; **34**; **35**; **36**; **38**; **39**; **41**; **42**; **43**; **45**; **46**; **47**; **48**; **49**; **50**]. We do not give references here to papers on the abstract Cauchy problem, semigroups, or the construction of evolution operators unless they explicitly deal with abstract equations of the form (1.1) or (1.2); similarly we have not tried to list the recent work on hyperbolic systems due to Friedrichs, Phillips, Lax, Sarason, Crandall, etc., nor have we given references to hyperbolic situations treated by the methods of singular integral operators (see [**9**] for bibliography in all these cases).

In section two we reformulate a simplified version of (1.2) as a first order system (problem (2.2)). This version involves $B(t) = 0$ and $u_1 = 0$ with $a(t, \cdot, \cdot)$ selfadjoint and coercive, which is sufficient to indicate the main features of the theory developed here. (Extensions of this theory to include forms $a(t, \cdot, \cdot)$ which are perturbations of selfadjoint coercive $p(t, \cdot, \cdot)$ as in section three and to include operators $B(t)$ of the form indicated below seem to be straightforward and we will not spell out the details (cf. also remark 2.9).) One considers a parabolically regularized strong problem (problem 2.5), associated with problem 2.2, to which solutions $\mathbf{u}^\epsilon$ can be found, and, after obtaining suitable estimates on the $\mathbf{u}^\epsilon$, limits can be taken

†See also an important recent paper: J. Cooper and C. Bardos, *A nonlinear wave equation in a time dependent domain*, TR 71-25, University of Maryland, 1971.

in the related regularized weak problem which leads to a solution of problem 2.2. The resulting theorem (Theorem 2.7) seems quite strong but we remark that it is an abstract theorem in the sense that the hypothesis (2.17) is difficult to verify in practice (cf., however, remark 3.8 for some interpretation of (2.17)). In section three we work with (1.2) directly using a version of a technique of State (cf. [**13**; **44**]) which involves the introduction of a new space of test functions. The ensuing result is essentially weaker than Theorem 2.7 but, as indicated in section three, the technique is of independent interest, additional terms are included, and moreover it illustrates again the role of (2.17); remark 3.7 provides additional reasons for exhibiting this technique. The results have been announced in [**19**].

2. In general one expects results with forms $a(t, \cdot, \cdot)$ which are perturbations of selfadjoint forms $p(t, \cdot, \cdot)$ (i.e. $p(t, x, y) = \overline{p(t, y, x)}$) where $p(t, y, x) \geqq \alpha\|x\|_t^2 - \delta|x|^2$; cf. [**1**; **2**; **32**; **44**], and section three of this paper). In this section we simply take a coercive selfadjoint family $a(t, \cdot, \cdot)$ (i.e. $a(t, x, x) \geqq \alpha_t\|x\|_t^2$) with $|a(t, x, y)| \leqq c_t\|x\|_y\|y\|_t$ in order to illustrate an application of the method of parabolic regularization to a weak abstract variable domain hyperbolic situation (for parabolic regularization see, e.g., [**4**; **33**; **35**]). The ensuing result can evidently be improved upon to include additional terms etc. as in (1.2) but we will not dwell on this (cf. comments in section one and hypotheses in section three). Now such coercive selfadjoint forms $a(t, \cdot, \cdot)$ induce a norm topology on $V(t)$ equivalent to its original topology since $\alpha_t\|x\|_t^2 \leqq a(t, x, x) \leqq c_t\|x\|_t^2$. Hence we introduce a new Hilbert structure on $V(t)$ with scalar product $((x, y))_t = a(t, x, y)$, and use this structure exclusively in the remainder of this section (with the same notation $((\ ,\ ))_t$ and $\|\ \|_t$). Note that when $\mathfrak{A}(t)$ is the identity, $\mathfrak{A}(t)$ is automatically a measurable family in our new W and neither uniform boundedness nor coercivity relative to the old norms is required.

Now $a(t, x, y) = \langle A(t)x, y\rangle$ where $A(t)\colon V(t) \to V'(t)$ is linear and continuous (recall that $V'(t)$ is the antidual). Note also that $\langle\ ,\ \rangle$ is linear in the first argument and antilinear in the second and one can thank of $V(t) = V''(t)$ as the antidual of $V'(t)$ in formulas such as $\langle A(t)x, y\rangle = \overline{\langle A(t)y, x\rangle} = \langle x, A(t)y\rangle$. We define an (unbounded) operator in H, denoted also by $A(t)$, by stipulating that x in $V(t)$ belongs to $D(A(t))$ if $A(t)x \in H$. Thus, for $x \in D(A(t))$, $a(t, x, y) = \langle A(t)x, y\rangle = (A(t)x, y)$ and this is the same as specifying that x in $V(t)$ shall belong to the domain of a linear operator $A(t)$ if the map $y \to a(t, x, y)\colon V(t) \to \mathbf{C}$ is continuous in the topology of H (cf. [**9**; **32**]). Consequently, since $(A(t)x, y) = a(t, x, y) = \overline{a(t, y, x)} = \overline{(A(t)y, x)} = (x, A(t)y)$ for $x, y \in D(A(t))$, we see that $A(t)$ is a selfadjoint operator in H (cf. [**9**; **32**]); in particular, $A(t)$ is closed and densely defined. We recall next (cf. [**9**]) that there is an isometric isomorphism

$$\theta(t)\colon V'(t) \to V(t)$$

determined by $\langle x, y\rangle = ((\theta(t)x, y))_t$; hence

$$\langle x, y\rangle = a(t, \theta(t)x, y) = \langle A(t)\theta(t)x, y\rangle.$$

This formula, together with $\langle A(t)v, y\rangle = ((v, y))_t = \langle\theta^{-1}(t)v, y\rangle$, shows that in fact $A(t) = \theta^{-1}(t)$. We recall next from [**16**] that $\theta(t) = S^{-2}(t)$ when restricted to H. Indeed for $w \in H$ and $v \in V(t)$ one has

$$\langle w, v\rangle = (S(t)\theta(t)w, S(t)v)$$

so $\langle w, v\rangle = (w, v)$ means that $S(t)\theta(t)w \in D(S(t))$ with $S^2(t)\theta(t)w = w$. Thus, when $x \in D(A(t))$ with $A(t)x = y \in H$ we have $\theta(t)y = S^{-2}(t)y = x$ so $S^2(t)x = A(t)x$ for $x \in D(A(t))$; this means that $S(t) = A^{\frac{1}{2}}(t)$. We summarize some of this as a lemma.

LEMMA 2.1. *With the Hilbert structure on $V(t)$ defined by $((x, y))_t = a(t, x, y)$ it follows that $A(t) = \theta^{-1}(t)$: $V(t) \to V'(t)$ and, as an operator in H, $A(t) = S^2(t)$ is selfadjoint.*

Measurability properties and bounds will be indicated below. Now we will change (1.1) into a first order system in order to exploit some results of [**15**; **16**] (cf. also [**4**; **33**; **35**] for somewhat different formulations). The novelty here is that an additional term must be added in the weak problem to take into account the variation of the $V(t)$. Thus consider formally $u_1 = A^{\frac{1}{2}}u = Su$ and $u_2 = u'$ where $u \in W$ and $u' \in L^2(H)$ ($'$ in $\mathfrak{D}'(H)$ on $(0, T)$; cf. [**18**] for similar reductions). We shall assume from now on that $S^{-1}(\cdot)$ is weakly C^1 (i.e. that $(S^{-1}(\cdot)h, k)$ is C^1 for $h, k \in H$). It follows that there are selfadjoint operators $\dot{S}^{-1}(t) \in \mathfrak{L}(H)$ such that for $h, k \in H$ $(S^{-1}(t)h, k)' = (\dot{S}^{-1}(t)h, k)$ with $||S^{-1}(t)|| \leqq c_1$ and $||\dot{S}^{-1}(t)|| \leqq c_4$ while $S^{-1}(\cdot)$ is Lipschitz continuous in norm (see [**9**, Lemma 4.5.7]). In particular $S^{-1}(\cdot)h$ is measurable in H for $h \in H$ and $W \subset L^2(H)$. It follows that in $\mathfrak{D}'(H)$ (suppressing the t argument for convenience)

$$u' = (S^{-1}u_1)' = \dot{S}^{-1}u_1 + S^{-1}u_1' \tag{2.1}$$

provided u_1' makes sense, and one is led to pose the following weak problem.

Problem 2.2. Find

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathfrak{H} = L^2(H) \times L^2(H)$$

such that

$$-(\mathbf{u}, \mathbf{v}') + \lambda(\mathbf{u}, \mathbf{v}) + (\mathfrak{A}\mathbf{u}, S\mathbf{v}) = (\mathbf{f}, \mathbf{v}) \tag{2.2}$$

for all

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in W \times W = \mathfrak{W} \quad \text{with} \quad \mathbf{v}' \in \mathfrak{H}$$

and $\mathbf{v}(T) = 0$ where

$$\mathfrak{A} = \begin{pmatrix} \dot{S}^{-1} & -1 \\ 1 & 0 \end{pmatrix}; \quad \mathbf{f} = \begin{pmatrix} 0 \\ f \end{pmatrix} \tag{2.3}$$

with $f \in L^2(H)$ and λ is an arbitrary real number.

Existence (or uniqueness) is not affected by the λ term (cf. [**9**]) and λ will be chosen large enough to aid in obtaining estimates later. We display the equations (2.2) also in the form

$$-\int_0^T (u_1, v_1')dt + \lambda \int_0^T (u_1, v_1)dt - \int_0^T (u_2, Sv_1)dt + \int_0^T (\dot{S}^{-1}u_1, Sv_1)dt = 0; \tag{2.4}$$

$$-\int_0^T (u_2, v_2')dt + \lambda \int_0^T (u_2, v_2)dt + \int_0^T (u_1, Sv_2)dt = \int_0^T (f, v_2)dt. \tag{2.5}$$

Thus setting $v_1 = S^{-1}\varphi$ for $\varphi \in C_0^\infty(H)$ on $(0, T)$ with $v_1' = \dot{S}^{-1}\varphi + S^{-1}\varphi'$ and taking $\lambda = 0$, (2.4) leads to

$$-\int_0^T (S^{-1}u_1, \varphi')dt = \int_0^T (u_2, \varphi)dt. \tag{2.6}$$

This implies that $u_2 = (S^{-1}u_1)'$ in $\mathfrak{D}'(H)$ and hence given a solution $\mathbf{u}$ of problem 2.2 (with $\lambda = 0$) we define $u = S^{-1}u_1$ so that $u_2 = u'$ and $(u_1, Sv_2) = (Su, Sv_2) = a(t, u, v_2)$. Hence (2.5) becomes

$$-\int_0^T (u', v_2')dt + \int_0^T a(t, u, v_2)dt = \int_0^T (f, v_2)dt. \tag{2.7}$$

In a certain weak sense (2.2) corresponds to initial values $\mathbf{u}(0) = 0$, but we will not dwell on this; the technique we use gives $\mathbf{u} = \lim \mathbf{u}^\epsilon$ weakly in $\mathfrak{H}$ with $\mathbf{u}^\epsilon(0) = 0$.

We recall next from [**16**] that one can write $W' = L^2(V(t))' = L^2(V'(t))$, where $L^2(V'(t))$ is defined as $\theta^{-1}W = \{\theta^{-1}(\cdot)v(\cdot) | v \in W\}$. Following [**16**], we define L: $W \to W'$ by $Lu = u'$ with $D(L) = \{u \in W | u' \in L^2(H); u(0) = 0\}$ and L': $W \to W'$ by $L'u = -u'$ with $D(L') = \{u \in W | u' \in L^2(H); u(T) = 0\}$. By [**16**, remark 2.3], L and L' are densely defined and by [**16**, Theorem 4.7 (or 4.8)], $L_s = \bar{L} = (L')^* = L_w$; both results are a consequence of $S^{-1}(\cdot)$ being weakly C^1 ($\bar{L}$ denotes the closure of L).

Remark 2.3. One sees easily that, following the presentation of [**16**], the present theory can be developed for $W = L^p(V(t))$

$$\left(\text{with } W' = L^q(V'(t)) \text{ for } \frac{1}{p} + \frac{1}{q} = 1\right)$$

when $p \geqq 2$. Also, nonlinear A might be envisioned. However, we confine ourselves to the L^2 case and linear A for simplicity.

We recall now that a (nonlinear) map $Q: W \to W'$ (W being, for example, a reflexive Banach space) is monotone if

$$\operatorname{Re}\langle Q(u) - Q(v), u - v\rangle \geqq 0 \tag{2.8}$$

for $u, v \in D(Q) \subset W$. When Q is linear and monotone with $D(Q)$ dense we say that Q is maximal monotone if it is not the proper restriction of another linear monotone operator. A (nonlinear) operator $Q: W \to W'$ with $D(Q) = W$ is bounded if it takes bounded sets into bounded sets; such a Q is hemicontinuous if it is continuous from lines in W to the weak topology of W' and it is coercive (in a general sense) if $\operatorname{Re}\langle Qx, x\rangle \geqq \varphi(\|x\|)\|x\|$ where $\varphi(x) \to \infty$ as $x \to \infty$ (with $\varphi(x)$ possibly negative for small x). Now, as in [**16**], by a result of Brezis [**3**], L_s is maximal monotone since $L_s = L_w$ (this holds also for the reflexive Banach space $W = L^p(V(t))$, $p \geqq 2$) and we cite the following special case of a result of Browder [**5**] (only single valued maps are considered here).

THEOREM 2.4. *Assume that $\mathfrak{W}$ is a reflexive Banach space. Let $\mathfrak{L}: \mathfrak{W} \to \mathfrak{W}'$ be a (closed and densely defined) linear maximal monotone map and $\mathfrak{C}: \mathfrak{W} \to \mathfrak{W}'$ a monotone, hemicontinuous, bounded, and coercive map. Then $\mathfrak{L} + \mathfrak{C}$ maps $D(\mathfrak{L}) \subset \mathfrak{W}$ onto $\mathfrak{W}'$.*

We will apply these results to the regularized strong parabolic problem associated with (2.2). Thus we consider

Problem 2.5. Find $\mathbf{u}^\epsilon \in \mathfrak{W}$ such that

$$\mathfrak{L}\mathbf{u}^\epsilon + \lambda\mathbf{u}^\epsilon + \mathfrak{C}\mathbf{u}^\epsilon + \epsilon\mathfrak{B}\mathbf{u}^\epsilon = \mathbf{f} \tag{2.9}$$

where

$$\mathfrak{L} = \begin{pmatrix} L_s & 0 \\ 0 & L_s \end{pmatrix}; \tag{2.10}$$

$$\mathfrak{B} = \begin{pmatrix} \theta^{-1} & 0 \\ 0 & \theta^{-1} \end{pmatrix};$$

$$\mathfrak{C} = \begin{pmatrix} \theta^{-1}S^{-1}\dot{S}^{-1} & -S \\ S & 0 \end{pmatrix}.$$

We check first that a solution of (2.9) satisfies (2.2) with a suitable ϵ term added. Thus let $\mathbf{u}^\epsilon$ satisfy (2.9) and let $\mathbf{v} \in \mathfrak{W}$ with $\mathbf{v}' \in \mathfrak{H}$ and $\mathbf{v}(T) = 0$. Then $\mathbf{v} \in D(\mathfrak{L}')$ in an obvious notation and taking $\mathfrak{W} - \mathfrak{W}'$ brackets in (2.9) with $\mathbf{v}$ we obtain

$$-(\mathbf{u}^\epsilon, \mathbf{v}')_{\mathfrak{H}} + \lambda(\mathbf{u}^\epsilon, \mathbf{v})_{\mathfrak{H}} + (\mathfrak{A}\mathbf{u}^\epsilon, S\mathbf{v})_{\mathfrak{H}} + \epsilon(S\mathbf{u}^\epsilon, S\mathbf{v})_{\mathfrak{H}} = (\mathbf{f}, \mathbf{v})_{\mathfrak{H}}. \tag{2.11}$$

Indeed the individual equations in (2.9) are

$$L_s u_1^\epsilon + \lambda u_1^\epsilon + \theta^{-1}S^{-1}\dot{S}^{-1}u_1^\epsilon - Su_2^\epsilon + \epsilon\theta^{-1}u_1^\epsilon = 0 \tag{2.12}$$

$$L_s u_2^\epsilon + \lambda u_2^\epsilon + Su_1^\epsilon + \epsilon\theta^{-1}u_2^\epsilon = f \tag{2.13}$$

and one notes here that if $x \in H$ and $v \in V(t)$ then $\langle x, v\rangle = (x, v)$ whereas if $x \in V'(t)$ and $v \in V(t)$ then

$$\langle x, v\rangle = ((\theta x, v)) = (S\theta x, Sv). \tag{2.14}$$

Hence, in appropriate spaces, we have $\langle \theta^{-1}S^{-1}\dot{S}^{-1}u_1^\epsilon, v_1\rangle = (\dot{S}^{-1}u_1^\epsilon, Sv_1)$, $-\langle Su_2^\epsilon, v_1\rangle = -(Su_2^\epsilon, v_1) = -(u_2^\epsilon, Sv_1)$, $\langle \theta^{-1}u_1^\epsilon, v_1\rangle = (Su_1^\epsilon, Sv_1)$, etc., so $\langle \mathfrak{E}\mathbf{u}^\epsilon, \mathbf{v}\rangle = (\mathfrak{A}\mathbf{u}^\epsilon, S\mathbf{v})$, $\langle \mathfrak{B}\mathbf{u}^\epsilon, \mathbf{v}\rangle = (S\mathbf{u}^\epsilon, S\mathbf{v})$, etc. Now the idea in what follows is to find solutions $\mathbf{u}^\epsilon$ of (2.9) with $|\mathbf{u}^\epsilon|_{\mathfrak{H}}$ and $\epsilon^{\frac{1}{2}}||\mathbf{u}^\epsilon||_{\mathfrak{W}}$ bounded (note that one does *not* expect $||\mathbf{u}^\epsilon||_{\mathfrak{W}}$ to be bounded; cf. [**35**]). Then by weak compactness (or weak sequential compactness) we can take limits in (2.11) as $\epsilon \to 0$, which leads to a solution of (2.2).

By the previous discussion $\mathfrak{L}$ is obviously maximal monotone and it remains to see when $\mathfrak{C} = \mathfrak{E} + \lambda + \epsilon\mathfrak{B}$ satisfies the conditions of Theorem 2.4. First one sees that $\mathfrak{C}$ is obviously continuous from $\mathfrak{W}$ to $\mathfrak{W}'$ since the injections $\mathfrak{W} \twoheadrightarrow \mathfrak{H} \twoheadrightarrow \mathfrak{W}'$ are continuous and θ^{-1} is an isometric isomorphism $\mathfrak{W} \to \mathfrak{W}'$. Therefore $\mathfrak{C}$ is trivially hemicontinuous and, since it is continuous and linear, $\mathfrak{C}$ is also a bounded map. We need consider therefore only $\mathrm{Re}\langle \mathfrak{C}\mathbf{u}, \mathbf{u}\rangle = \Xi$ which, using (2.12) and (2.13), is the sum of two terms:

$$\Xi_1 = \lambda|u_1|^2_{L^2(H)} + \mathrm{Re}(\dot{S}^{-1}u_1, Su_1)_{L^2(H)} - \mathrm{Re}(Su_2, u_1)_{L^2(H)} + \epsilon||u_1||_W{}^2 \tag{2.15}$$

$$\Xi_2 = \lambda|u_2|^2_{L^2(H)} + \epsilon||u_2||_W{}^2 + \mathrm{Re}(Su_1, u_2)_{L^2(H)}. \tag{2.16}$$

Since $\mathrm{Re}(Su_2, u_1) = \mathrm{Re}(Su_1, u_2)$, in order to have coercivity in the form $\Xi \geqq \epsilon||\mathbf{u}||^2_{\mathfrak{W}}$ (and thus also monotonicity since $\mathfrak{C}$ is linear), it suffices to assume that

$$\mathrm{Re}(x, \dot{S}^{-1}(t)S(t)x)_H \geqq -\beta|x|_H{}^2 \tag{2.17}$$

for $x \in V(t)$ and then we take $\lambda > \beta$ (the hypotheses (2.17) will also arise in section three); if $\beta \leqq 0$ we simply neglect this term. Thus under these circumstances we can apply Theorem 2.4 to obtain

THEOREM 2.6. *Let $a(t, \cdot, \cdot)$ be a family of continuous sesquilinear coercive selfadjoint forms, and put on $V(t)$ the corresponding Hilbert structure, so that $A(t) = \theta^{-1}(t)$ with $S^2(t) = A(t)$ in H as in Lemma* 2.1. *Assume that $S^{-1}(\cdot)$ is weakly C^1 and suppose that* (2.17) *holds. Then for $\lambda > \beta$ there exists a solution $\mathbf{u}^\epsilon$ of problem* 2.5.

Now our solutions $\mathbf{u}^\epsilon$ satisfy (2.12) and (2.13) and we recall from [**16**] that $\mathrm{Re}\langle L_s u, u\rangle \geqq 0$. Thus, using (2.15), (2.16), and (2.17), one obtains

$$\begin{aligned}(2.18)\qquad \epsilon\|\mathbf{u}^\epsilon\|_{\mathfrak{W}}^2 + (\lambda - \beta)|\mathbf{u}^\epsilon|_{\mathfrak{H}}^2 &\leqq \mathrm{Re}\langle f, u_2^\epsilon\rangle \\ &= \mathrm{Re}(f, u_2^\epsilon) \\ &\leqq |f|_{L^2(H)}|u_2^\epsilon|_{L^2(H)} \\ &\leqq \delta|u_2^\epsilon|^2_{L^2(H)} + |f|^2_{L^2(H)}/\delta.\end{aligned}$$

Consequently, taking $\delta < \lambda - \beta$ we see that, for $\lambda > \beta$, $|\mathbf{u}^\epsilon|_{\mathfrak{H}} \leqq c_5$ and $\epsilon^{\frac{1}{2}}\|\mathbf{u}^\epsilon\|_{\mathfrak{W}} \leqq c_6$. Take now for simplicity $\epsilon = \epsilon_n \to 0$, so by weak sequential compactness there exists a subsequence $\mathbf{u}^{\epsilon_n}$ (again denoted by ϵ_n) with $\mathbf{u}^{\epsilon_n} \to \mathbf{u}$ weakly in $\mathfrak{H}$. Since $\mathfrak{A}$ is linear and continuous from $\mathfrak{H}$ to $\mathfrak{H}$ it is weakly continuous (cf. [**9**]) and we can take limits in (2.11) to obtain (2.2); note here that

$$(2.19)\qquad \epsilon_n|(S\mathbf{u}^{\epsilon_n}, S\mathbf{v})_{\mathfrak{H}}| \leqq c_6\epsilon_n^{\frac{1}{2}}|S\mathbf{v}|_{\mathfrak{H}} \to 0$$

because $|S\mathbf{u}^{\epsilon_n}|_{\mathfrak{H}} = \|\mathbf{u}^{\epsilon_n}\|_{\mathfrak{W}}$. This proves

THEOREM 2.7. *Under the hypotheses of Theorem* 2.6, *there exists a solution of problem* 2.2.

Remark 2.8. When $V(t) = V$ we note that in general $S^{-1}(t)$ will still depend on t since $a(t, x, y)$ will usually depend on t for $x, y \in V$. Thus, some differentiability relative to the forms $a(t, x, y)$ remains in the hypotheses, as one expects from [**32**].

Remark 2.9. It is clear that by suitably increasing λ the preceding theory applies to selfadjoint forms $a(t, x, y)$ where $a(t, x, x) \geqq \alpha_t\|x\|_t^2 - \tilde{\beta}|x|^2$ (cf. (2.4) and (2.5) and the statement immediately following). One would then take $((x, y))_t = a(t, x, y) + \tilde{\beta}(x, y)$.

3. One has the impression from section two and previous work of State [**44**] that whenever $a(t, x, y) = p(t, x, y) + r(t, x, y)$, with $r(t, x, y)$ a small perturbation of a selfadjoint coercive $p(t, x, y)$, then the introduction of a new scalar product $((x, y))_t = p(t, x, y)$ on $V(t)$ allows one to deal at once with a more meaningful $S(t)$; thus a certain economy in the hypotheses is promised. In view of remark 2.9 we need only consider coercive selfadjoint $p(t, x, y)$. In this section we give a version of a technique of State [**44**] in which the new scalar product is now used. This technique involves working with (1.2) directly by introducing a new space of test functions and was reported on briefly in [**13**]; the hypotheses indicated in [**13**] imply that $V(t) = V$, as pointed out by T. Kato (cf. [**16**]), but the technique is flexible and leads to a result using (2.17) (cf. [**44**]). The result is weaker in certain respects than Theorem 2.7 but it contains terms not in (2.2) and it illustrates again the role of (2.17); we include it also to display the technique of using new spaces of test functions and remark 3.7 provides additional motivation.

This kind of technique has led recently to some existence results for first order problems where regularity of data is systematically inserted into the problem and then an equivalent problem with new test functions is solved (see [**12**; **17**; **37**]); we consider such techniques an interesting complement to the Lions Projection Theorem.

Thus we take $a = p + r$ as indicated below with $|p(t, x, y)| \leqq c_7{}^t||x||_t||y||_t$. Then given that $p(t, \cdot, \cdot)$ is selfadjoint with $p(t, x, x) \geqq \tilde{\alpha}_t||x||_t{}^2$ we put on $V(t)$ the new scalar product $((x, y))_t = p(t, x, y)$ and use this from now on. As before, writing $((x, y))_t = p(t, x, y) = ((\mathfrak{P}(t)x, y))_t$, we have $\mathfrak{P}(t)$ is the identity and no measurability assumption is required since we work in the new W. We write also $r(t, x, y) = (R(t)x, y)$ and we assume that $Ru \in L^2(H)$ for $u \in W$ with $|r(t, x, y)| \leqq c_8||x||_t|y|$ (cf. [**32**]). Operators $\theta(t)$ and $S(t)$ are determined as before and we write $p(t, x, y) = \langle P(t)x, y\rangle$ with $P(t) = S^2(t)$ as an operator in H.

The idea now is to work with (1.2) directly and use the Lions Projection Theorem applied to suitable spaces. Thus let $W = L^2(V(t))$, emphasizing that this is the new W determined by the $p(t, \cdot, \cdot)$, and set

$$F = \{u \in W | u' \in L^2(H); u(0) = 0\},$$

$$K = \{v \in W | v' \in L^2(H); v(T) = 0\},$$

and
$$\Phi = \left\{\varphi | \varphi(t) = S^{-1}(t)\int_0^t e^{2\lambda\gamma}S(\lambda)v(\lambda)d\lambda \text{ for } v \in K\right\}$$

where $\gamma \geqq 0$ is real. If we call the left side of (1.2) (Eu, v) and the right side $L(v)$, then the problem to solve is

Problem 3.1. Given $f \in L^2(H)$ and $u_1 \in H$ find $u \in F$ such that $E(u, v) = L(v)$ for all $v \in K$.

Now we assume again that $S^{-1}(\cdot)$ is weakly C^1 and, if $\varphi \in \Phi$, it follows that

$$\varphi' - \dot{S}^{-1}S\varphi = e^{2\gamma t}v. \tag{3.1}$$

Therefore we can write (note that $\varphi(0) = 0$)

$$\tilde{E}(u, \varphi) = E(u, v) = E(u, e^{-2\gamma t}(\varphi' - \dot{S}^{-1}S\varphi)); \tag{3.2}$$

$$\tilde{L}(\varphi) = L(v) = (u_1, \varphi'(0)) + \int_0^T (f, e^{-2\gamma t}(\varphi' - \dot{S}^{-1}S\varphi))dt. \tag{3.3}$$

Then solving problem 3.1 is equivalent to solving

Problem 3.2. Find $u \in F$ such that $\tilde{E}(u, \varphi) = \tilde{L}(\varphi)$ for all $\varphi \in \Phi$.

We check first that $\Phi \subset F$ algebraically. Clearly $\varphi(0) = 0$ and $\varphi(t) \in V(t)$. Then, using Hölder's inequality, one has

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$$(3.4)\qquad \|\varphi\|_W^2 = |S\varphi|^2_{L^2(H)} = \int_0^T \left| \int_0^t e^{2\gamma\lambda} S(\lambda)v(\lambda)d\lambda \right|^2 dt$$
$$\leqq c_{11} \int_0^T \left(\int_0^t |Sv|^2 d\lambda \right) dt \leqq c_{11} T \|v\|_W^2.$$

Finally from (3.1), we see that $\varphi' \in L^2(H)$ since $v \in W \subset L^2(H)$, and

$$(3.5)\qquad |\dot{S}^{-1} S\varphi|_{L^2(H)} \leqq c_4 |S\varphi|_{L^2(H)} = c_4 \|\varphi\|_W$$

where $\|\dot{S}^{-1}(t)\| \leqq c_4$. We introduce a scalar product on F by

$$(3.6)\qquad ((u, v))_F = ((u, v))_W + (u', v')_{L^2(H)},$$

whereas for Φ we write

$$(3.7)\qquad ((\varphi, \psi))_\Phi = ((\varphi, \psi))_F + (\varphi'(0), \psi'(0)).$$

It follows easily that F is a Hilbert space. Note that from

$$u_n(t) = \int_0^t u_n'(\xi) d\xi$$

there results

$$|u_n(t)| \leqq t^{\frac{1}{2}} \left(\int_0^t |u_n'(\xi)|^2 d\xi \right)^{\frac{1}{2}}$$

and hence $u(0) = 0$ is preserved in taking limits. Since the injection $\Phi \to F$ is obviously continuous we can state

LEMMA 3.3. *F is a Hilbert space and $\Phi \subset F$ is a pre-Hilbert space with continuous injection.*

This serves as context for the Lions Projection Theorem (see [**32**]) which we cite as

THEOREM 3.4. *Let $\Phi \subset F$ be a pre-Hilbert space contained in a Hilbert space F with continuous injection (Φ not necessarily dense or complete). Assume that $\tilde{E}(u, \varphi)$ is a sesquilinear form on $F \times \Phi$ with $u \to \tilde{E}(u, \varphi)$: $F \to \mathbf{C}$ continuous and $|\tilde{E}(\varphi, \varphi)| \geqq \epsilon \|\varphi\|_\Phi^2$. Let $\tilde{L}$ be a continuous conjugate linear form on Φ. Then there exists $u \in F$ such that $\tilde{E}(u, \varphi) = \tilde{L}(\varphi)$ for all $\varphi \in \Phi$.*

Now, looking at the left side of (2.1), it is obvious that $u \to E(u, v) = \tilde{E}(u, \varphi)$: $F \to \mathbf{C}$ is continuous so, in order to apply Theorem 3.4, it remains only to check that $\tilde{L}$ is continuous and that $|\tilde{E}(\varphi, \varphi)| \geqq \epsilon \|\varphi\|_\Phi^2$ for $\varphi \in \Phi$.

LEMMA 3.5. *The map $\varphi \to \tilde{L}(\varphi)$: $\Phi \to \mathbf{C}$ is continuous.*

Proof. First, referring to (3.3), we note that

$$|(u_1, \varphi'(0))| \leqq |u_1| \, |\varphi'(0)| \leqq c_{12} |\varphi'(0)|.$$

Further, since $|a + b|^2 \leqq 2(a^2 + b^2)$ and $e^{-2\gamma t} \leqq 1$, we have

$$(3.8)\quad \left| \int_0^T (f, e^{-2\gamma t}(\varphi' - \dot{S}^{-1}S\varphi))dt \right|$$
$$\leqq \int_0^T e^{-2\gamma t}|f|\,|\varphi' - \dot{S}^{-1}S\varphi|dt$$
$$\leqq \sqrt{2}\int_0^T |f|(|\varphi'|^2 + |\dot{S}^{-1}S\varphi|^2)^{\frac{1}{2}}dt$$
$$\leqq \sqrt{2}\left(\int_0^T |f|^2dt\right)^{\frac{1}{2}}\left(\int_0^T (|\varphi'|^2 + |\dot{S}^{-1}S\varphi|^2)dt\right)^{\frac{1}{2}}$$
$$\leqq c_{13}\left(\int_0^T (|\varphi'|^2 + c_4{}^2|S\varphi|^2)dt)\right)^{\frac{1}{2}} \leqq c_{14}||\varphi||_\Phi.$$

Now write

$$(3.9)\qquad X = \operatorname{Re}\int_0^T a(t, \varphi, \varphi' - \dot{S}^{-1}S\varphi)e^{-2\gamma t}dt,$$

$$(3.10)\qquad Y = \operatorname{Re}\int_0^T (B(t)\varphi', \varphi' - \dot{S}^{-1}S\varphi)e^{-2\gamma t}dt,$$

$$(3.11)\qquad Z = -\operatorname{Re}\int_0^T (\varphi', (e^{-2\gamma t}(\varphi' - \dot{S}^{-1}S\varphi))')dt,$$

so $X + Y + Z = \operatorname{Re} \tilde{E}(\varphi, \varphi)$. We note first that if $\varphi \in \Phi$ with

$$h = S\varphi = \int_0^t e^{2\gamma\lambda}S(\lambda)v(\lambda)d\lambda$$

then

$$(3.12)\quad \frac{d}{dt}p(t, \varphi, \varphi) = \frac{d}{dt}((\varphi, \varphi))_t = (h, h)' = 2\operatorname{Re}(h, h')$$
$$= 2\operatorname{Re}(S\varphi, S(\varphi' - \dot{S}^{-1}S\varphi)) = 2\operatorname{Re} p(t, \varphi, \varphi' - \dot{S}^{-1}S\varphi).$$

Furthermore one has easily

$$(3.13)\quad \left|2\operatorname{Re}\int_0^T r(t, \varphi, \varphi' - \dot{S}^{-1}S\varphi)e^{-2\gamma t}dt\right|$$
$$\leqq 2c_8\int_0^T ||\varphi||_t(|\varphi'| + |\dot{S}^{-1}S\varphi|)e^{-2\gamma t}dt$$
$$\leqq 2c_8\left\{\frac{1}{2}\int_0^T (||\varphi||_t^2 + |\varphi'|^2)e^{-2\gamma t}dt + c_4\int_0^T ||\varphi||_t^2e^{-2\gamma t}dt\right\}$$
$$\leqq c_8(1 + 2c_4)\int_0^T ||\varphi||_t^2e^{-2\gamma t}dt + c_8\int_0^T |\varphi'|^2e^{-2\gamma t}dt.$$

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Consequently there results

$$(3.14)\quad 2X = \int_0^T \frac{d}{dt}\, p(t, \varphi, \varphi)e^{-2\gamma t}dt + \int_0^T 2\,\mathrm{Re}\, r(t, \varphi, \varphi' - \dot{S}^{-1}S\varphi)e^{-2\gamma t}dt$$

$$\geqq ||\varphi(T)||^2 e^{-2\gamma T} - c_8 \int_0^T |\varphi'|^2 e^{-2\gamma t}dt$$

$$+ (2\gamma - c_8 - 2c_8c_4)\int_0^T ||\varphi||_t^2 e^{-2\gamma t}dt.$$

It is evident that

$$(3.15)\quad 2Y \geqq -2c_3 \int_0^T |\varphi'|^2 e^{-2\gamma t}dt - c_3c_4 \int_0^T (|\varphi'|^2 + ||\varphi||_t^2)e^{-2\gamma t}dt$$

so it remains to find estimates for Z compatible with (3.14) and (3.15). The procedure we use involves the technical assumption that S^{-1} is weakly C^2 and in view of Theorem 2.7 this seems excessive in general (cf. remarks in sections one and two about extensions of problem 2.2 to include additional terms). Thus, techniques based on new spaces of test functions and Theorem 3.4 may not always give the best possible results, but they have provided already a number of general abstract results in areas where no such theorems were previously available (see [**12**; **17**; **37**; **44**]). In particular such techniques are seen to be a useful adjunct to the Lions Projection Theorem.

Now to estimate Z, we recall that $\varphi(0) = 0$ and $v(T) = 0$ implies that $(\varphi' - \dot{S}^{-1}S\varphi)(T) = 0$. Therefore

$$(3.16)\quad 2Z = -2\,\mathrm{Re}\int_0^T ((\varphi' - \dot{S}^{-1}S\varphi)e^{-2\gamma t}, (e^{-2\gamma t}(\varphi' - \dot{S}^{-1}S\varphi))')e^{2\gamma t}dt$$

$$- 2\,\mathrm{Re}\int_0^T (\dot{S}^{-1}S\varphi, (e^{-2\gamma t}(\varphi' - \dot{S}^{-1}S\varphi))')dt$$

$$= |\varphi'(0)|^2 + 2\gamma \int^T |\varphi' - \dot{S}^{-1}S\varphi|^2 e^{-2\gamma t}dt$$

$$+ 2\,\mathrm{Re}\int_0^T ((\dot{S}^{-1}S\varphi)', \varphi' - \dot{S}^{-1}S\varphi)e^{-2\gamma t}dt.$$

However we can write $(\dot{S}^{-1}S\varphi)' = \ddot{S}^{-1}S\varphi + \dot{S}^{-1}(S\varphi)'$ and one notes that $\varphi' - \dot{S}^{-1}S\varphi = S^{-1}(S\varphi)'$ since $\varphi' = (S^{-1}S\varphi)' = \dot{S}^{-1}S\varphi + S^{-1}(S\varphi)'$. Now $||\ddot{S}^{1}(t)|| \leqq c_{15}$ and, looking at individual terms, we have

$$(3.17)\quad \left|2\,\mathrm{Re}\int_0^T (\ddot{S}^{-1}S\varphi, \varphi' - \dot{S}^{-1}S\varphi)e^{-2\gamma t}dt\right|$$

$$\leqq (c_{15}^2 + 2c_{15}c_4)\int_0^T ||\varphi||_t^2 e^{-2\gamma t}dt + \int_0^T |\varphi'|^2 e^{-2\gamma t}dt.$$

Further, using (2.17) we obtain

$$(3.18)\qquad 2\,\mathrm{Re}\int_0^T (\dot{S}^{-1}(S\varphi)', S^{-1}(S\varphi)')e^{-2\gamma t}dt \geqq -\beta\int_0^T |S^{-1}(S\varphi)'|^2 e^{-2\gamma t}dt$$
$$\geqq -\beta\int_0^T |\varphi' - \dot{S}^{-1}S\varphi|^2 e^{-2\gamma t}dt.$$

Consequently, we obtain from (3.16) and (3.18)

$$(3.19)\quad 2Z \geqq |\varphi'(0)|^2 + (2\gamma-\beta)\int_0^T |\varphi' - \dot{S}^{-1}S\varphi|^2 e^{-2\gamma t}dt$$
$$- c_{15}(c_{15}+2c_4)\int_0^T ||\varphi||_t^2 e^{-2\gamma t}dt - \int_0^T |\varphi'|^2 e^{-2\gamma t}dt.$$

Finally, since $ab \leqq \delta a^2 + b^2/\delta$, where we take $0 < \delta < 1$, there results

$$(3.20)\quad \int_0^T |\varphi' - \dot{S}^{-1}S\varphi|^2 e^{-2\gamma t}dt \geqq (1-\delta)\int_0^T |\varphi'|^2 e^{-2\gamma t}dt$$
$$-\left(\frac{1}{\delta}-1\right)\int_0^T |\dot{S}^{-1}S\varphi|^2 e^{-2\gamma t}dt.$$

Therefore (3.19) becomes

$$(3.21)\quad 2Z \geqq |\varphi'(0)|^2 - c_{16}\int_0^T ||\varphi||_t^2 e^{-2\gamma t}dt + c_{17}\int_0^T |\varphi'|^2 e^{-2\gamma t}dt,$$

where

$c_{16} = (2\gamma-\beta)(1/\delta - 1)c_4^2 - c_{15}^2 - 2c_{15}c_4$ and $c_{17} = (2\gamma-\beta)(1-\delta) - 1.$

Combining (3.14), (3.15), and (3.21) it follows that

$$(3.22)\quad 2\,\mathrm{Re}\,\tilde{E}(\varphi,\varphi) \geqq |\varphi'(0)|^2$$
$$+ (2\gamma - c_8 - 2c_8c_4 - c_3c_4 - c_{16})\int_0^T ||\varphi||_t^2 e^{-2\gamma t}dt$$
$$+ (c_{17} - c_3c_4 - 2c_3 - c_8)\int_0^T |\varphi'|^2 e^{-2\gamma t}dt.$$

We write $2\gamma - c_8 - 2c_8c_4 - c_3c_4 - c_{16} = 2\gamma(1 - c_4^2(1/\delta - 1)) - c_{18}$ and $c_{17} - c_3c_4 - 2c_3 - c_8 = (2\gamma-\beta)(1-\delta) - c_{19}$. Now first pick δ so that $1 > c_4^2(1/\delta - 1)$ (i.e. $c_4^2/(1+c_4^2) < \delta < 1$) and then pick γ large enough so that $(2\gamma-\beta)(1-\delta) > c_{19}$ with $2\gamma(1 - c_4^2(1/\delta-1)) > c_{18}$. Under these circumstances $2\,\mathrm{Re}\,\tilde{E}(\varphi,\varphi) \geqq \epsilon\,||\varphi||_\Phi^2$ and we have

THEOREM 3.6. *Let $a(t,x,y) = p(t,x,y) + r(t,x,y)$ where $p(t,\cdot,\cdot)$ is a family of continuous, selfadjoint, coercive, sesquilinear forms on $V(t) \times V(t)$ and put on $V(t)$ the corresponding Hilbert structure so that $P(t) = \theta^{-1}(t)$ with $S^2(t) = P(t)$ as an operator in H. Let $|r(t,x,y)| \leqq c_8||x||_t|y|$ for $x, y \in V(t)$*

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and assume that $Ru \in L^2(H)$ *for* $u \in W$ *where* $r(t, x, y) = (R(t)x, y)$. *Let* $B(t) \in \mathfrak{L}(H)$ *with* $\|B(t)\| \leqq c_3$ *and* $B(\cdot)h$ *measurable for* $h \in H$. *Assume that* $S^{-1}(\cdot)$ *is weakly* C^2 *and suppose that* (2.17) *holds. Then there exists a solution of problem* 3.1.

Remark 3.7. In [**44**] State proves a version of Theorem 3.6 without introducing the new Hilbert structure determined by $p(t, \cdot, \cdot)$. This formulation requires one to express the interaction of $p(t, \cdot, \cdot)$ with the way $V(t)$ varies in order to describe $d/dt\, p(t, u, v)$ and this originally led to the introduction of the space Φ. Such theorems are indeed necessary since one may know how the $V(t)$ vary in their original description by $S^{-1}(t)$ but not know a priori much about the new $S^{-1}(t)$ determined by $p(t, \cdot, \cdot)$. State's result in [**44**] connects these descriptions and the linking hypothesis is that $S(t)\mathfrak{P}(t)S^{-1}(t)$ be weakly C^1 where $p(t, x, y) = ((\mathfrak{P}(t)x, y))$ and $S(t)$ is the original standard operator (cf. [**13**]). We have developed the technique here in section three in terms of the new $S(t)$ for purposes of comparison with section two and the application of the method of section two using the original $S(t)$ is developed in [**44**].

Remark 3.8. Suppose that for $h \in H$ fixed we have

$$\frac{d}{dt}|S^{-1}(t)h|^2 \geqq -2\beta|S^{-1}(t)h|^2. \tag{3.23}$$

Then $\operatorname{Re}(S^{-1}(t)h, \dot{S}^{-1}(t)h) \geqq -\beta|S^{-1}(t)h|^2$ and setting $x = S^{-1}(t)h$ we have $\operatorname{Re}(x, \dot{S}^{-1}(t)S(t)x) \geqq -\beta|x|^2$; since any $x \in V(t)$ is of the form $x = S^{-1}(t)h$ we obtain (2.17). Thus (3.23) implies (2.17) (and conversely) and in order to elucidate (3.23) we set first $y = |S^{-1}h|^2$. Then from $y' + 2\beta y \geqq 0$ one has $(y \exp 2\beta t)' \geqq 0$ and thus upon integration $y(t) \exp 2\beta t \geqq y(s) \exp 2\beta s$, for $t \geqq s$. Therefore, setting $S^{-1}(t)h = z$, one obtains for $z \in V(t)$ (taking square roots)

$$|S^{-1}(s)S(t)z| \leqq |z| \exp \beta(t - s). \tag{3.24}$$

Consequently, $S^{-1}(s)S(t)$ extends by continuity from the dense set $V(t)$ to all of H as a bounded operator T for $t \geqq s$. Thus T^* is defined in all of H, so for $x \in V(t)$ and $y \in H$

$$(x, T^*y) = (Tx, y) = (S^{-1}(s)S(t)x, y) = (S(t)x, S^{-1}(s)y). \tag{3.25}$$

This shows that $x \to (S(t)x, S^{-1}(s)y)\colon V(t) \to \mathbf{C}$ is continuous in the topology of H, which implies that $S^{-1}(s)y \in D(S(t))$ (recall $S(t)$ is selfadjoint); it follows that $T^*y = S(t)S^{-1}(s)y$ for any $y \in H$ and $S(t)S^{-1}(s)$ is therefore a bounded operator for $t \geqq s$. In particular, $V(s) \subset V(t)$ for $t \geqq s$.

We note also that if $V(s) \subset V(t)$ for $t \geqq s$ then $P = S(t)S^{-1}(s)$ is a bounded operator in H (cf. [**9**]) and for $y \in V(t)$ with $x \in H$ one has

$$(y, Px) = (S^{-1}(s)S(t)y, x).$$

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Thus, for $y \in V(t)$, $P^*y = S^{-1}(s)S(t)y$ and hence $S^{-1}(s)S(t)$ extends to be a bounded operator $T = P^*$ in H. We do not know however if (3.23) holds in this case; it seems probable that it is not so in general. A concrete example where (3.23) holds is provided by $V(t) = H^{\alpha(t)}(-\infty, \infty)$ where $\alpha(t) = (2T - t)/2T$ (for details see [**44**]).

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