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Abstract

We analyse the mod p étale cohomology of the Lubin–Tate tower both with compact support and without support. We prove that there are no supersingular representations in the H^1_c of the Lubin–Tate tower. On the other hand, we show that in H^1 of the Lubin–Tate tower appears the mod p local Langlands correspondence and the mod p local Jacquet–Langlands correspondence, which we define in the text. We discuss the local-global compatibility part of the Buzzard–Diamond–Jarvis conjecture which appears naturally in this context.

1. Introduction

In recent years, the mod p and p-adic local Langlands correspondences emerged as a form of refinement of the l-adic Langlands correspondence for $l = p$. This program was basically started by Christophe Breuil and then, by the work of many people, the p-adic local Langlands correspondence was established for $\mathrm{GL}_2(\mathbb{Q}_p)$ (see [Col10] for a final step). Unfortunately, as for now, it is hard to predict how the conjectures should look like for $\mathrm{GL}_2(F)$ where F is a finite extension of $\mathbb{Q}_p$ (or other groups) basically because there are too many objects on the automorphic side (see [BP12]), so that the pure representation theory does not indicate which representations of GL_2 we should choose for our correspondence. A possible remedy to this situation might come by looking at the mod p and completed p-adic cohomology of the Lubin–Tate tower. Let us remind the reader that the classical local Langlands correspondence was also firstly proved for GL_2 by representation-theoretic methods. Only afterwards by using geometric arguments and finding the correspondence in the l-adic cohomology of the Lubin–Tate tower, it was proved for GL_n by Harris and Taylor in [HT01]. Our aim is to do a step in this direction in the $l = p$ setting, hoping that this will give us an insight how to define a correspondence for other groups than $\mathrm{GL}_2(\mathbb{Q}_p)$.

By the recent work of Emerton (see [Eme11]) we know that the p-adic completed (respectively mod p) cohomology of the tower of modular curves realizes the p-adic (respectively mod p) local Langlands correspondence. In this article we will obtain an analogous but weaker result for the mod p cohomology of the Lubin–Tate tower over $\mathbb{Q}_p$. The main result of Emerton from [Eme11] will be a key ingredient in our proof. We will analyse both the cohomology with compact support and the cohomology without support of the Lubin–Tate tower. Here are the two main results which we prove.

(1) In the first cohomology group $H^1_{\mathrm{LT}, \bar{\mathbb{F}}_p}$ of the Lubin–Tate tower, there appears the mod p local Langlands correspondence and the naive mod p Jacquet–Langlands correspondence, meaning that there is an injection of representations

$$\pi \otimes \bar{\rho} \hookrightarrow H^1_{\mathrm{LT}, \bar{\mathbb{F}}_p}$$

and $\sigma \otimes \pi \otimes \bar{\rho}$ appears as a subquotient in $H^1_{\mathrm{LT}, \bar{\mathbb{F}}_p}$, where π is a supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, $\bar{\rho}$ is its associated local mod p Galois representation and σ is the naive mod p Jacquet–Langlands correspondence (for details, see § 8).

(2) The first cohomology group $H^1_{\mathrm{LT}, c, \bar{\mathbb{F}}_p}$ with compact support of the Lubin–Tate tower does not contain any supersingular representations. This surprising result shows that the mod p situation is much different from its mod l analogue. It also permits us to show that the mod p local Langlands correspondence appears in H^1 of the ordinary locus; again a fact which is different from the l-adic setting for supercuspidal representations.

Before sketching how we obtain the above results, let us outline the first main difference with the non-abelian Lubin–Tate theory in the l-adic case. When $l \neq p$ the comparison between the Lubin–Tate tower and the modular curve tower is made via vanishing cycles. For that, we need to know that the stalks of vanishing cycles gives the cohomology of the Lubin–Tate tower, or in other words we need an analogue of the theorem proved by Berkovich in [Ber96]. However, when $l = p$, the statement does not hold anymore (see [Ber96, Remark 3.8(iv)]) and hence we cannot imitate directly the arguments from the l-adic theory.

To circumvent this difficulty, we work from the beginning at the rigid-analytic level and consider embeddings from the ordinary and the supersingular tubes into modular curves. This gives two long exact sequences of cohomology, depending on whether we take compact support or a support in the ordinary locus and we start our analysis by resuming facts about the geometry of modular curves. We recall a decomposition of the ordinary locus, which proves that its cohomology is induced from some proper parabolic subgroup of GL_2. We use this fact several times in order to have vanishing of the cohomology of ordinary locus after localising at a supersingular representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$. We then recall standard facts about admissible representations and review the functor of localisation at π which comes out of the work of Paškūnas.

We then turn to the analysis of the supersingular locus. In this context, there naturally appears a quaternion algebra $D^\times/\mathbb{Q}$ which is ramified exactly at p and ∞. We define the local fundamental representation of Deligne in our setting (which appeared for the first time in the letter of Deligne [Del73]) and we show a decomposition of the cohomology of supersingular locus. At this point we will be able to show that H^1 of the tower of modular curves injects into H^1 of the Lubin–Tate tower hence proving part of result (1).

Having established this result, we start analysing mod p representations of the p-adic quaternion algebra and define a candidate for the mod p Jacquet–Langlands correspondence $\sigma_{\mathfrak{m}}$ which we later show to appear in the cohomology. It will *a priori* depend on a global input, namely a maximal ideal $\mathfrak{m}$ of a Hecke algebra corresponding to some modular mod p Galois representation $\bar{\rho}$, but we conjecture that it is independent of $\mathfrak{m}$. This is reasonable as it would follow from the local-global compatibility part of the Buzzard–Diamond–Jarvis conjecture. After further analysis of $\sigma_{\mathfrak{m}}$ we are able to finish the proof of result (1).

Using similar techniques, we start analysing the cohomology with compact support of the Lubin–Tate tower. By using the Hochschild–Serre spectral sequence, we are able to reduce result (2) to the question of whether supersingular Hecke modules of the mod p Hecke algebra

at the pro-p Iwahori level appear in the H_c^1 of the Lubin–Tate tower at the pro-p Iwahori level. We resolve this question by explicitly computing some cohomology groups.

While proving the above theorems, we will also prove that the first cohomology group of the Lubin–Tate tower and the first cohomology group of the ordinary locus are non-admissible smooth representations. In particular, they are much harder to describe than their mod l analogues. Moreover our model for the mod p Jacquet–Langlands correspondence $\sigma_{\mathfrak{m}}$ (actually we propose three candidates for the correspondence which we discuss in § 7.4) is a representation of $D^\times(\mathbb{Q}_p)$ of infinite length. This indicates that already for $D^\times(\mathbb{Q}_p)$ the mod p Langlands correspondence is complicated (as in the work of [BP12], representations in question are not of finite length). On the other hand, the case of $D^\times(\mathbb{Q}_p)$ is much simpler than that of $\mathrm{GL}_2(F)$ for F a finite extension of $\mathbb{Q}_p$, and hence we might be able to describe $\sigma_{\mathfrak{m}}$ precisely. The natural question in this discussion is the local-global compatibility part of the Buzzard–Diamond–Jarvis conjecture (see [BDJ10], Conjecture 4.7]) which says that we have an isomorphism

$$\mathbf{F}[\mathfrak{m}] \simeq \sigma_{\mathfrak{m}} \otimes \pi^p(\bar{\rho})$$

where $\mathbf{F}$ denotes locally constant functions on $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)$ with values in $\bar{\mathbb{F}}_p$ and $\pi^p(\bar{\rho})$ is a representation of $\mathrm{GL}_2(\mathbb{A}_f^p)$ associated to $\bar{\rho}$ by the modified Langlands correspondence.

At the end, we remark that our arguments work well in the $l \neq p$ setting and omit the use of vanishing cycles. As some of our arguments are geometric, we can also get similar results in the p-adic setting. We hope to return to this issue in our future work. Also, the geometry of modular curves is very similar to the geometry of Shimura curves and hence we hope that some of the reasonings in this article will give an insight into the nature of the mod p local Langlands correspondence of $\mathrm{GL}_2(F)$ for F a finite extension of $\mathbb{Q}_p$.

2. Geometry of modular curves

Throughout this text we shall assume that $p > 2$. This is needed for Theorem 6.2 of Emerton.

Let $X(Np^m)$ be the Katz–Mazur compactification of the modular curve associated to the moduli problem $(\Gamma(p^m), \Gamma_1(N))$ (see [KM85]) which is defined over $\mathbb{Z}[1/N, \zeta_{p^m}]$, where ζ_{p^m} is a primitive p^mth root of unity, that is $X(Np^m)$ parametrizes (up to isomorphism) triples (E, ϕ, α), where E is an elliptic curve, $\phi : (\mathbb{Z}/p^m\mathbb{Z})^2 \to E[p^m]$ is a Drinfeld level structure and $\alpha : \mathbb{Z}/N\mathbb{Z} \to E[N]$ is a $\Gamma_1(N)$-structure. We consider the integral model of it defined over $\mathbb{Z}_p^{nr}[\zeta_{p^m}]$, where $\mathbb{Z}_p^{nr}$ is the maximal unramified extension of $\mathbb{Z}_p$, which we will denote also by $X(Np^m)$. Let us denote by $X(Np^m)^{an}$ the analytification of $X(Np^m)$ which is a Berkovich space.

Recall that there exists a reduction map $\pi : X(Np^m)^{an} \to \overline{X(Np^m)}$, where $\overline{X(Np^m)}$ is the special fiber of $X(Np^m)$. We define $\overline{X(Np^m)_{ss}}$ (respectively $\overline{X(Np^m)_{ord}}$) to be the set of supersingular (respectively ordinary) points in $\overline{X(Np^m)}$. Define the tubes $X(Np^m)_{ss} = \pi^{-1}(\overline{X(Np^m)_{ss}})$ and $X(Np^m)_{ord} = \pi^{-1}(\overline{X(Np^m)_{ord}})$ inside $X(Np^m)^{an}$ of supersingular and ordinary points respectively.

2.1 Two exact sequences
In what follows, we shall use a shorthand notation by writing simply $H^i(X, F)$ for the étale cohomology group $H_{et}^i(X_{\mathbb{C}_p}, F)$ of a rigid-analytic space X after base change to $\mathbb{C}_p$ with coefficients in an étale sheaf F.

We know that $X(Np^m)_{ss}$ is an open analytic subspace of $X(Np^m)^{an}$ isomorphic to some copies of Lubin–Tate spaces, where the number of copies is equal to the number of points in $\overline{X(Np^m)_{ss}}$ (see [Buz03, § 3]). We have a decomposition $X(Np^m)^{an} = X(Np^m)_{ss} \cup X(Np^m)_{ord}$

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and we put $j : X(Np^m)_{ss} \hookrightarrow X(Np^m)^{an}$ and $i : X(Np^m)_{ord} \to X(Np^m)^{an}$. We remark that j is an open immersion of Berkovich spaces. Let F be a sheaf in the étale topoi of $X(Np^m)^{an}$. By the general formalism of six operations (due in this setting to Berkovich, see [Ber93]) we have a short exact sequence

$$0 \to j_! j^* F \to F \to i_* i^* F \to 0$$

which gives a long exact sequence of étale cohomology groups:

$$\cdots \to H^0(X(Np^m)_{ord}, i^* F) \to H^1_c(X(Np^m)_{ss}, F)$$
$$\to H^1(X(Np^m)^{an}, F) \to H^1(X(Np^m)_{ord}, i^* F) \to \cdots.$$

On the other hand, we can consider a similar exact sequence for the cohomology without compact support, but instead considering support on the ordinary locus. This results in the long exact sequence

$$\cdots \to H^1_{X_{ord}}(X(Np^m)^{an}, F) \to H^1(X(Np^m)^{an}, F)$$
$$\to H^1(X(Np^m)_{ss}, i^* F) \to H^2_{X_{ord}}(X(Np^m)^{an}, F) \to \cdots,$$

where we have denoted by $H^1_{X_{ord}}(X(Np^m)^{an}, F)$ the étale cohomology of $X(Np^m)^{an}$ with support on $X(Np^m)_{ord}$. Because of the vanishing of the cohomology with compact support of the supersingular locus localised at π (see the explanation in the next sections), this exact sequence will be of more importance to us later on. We will analyse those two exact sequences simultaneously.

2.2 Decomposition of ordinary locus

Let us recall that we have the Weil pairing on elliptic curves

$$e_{p^m} : E[p^m] \times E[p^m] \to \mu_{p^m}.$$

Denote by ζ_{p^m} a p^mth primitive root of unity. For $a \in (\mathbb{Z}/p^m\mathbb{Z})^*$ we define a substack $X(Np^m)_a$ of $X(Np^m)$ as the moduli problem which classifies elliptic curves E with level structures (ϕ, α) such that $e_{p^m}(\phi\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \phi\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)) = \zeta_{p^m}^a$. This moduli problem is representable by a scheme over $\mathbb{Z}_p^{nr}[\zeta_{p^m}]$ (see [KM85, ch. 9]). Moreover the coproduct $\coprod_a X(Np^m)_a$ is a regular model of $X(Np^m)$ over $\mathbb{Z}_p^{nr}[\zeta_{p^m}]$.

Let us denote by $\overline{X(Np^m)}_{a,ord}$ the ordinary locus of the reduction of $X(Np^m)_a$. We recall (see for example [KM85, ch. 13]) that the set of irreducible components of $\overline{X(Np^m)}_{ord}$ consists of smooth curves $C_{a,b}(Np^m)$ defined on points by

$$C_{a,b}(Np^m)(S) = \left\{ (E, \phi, \alpha) \in \overline{X(Np^m)}_{a,ord}(S) \mid e_{p^m}\left(\phi\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \phi\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\right) = \zeta_{p^m}^a \text{ and } \operatorname{Ker}\phi = b \right\}$$

where $a \in (\mathbb{Z}/p^m\mathbb{Z})^*$ and $b \in \mathbb{P}^1(\mathbb{Z}/p^m\mathbb{Z})$ is regarded as a line in $\mathbb{Z}/p^m\mathbb{Z} \times \mathbb{Z}/p^m\mathbb{Z}$. We observe that $\zeta_{p^m}^a = 1$ modulo p.

We are interested in lifting $C_{a,b}(Np^m)$ to characteristic zero and so we put

$$\mathbb{X}_{a,b}(Np^m) = \pi^{-1}(C_{a,b}(Np^m)).$$

Hence $\{\mathbb{X}_{a,b}(Np^m)\}$ for $a \in (\mathbb{Z}/p^m\mathbb{Z})^*$, $b \in \mathbb{P}^1(\mathbb{Z}/p^m\mathbb{Z})$ form a decomposition of the ordinary locus $X(Np^m)_{ord}$ because different $C_{a,b}(Np^m)$ intersect only at supersingular points. The spaces $\mathbb{X}_{a,b}(Np^m)$ may be regarded as analytifications of lifts of Igusa curves. For a detailed discussion,

see [Col05]. We do not determine here whether $\mathbb{X}_{a,b}(Np^m)$ are precisely the connected components of $X(Np^m)_{ord}$. We remark also that one can give a moduli description of each $\mathbb{X}_{a,b}(Np^m)$.

There is an action of $\mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})$ on $X(Np^m)^{an}$ which is given on points by

$$(E, \phi, \alpha) \cdot g = (E, \phi \circ g, \alpha)$$

for $g \in \mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})$. Observe that if $e_{p^m}(\phi(\begin{smallmatrix}1\\0\end{smallmatrix}), \phi(\begin{smallmatrix}0\\1\end{smallmatrix})) = \zeta_{p^m}^a$, then

$$e_{p^m}\big((\phi \circ g)(\begin{smallmatrix}1\\0\end{smallmatrix}), (\phi \circ g)(\begin{smallmatrix}0\\1\end{smallmatrix})\big) = \zeta_{p^m}^{a\cdot\det g} \quad \text{for } g \in \mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})$$

and so g induces an isomorphism between $\mathbb{X}_{a,b}(Np^m)$ and $\mathbb{X}_{a\cdot\det g, g^{-1}\cdot b}(Np^m)$.

For $b \in \mathbb{P}^1(\mathbb{Z}/p^m\mathbb{Z})$ there is a Borel subgroup $B_m(b)$ in $\mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})$ which fixes b and hence the Borel subgroup $B_m(b)^+ = B_m(b) \cap \mathrm{SL}_2(\mathbb{Z}/p^m\mathbb{Z})$ in $\mathrm{SL}_2(\mathbb{Z}/p^m\mathbb{Z})$ stabilises $\mathbb{X}_{a,b}(Np^m)$.

Let $b = \infty = (\begin{smallmatrix}1\\0\end{smallmatrix}) \in \mathbb{P}^1(\mathbb{Z}/p^m\mathbb{Z})$. By the above considerations we have

$$H^i(X(Np^m)_{ord}, i^*F) = \bigoplus_{a,b} H^i(\mathbb{X}_{a,b}(Np^m), (i^*F)_{|\mathbb{X}_{a,b}(Np^m)})$$

$$\simeq \mathrm{Ind}_{B_m(\infty)}^{\mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})}\bigg(\bigoplus_a H^i(\mathbb{X}_{a,\infty}(Np^m), (i^*F)_{|\mathbb{X}_{a,\infty}})\bigg)$$

and also

$$H^1_{X_{ord}}(X(Np^m)^{an}, F) \simeq \mathrm{Ind}_{B_m(\infty)}^{\mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})}\bigg(\bigoplus_a H^1_{\mathbb{X}_{a,\infty}(Np^m)}(X(Np^m)^{an}, F)\bigg).$$

Those results will be extremely useful for us later on, when we introduce the localisation at a given supersingular representation.

2.3 Supersingular points

Let us denote by D the quaternion algebra over $\mathbb{Q}$ which is ramified precisely at p and at ∞. We recall the description of supersingular points $\overline{X(Np^m)}_{ss}$ which has appeared in [Del73] and then was explained in [Car86], §§ 9.4 and 10.4. Fix a supersingular elliptic curve $\overline{E}$ over $\mathbb{F}_p$ and a two-dimensional vector space V over $\mathbb{Q}_p$. Let $\det(\overline{E}) = \mathbb{Z}$ be the determinant of $\overline{E}$. Denote by $W(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ the Weil group of $\mathbb{F}_p$ and put

$$\Delta = (W(\overline{\mathbb{F}}_p/\mathbb{F}_p) \times \mathrm{Isom}(\det(\overline{E}) \otimes_{\mathbb{Z}} \mathbb{Q}_p(1), \wedge^2 V))/\sim$$

where $\sim$ is defined by $(\sigma, \beta) \sim (\sigma \,\mathrm{Frob}^k, p^{-k}\beta)$ for $k \in \mathbb{Z}$, where $\mathrm{Frob}: x \mapsto x^p$ is a Frobenius map. We define K_m to be the kernel of $D^\times(\mathbb{Z}_p) \to D^\times(\mathbb{Z}/p^m\mathbb{Z})$ and we let $K(N) = \{(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}) \in \mathrm{GL}_2(\mathbb{Z}) \mid a \equiv 1 \bmod N$ and $c \equiv 0 \bmod N\}$, viewed as a subgroup of $\mathrm{GL}_2(\mathbb{A}_f^p)$ by the diagonal embedding. Then

$$\overline{X(Np^m)}_{ss} = \Delta/K_m \times_{D^\times(\mathbb{Q})} \mathrm{GL}_2(\mathbb{A}_f^p)/K(N).$$

Every $\delta \in \Delta/K_m$ furnishes a supersingular elliptic curve $E(\delta)$, so that for every $\delta \in \Delta$ we can consider the Lubin–Tate tower $\mathrm{LT}_\delta = \varprojlim_m \mathrm{LT}_\delta(p^m)$, which is the generic fiber of the deformation space of the formal group attached to $E(\delta)$ and where $\mathrm{LT}_\delta(p^m)$ denotes the generic fiber of the deformation space of formal groups with p^m-level structure (see [Dat12] for details on the Lubin–Tate tower). Let us denote by $\mathbb{E}(\delta)$ the universal formal group deforming the formal group attached to $E(\delta)$ and let $\mathbb{E}(\Delta) = \coprod_{\delta \in \Delta} \mathbb{E}(\delta)$. By [Car86, § 9.4], the universal formal group over $\varprojlim_{N,p^m} \overline{X(Np^m)}_{ss}$ is isomorphic to $\mathbb{E}(\Delta) \times_{D^\times(\mathbb{Q})} \mathrm{GL}_2(\mathbb{A}_f^p)$, and hence we conclude that

$$\varprojlim_{Np^m} X(Np^m)_{ss} \simeq \mathrm{LT}_\Delta \times_{D^\times(\mathbb{Q})} \mathrm{GL}_2(\mathbb{A}_f^p)$$

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where $\mathrm{LT}_\Delta = \coprod_{\delta \in \Delta} \mathrm{LT}_\delta$. We also get a description at a finite level

$$X(Np^m)_{ss} \simeq \mathrm{LT}_{\Delta/K_m} \times_{D^\times(\mathbb{Q})} \mathrm{GL}_2(\mathbb{A}_f^p)/K(N)$$

where $\mathrm{LT}_{\Delta/K_m} = \coprod_{\delta \in \Delta/K_m} \mathrm{LT}_\delta(p^m)$.

These results will allow us later on to define the local fundamental representation and analyse the action of the quaternion algebra $D^\times$.

3. Admissibility of cohomology groups

In this section we will recall the notion of admissibility in the context of mod p representations. It will be crucial in our study of cohomology.

3.1 General facts and definitions

We start with general facts about admissible representations. In our definitions, we will follow [Eme10]. Let k be a field of characteristic p and let G be a connected reductive group over $\mathbb{Q}_p$.

DEFINITION 3.1. Let V be a representation of G over k. A vector $v \in V$ is smooth if v is fixed by some open subgroup of G. Let V_{sm} denote the subset of smooth vectors of V. We say that a G-representation V over k is smooth if $V = V_{sm}$.

A smooth G-representation V over k is admissible if V^H is finitely generated over k for every open compact subgroup H of G.

PROPOSITION 3.2. *The category of admissible k-representations is abelian.*

Proof. This category is (anti-)equivalent to the category of finitely generated augmented modules over certain completed group rings. See [Eme10, Propositions 2.2.13 and 2.4.11]. □

Now, we will prove an analogue of Lemma 13.2.3 from [Boy99] in the $l = p$ setting. We will later apply this lemma to the cohomology of the ordinary locus to force its vanishing after localisation at a supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$.

LEMMA 3.3. *For any smooth admissible representation (π, V) of the parabolic subgroup $P \subset G$ over k, the unipotent radical U of P acts trivially on V.*

Proof. Let L be a Levi subgroup of P, so that $P = LU$. Let $v \in V$ and let $K_P = K_L K_U$ be a compact open subgroup of P such that $v \in V^{K_P}$. We choose an element z in the centre of L such that

$$z^{-n} K_P z^n \subset \cdots \subset z^{-1} K_P z \subset K_P \subset z K_P z^{-1} \subset \cdots \subset z^n K_P z^{-n} \subset \cdots$$

and $\bigcup_{n \geqslant 0} z^n K_P z^{-n} = K_L U$. For every n and m, modules $V^{z^{-n} K_P z^n}$ and $V^{z^{-m} K_P z^m}$ are of the same length, as they are isomorphic via $\pi(z^{n-m})$, and hence we have not only an isomorphism but an equality $V^{z^{-n} K_P z^n} = V^{z^{-m} K_P z^m}$. Thus for every $x \in V^{K_P}$ we have $x \in V^{K_P} = V^{z^{-n} K_P z^n} = V^{K_L U}$, which is contained in V^U. □

We also record the following result of Emerton for the future use.

LEMMA 3.4. *Let $V = \mathrm{Ind}_P^G W$ be a parabolic induction. If V is a smooth admissible representation of G over k, then W is a smooth admissible representation of P over k.*

Proof. This follows from [Eme10, Theorem 4.4.6]. □

3.2 Cohomology and admissibility

In [Eme06], Emerton has introduced the completed cohomology, which plays a crucial role in the p-adic Langlands program. The most important thing for us right now is the fact that those cohomology groups for modular curves are admissible as $\mathrm{GL}_2(\mathbb{Q}_p)$-representations. We have the following proposition.

PROPOSITION 3.5. *The $\mathrm{GL}_2(\mathbb{Q}_p)$-representation*

$$\widehat{H}^1(X(N), \bar{\mathbb{F}}_p) = \varinjlim_m H^1(X(Np^m)^{an}, \bar{\mathbb{F}}_p)$$

is admissible.

Proof. This proposition is [Eme06, Theorem 2.1.5] (see also [CE13, Theorem 1.16]). □

By formal properties of the category of admissible representations, which form a Serre subcategory of the category of smooth representations (see [Eme10, Proposition 2.2.13]), the above result permits us to deduce admissibility for other cohomology groups which are of interest to us. Let us remark that we can define also the cohomology of the Lubin–Tate tower with compact support.

Remark 3.6. *A priori*, cohomology with compact support is a covariant functor. However, using the adjunction map

$$\Lambda \to \pi_* \pi^* \Lambda \simeq \pi_! \pi^! \Lambda$$

where Λ is a constant sheaf and $\pi : X(Np^{m+1})_{ss} \to X(Np^m)_{ss}$ is finite (hence $\pi_* = \pi_!$) and étale (hence $\pi^! = \pi^*$) by the properties of the Lubin–Tate tower, we get maps $H_c^i(X(Np^m)_{ss}, \Lambda) \to H_c^i(X(Np^{m+1})_{ss}, \Lambda)$ compatible with $H^i(X(Np^m)^{an}, \Lambda) \to H^i(X(Np^{m+1})_{ss}, \Lambda)$.

We start firstly by analysing cohomology groups which appear in the exact sequence for the cohomology with compact support. We have the following proposition.

PROPOSITION 3.7. *The $\mathrm{GL}_2(\mathbb{Q}_p)$-representation*

$$\widehat{H}^0(X(N)_{ord}, \bar{\mathbb{F}}_p) = \varinjlim_m H^0(X(Np^m)_{ord}, \bar{\mathbb{F}}_p)$$

is admissible.

Proof. The number of connected components of $X(Np^m)_{ord}$ is finite; let $d(Np^m)$ be their number. For $s > 0$, we have

$$H^0(X(Np^m)_{ord}, \bar{\mathbb{F}}_p) = (\bar{\mathbb{F}}_p)^{d(Np^m)},$$

and hence $\varinjlim_m H^0(X(Np^m)_{ord}, \bar{\mathbb{F}}_p)$ is admissible. □

We deduce the following proposition.

PROPOSITION 3.8. *The $\mathrm{GL}_2(\mathbb{Q}_p)$-representation*

$$\widehat{H}_c^1(X(N)_{ss}, \bar{\mathbb{F}}_p) = \varinjlim_m H_c^1(X(Np^m)_{ss}, \bar{\mathbb{F}}_p)$$

is admissible.

Proof. We consider the exact sequence from § 2.1,

$$\cdots \to \widehat{H}^0(X(N)_{ord}, \bar{\mathbb{F}}_p) \to \widehat{H}^1_c(X(N)_{ss}, \bar{\mathbb{F}}_p) \to \widehat{H}^1(X(N)^{an}, \bar{\mathbb{F}}_p) \to \widehat{H}^1(X(N)_{ord}, \bar{\mathbb{F}}_p) \to \cdots,$$

and finish by using the fact that admissible representations form a Serre subcategory of smooth representations and the propositions proved above. □

We remark that the cohomology with compact support of the Lubin–Tate tower is much easier to work with than the cohomology without the support. This is because the latter will turn out to be non-admissible.

We finish this section with the following proposition.

PROPOSITION 3.9. *The* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation*

$$\widehat{H}^1_{X_{ord}}(X(N), \bar{\mathbb{F}}_p) = \varinjlim_m H^1_{X_{ord}}(X(Np^m)^{an}, \bar{\mathbb{F}}_p)$$

is admissible.

Proof. This follows from the exact sequence (we use the notations from the previous section)

$$H^0(X(Np^m)_{ss}, \bar{\mathbb{F}}_p) \to H^1_{X_{ord}}(X(Np^m)^{an}, \bar{\mathbb{F}}_p) \to H^1(X(Np^m)^{an}, \bar{\mathbb{F}}_p)$$

and Proposition 3.5. □

4. Supersingular representations

In this section we recall results on the structure of admissible representations and we apply them to the exact sequence of cohomology groups that we have introduced before, getting the first comparison between the cohomology of the Lubin–Tate tower and the cohomology of the tower of modular curves. We will start with a reminder on the mod p local Langlands correspondence. We refer to [Ber11] for references to proofs of cited facts.

4.1 The mod p local Langlands correspondence

Let ω_n be the fundamental character of Serre of level n which is defined on inertia group I via $\sigma \mapsto \sigma(p^{1/n})/p^{1/n}$. Let ω be the mod p cyclotomic character. For $h \in \mathbb{N}$, we write $\mathrm{Ind}\,\omega_n^h$ for the unique semisimple $\bar{\mathbb{F}}_p$-representation of $G_{\mathbb{Q}_p}$ which has determinant ω^h and whose restriction to I is isomorphic to $\omega_n^h \oplus \omega_n^{ph} \oplus \cdots \oplus \omega_n^{p^{n-1}h}$. If $\chi : G_{\mathbb{Q}_p} \to k^\times$ is a character, we will denote by $\rho(r, \chi)$ the representation $\mathrm{Ind}(\omega_2^{r+1}) \otimes \chi$ which is absolutely irreducible if $r \in \{0, \ldots, p-1\}$. In fact, any absolutely irreducible representation of $G_{\mathbb{Q}_p}$ of dimension 2 is isomorphic to some $\rho(r, \chi)$ for $r \in \{0, \ldots, p-1\}$. We remark that $\mathrm{Ind}\,\omega_2^{r+1}$ is not isomorphic to the induced representation $\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1}$, because of the condition which we put on the determinant. In fact, computing the determinant of $\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1}$, one sees that

$$\mathrm{Ind}\,\omega_2^{r+1} = \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} (\omega_2^{r+1} \cdot \mathrm{sgn})$$

where sgn is the $\bar{\mathbb{F}}_p$-character of $G_{\mathbb{Q}_{p^2}}$ which factors through $\mathbb{F}_{p^2}^\times \times \mathbb{Z}$ and which is trivial on $\mathbb{F}_{p^2}^\times$ and takes the Frobenius automorphism of $G_{\mathbb{Q}_{p^2}}$ to -1 in $\mathbb{Z}$ (we have to make a choice of a uniformiser to have the map $G_{\mathbb{Q}_{p^2}} \twoheadrightarrow \mathbb{F}_{p^2}^\times \times \mathbb{Z}$).

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On the GL_2-side, one considers representations $\mathrm{Sym}^r k^2$ inflated to $\mathrm{GL}_2(\mathbb{Z}_p)$ and then extended to $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$ by making p acts by identity. We then consider the induced representation

$$\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^r k^2.$$

One can show that the endomorphism ring (a Hecke algebra) $\mathrm{End}_{k[\mathrm{GL}_2(\mathbb{Q}_p)]}(\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^r k^2)$ is isomorphic to $k[T]$, where T corresponds to the double class $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times \left(\begin{smallmatrix} p & 1 \\ 0 & 1 \end{smallmatrix}\right) \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$. For a character $\chi : G_{\mathbb{Q}_p} \to k^\times$ and $\lambda \in k$. we introduce representations:

$$\pi(r, \lambda, \chi) = \frac{\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^r k^2}{T - \lambda} \otimes (\chi \circ \det).$$

For $r \in \{0, \ldots, p-1\}$ such that $(r, \lambda) \notin \{(0, \pm 1), (p-1, \pm 1)\}$, the representation $\pi(r, \lambda, \chi)$ is irreducible. If $\lambda = \pm 1$, then $\pi(r, \lambda, \chi)$ appears as either a subrepresentation or a subquotient of the special representation Sp. One proves that $\chi \circ \det$, $\mathrm{Sp} \otimes(\chi \circ \det)$ and $\pi(r, \lambda, \chi)$ for $r \in \{0, \ldots, p-1\}$ and $(r, \lambda) \notin \{(0, \pm 1), (p-1, \pm 1)\}$ are all the smooth irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.

This explicit description gives a mod p correspondence by associating $\rho(r, \chi)$ to $\pi(r, 0, \chi)$.

4.2 Supersingular representations

Let us a fix a supersingular representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ on a $\bar{\mathbb{F}}_p$-vector space with a central character ξ. Recall the following result of Paškūnas.

PROPOSITION 4.1. *Let τ be an irreducible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ admitting a central character. If $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi, \tau) \neq 0$ then $\tau \simeq \pi$.*

Proof. See [Pas10] and [Pas11] for the case $p = 2$. □

This result permits us to conclude that the $\mathrm{GL}_2(\mathbb{Q}_p)$-block of any supersingular representation consists of one element: the supersingular representation itself. Here, by a $\mathrm{GL}_2(\mathbb{Q}_p)$-block we mean an equivalence class for a relation defined as follows. We write $\pi \sim \tau$ if there exists a sequence of irreducible smooth admissible $\bar{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(\mathbb{Q}_p)$: $\pi_0 = \pi, \pi_1, \ldots, \pi_n = \tau$ such that for each i one of the following conditions holds:

(1) $\pi_i \simeq \pi_{i+1}$;
(2) $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_i, \pi_{i+1}) \neq 0$;
(3) $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{i+1}, \pi_i) \neq 0$.

One can find a description of all $\mathrm{GL}_2(\mathbb{Q}_p)$-blocks in [Pas11] or [Pas13]. The general result of Gabriel on the block decomposition of locally finite categories gives the following proposition.

PROPOSITION 4.2. *We have a decomposition,*

$$\mathrm{Rep}_{\mathrm{GL}_2(\mathbb{Q}_p), \xi}^{adm}(\bar{\mathbb{F}}_p) = \mathrm{Rep}_{\mathrm{GL}_2(\mathbb{Q}_p), \xi}^{adm}(\bar{\mathbb{F}}_p)_{(\pi)} \oplus \mathrm{Rep}_{\mathrm{GL}_2(\mathbb{Q}_p), \xi}^{adm}(\bar{\mathbb{F}}_p)^{(\pi)},$$

where $\mathrm{Rep}_{\mathrm{GL}_2(\mathbb{Q}_p), \xi}^{adm}(\bar{\mathbb{F}}_p)$ is the (abelian) category of smooth admissible $\bar{\mathbb{F}}_p$-representations admitting a central character ξ, $\mathrm{Rep}_{\mathrm{GL}_2(\mathbb{Q}_p), \xi}^{adm}(\bar{\mathbb{F}}_p)_{(\pi)}$ (respectively $\mathrm{Rep}_{\mathrm{GL}_2(\mathbb{Q}_p), \xi}^{adm}(\bar{\mathbb{F}}_p)^{(\pi)}$) is the subcategory of it consisting of representations Π all of whose irreducible subquotients are (respectively are not) isomorphic to π.

Proof. See [Pas13, Proposition 5.32]. □

This result permits us to consider the localisation functor with respect to π

$$V \mapsto V_{(\pi)}$$

on the category of admissible representations such that all irreducible subquotients of $V_{(\pi)}$ are isomorphic to the fixed π.

Remark 4.3. We note that the condition on the existence of central characters is not important. Central characters always exist by the work of Berger [Ber12] in the mod p case.

4.3 Cohomology with compact support

We apply the localisation functor to the three admissible terms in the exact sequence obtained from §2.1,

$$\cdots \to \widehat{H}^0(X(N)_{ord}, \bar{\mathbb{F}}_p) \to \widehat{H}^1_c(X(N)_{ss}, \bar{\mathbb{F}}_p) \to \widehat{H}^1(X(N)^{an}, \bar{\mathbb{F}}_p) \to \widehat{H}^1(X(N)_{ord}, \bar{\mathbb{F}}_p) \to \cdots,$$

getting the exact sequence

$$\widehat{H}^0(X(N)_{ord}, \bar{\mathbb{F}}_p)_{(\pi)} \to \widehat{H}^1_c(X(N)_{ss}, \bar{\mathbb{F}}_p)_{(\pi)} \to \widehat{H}^1(X(N)^{an}, \bar{\mathbb{F}}_p)_{(\pi)}.$$

For $a \in \mathbb{Z}_p^\times$ let us define, in the light of 2.2,

$$\widehat{H}^1(\mathbb{X}_{a,\infty}(N), \bar{\mathbb{F}}_p) = \varinjlim_m H^1(\mathbb{X}_{a,\infty}(Np^m), \bar{\mathbb{F}}_p).$$

Recall now, that after §2.2, $\widehat{H}^0(X(N)_{ord}, \bar{\mathbb{F}}_p)$ is an admissible representation isomorphic to the induced representation

$$\mathrm{Ind}_{B_\infty(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \left(\bigoplus_a \widehat{H}^0(\mathbb{X}_{a,\infty}(N), \bar{\mathbb{F}}_p) \right)$$

where $B_\infty(\mathbb{Q}_p)$ is the Borel subgroup of upper triangular matrices in $\mathrm{GL}_2(\mathbb{Q}_p)$ and a goes over $\mathbb{Z}_p^\times$. On this representation unipotent group acts trivially by Lemma 3.3 (which we can use thanks to Lemma 3.4) and hence we see that it is induced from the tensor product of characters. This means that after localisation at π this representation vanishes

$$\mathrm{Ind}_{B_\infty(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \left(\bigoplus_a \widehat{H}^0(\mathbb{X}_{a,\infty}(N), \bar{\mathbb{F}}_p) \right)_{(\pi)} = 0$$

and we arrive at the following theorem.

THEOREM 4.4. *We have an injection of representations*

$$\widehat{H}^1_c(X(N)_{ss}, \bar{\mathbb{F}}_p)_{(\pi)} \hookrightarrow \widehat{H}^1(X(N), \bar{\mathbb{F}}_p)_{(\pi)}$$

By taking yet another direct limit, we define

$$\widehat{H}^1_{ss,c,\bar{\mathbb{F}}_p} = \varinjlim_N \widehat{H}^1_c(X(N)_{ss}, \bar{\mathbb{F}}_p),$$

$$\widehat{H}^1_{\bar{\mathbb{F}}_p} = \varinjlim_N \widehat{H}^1(X(N), \bar{\mathbb{F}}_p)$$

COROLLARY 4.5. *We have an injection of representations*

$$(\widehat{H}^1_{ss,c,\bar{\mathbb{F}}_p})_{(\pi)} \hookrightarrow (\widehat{H}^1_{\bar{\mathbb{F}}_p})_{(\pi)}.$$

We remark that later on in §8 using purely local methods we will prove that

$$(\widehat{H}^1_{ss,c,\bar{\mathbb{F}}_p})_{(\pi)} = 0$$

hence the above corollary will become obsolete.

We define also, for a future use,

$$\widehat{H}^1_{ord,\bar{\mathbb{F}}_p} = \varinjlim_N \varinjlim_m H^1(X(Np^m)_{ord}, \bar{\mathbb{F}}_p)$$

and, for $a \in \mathbb{Z}_p^\times$,

$$\widehat{H}^1_{a,\infty,\bar{\mathbb{F}}_p} = \varinjlim_N \varinjlim_m H^1(\mathbb{X}_{a,\infty}(Np^m), \bar{\mathbb{F}}_p)$$

4.4 Cohomology without support

We can apply similar reasoning as above to the situation without compact support. The roles of the ordinary locus and the supersingular locus are interchanged. By using again the decomposition of the ordinary locus and Lemmas 3.3 and 3.4, we get that the localisation of $\widehat{H}^1_{X_{ord}}$ vanishes

$$\widehat{H}^1_{X_{ord}}(X(N), \bar{\mathbb{F}}_p)_{(\pi)} = 0$$

and hence we get the following theorem.

THEOREM 4.6. *We have an injection of representations*

$$(\widehat{H}^1_{\bar{\mathbb{F}}_p})_{(\pi)} \hookrightarrow \widehat{H}^1_{ss,\bar{\mathbb{F}}_p}$$

where $\widehat{H}^1_{ss,\bar{\mathbb{F}}_p}$ is defined similarly as above.

Later on, we will show that $\widehat{H}^1_{ss,\bar{\mathbb{F}}_p}$ is a non-admissible representation, and this is why we cannot localise it a π. Let us finish by giving another definition for a future use (where $a \in \mathbb{Z}_p^\times$):

$$\widehat{H}^1_{\mathbb{X}_{a,\infty},\bar{\mathbb{F}}_p} = \varinjlim_N \varinjlim_m H^1_{\mathbb{X}_{a,\infty}(Np^m)}(X(Np^m)^{an}, \bar{\mathbb{F}}_p).$$

5. New vectors

Because there does not exist at the moment the Colmez functor in the context of quaternion algebras, which would be similar to the one considered for example in [Pas13], we are forced to give a global definition of the mod p Jacquet–Langlands correspondence. To do that, we prove an analogue of a classical theorem of Casselman in the context of the modified mod l Langlands correspondence of Emerton and Helm (see [EH]), which amounts to the statement that for any prime $l \neq p$, and for any local two-dimensional Galois representation ρ of $\mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$, there exists a compact, open subgroup $K_l \subset \mathrm{GL}_2(\mathbb{Z}_l)$ such that $\pi_l(\rho)^{K_l}$ has dimension 1, where $\pi_l(\rho)$ is the mod p representation of $\mathrm{GL}_2(\mathbb{Q}_l)$ associated to ρ by [EH].

Let $\mathfrak{b}$ be an ideal of $\mathbb{Z}_p$ and put $\Gamma_0(\mathfrak{b}) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \mathrm{GL}_2(\mathbb{Z}_p) | c \equiv 0 \bmod \mathfrak{b}\}$. Let us recall the classical result of Casselman (see [Cas73]).

THEOREM 5.1. *Let π be an irreducible admissible infinite-dimensional representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\bar{\mathbb{Q}}_l$-vector space and let ϵ be the central character of π. Let $c(\pi)$ be the conductor of π which is the largest ideal of $\mathbb{Z}_p$ such that the space of vectors v with $\pi(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})v = \epsilon(a)v$, for all $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_0(c(\pi))$, is non-zero. Then this space has dimension 1.*

We will prove that the result holds also modulo p for the modified mod l Langlands correspondence. For that we need to assume that our prime p is odd.

THEOREM 5.2. *Let* $\pi = \pi(\rho)$ *be the* mod p *admissible representation of* $\mathrm{GL}_2(\mathbb{Q}_l)$ *associated by the modified mod l Langlands correspondence to a Galois representation* $\rho : G_{\mathbb{Q}_l} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$. *Then there exists an open, compact subgroup K of* $\mathrm{GL}_2(\mathbb{Q}_l)$ *such that* $\dim_{\bar{\mathbb{F}}_p} \pi^K = 1$.

Proof. We recall the results of [EH] concerning the construction of the modified mod l Langlands correspondence. By of [EH, Proposition 5.2.1], the theorem is true when ρ^{ss} is not a twist of $1 \oplus |\cdot|$, by the reduction modulo p of the classical result of Casselman from [Cas73], which in the $l \neq p$ situation was proved by Vignéras in [Vig89b] (see Theorem 23 and Proposition 24). When this is not the case, we can suppose that in fact $\rho^{ss} = 1 \oplus |\cdot|$ and we go by case-by-case analysis of the possible forms of $\pi(\rho)$ as described in [EH] after Proposition 5.2.1 and in [Hel12]. The representations $\pi(\rho)$ which appear are mostly extensions of four kinds of representations (and some combinations of them): the trivial representation 1, $|\cdot| \circ \det$, the Steinberg St, and $\pi(1)$ of Vignéras (see [Vig89b]).

(1) Suppose $0 \to \pi(1) \to \pi(\rho) \to 1 \to 0$. In this case $l \equiv -1 \bmod p$. Let $\Gamma_0(p) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{GL}_2(\mathbb{Z}_p) | c \equiv 0 \bmod p, a \equiv d \equiv 1 \bmod p \}$. Then we have a long exact sequence associated with higher invariants by $\Gamma_0(p)$:

$$0 \to \pi(\rho)^{\Gamma_0(p)} \to 1 \to R^1\pi(1)^{\Gamma_0(p)}$$

as $\pi(1)^{\Gamma_0(p)} = 0$ by [Vig89b, Proposition 24]. We conclude by observing that $R^1\pi(1)^{\Gamma_0(p)} = 0$ because $|\Gamma_0(p)| = p^\infty$ and $l \nmid |\Gamma_0(p)|$ by our assumption.

(2) In the same way we deal with the situation when $\pi(\rho)$ is an extension of $|\cdot| \circ \det$ by $\pi(1)$ with the same assumption on l.

(3) When $l \equiv -1 \bmod p$, it is also possible to have $0 \to \pi(1) \to \pi(\rho) \to 1 \oplus |\cdot| \circ \det \to 0$. Look at $\mathrm{GL}_2(\mathbb{Z}_p)$-invariants. The associated long exact sequence is

$$0 \to \pi(\rho)^{\mathrm{GL}_2(\mathbb{Z}_p)} \to (1 \oplus |\cdot| \circ \det)^{\mathrm{GL}_2(\mathbb{Z}_p)} \to R^1\pi(1)^{\mathrm{GL}_2(\mathbb{Z}_p)} = \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{F}_p)}(1, \pi(1)).$$

Let us denote by $\mathcal{E}$ the extension of 1 by $\pi(1)$ which we get from $0 \to \pi(1) \to \pi(\rho) \to 1 \oplus |\cdot| \circ \det \to 0$. The last map in the above exact sequence is explicit:

$$(1 \oplus |\cdot| \circ \det)^{\mathrm{GL}_2(\mathbb{Z}_p)} \to \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{F}_p)}(1, \pi(1))$$
$$(a, b) \mapsto (a + b)\mathcal{E},$$

and we see that it gives a line in $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{F}_p)}(1, \pi(1))$ and hence the kernel, i.e. $\pi(\rho)^{\mathrm{GL}_2(\mathbb{Z}_p)}$, is one-dimensional as $(1 \oplus |\cdot| \circ \det)^{\mathrm{GL}_2(\mathbb{Z}_p)}$ has dimension 2.

(4) The last non-banal case with which we have to deal is the case when p is odd, $l \equiv 1 \bmod p$ and we have an extension

$$0 \to \mathrm{St} \to \pi(\rho) \to 1 \to 0.$$

In this case $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(1, \mathrm{St})$ is two-dimensional (see [Hel12, Lemma 4.2]). We look at the reduction map

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(1, \mathrm{St}) \to \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{F}_p)}(1, \mathrm{St}).$$

Let us denote by $\mathcal{E}$ the image of the class $[\pi(\rho)]$ of $\pi(\rho)$ in $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{F}_p)}(1, \mathrm{St})$ under the above reduction. We have two cases to consider. Suppose firstly that $\mathcal{E} = 0$. Then we claim that $K = \mathrm{GL}_2(\mathbb{Z}_p)$ works. Indeed we have in this case

$$0 \to \pi(\rho)^K \to 1^K \to \mathrm{Ext}^1_K(1, \mathrm{St})$$

and, as the image of 1^K in $\mathrm{Ext}^1_K(1, \mathrm{St})$ is $\mathcal{E}$, we finish by assumption.

Now let us suppose that $\mathcal{E} \neq 0$. Then we claim that the Iwahori subgroup $K = I$ works. We have

$$0 \to \mathrm{St}^K \to \pi(\rho)^K \to 1^K \to \mathrm{Ext}^1_K(1, \mathrm{St}).$$

The image of 1^K in $\mathrm{Ext}^1_K(1, \mathrm{St})$ is non-zero by assumption, because $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Z}_p)}(1, \mathrm{St}) \hookrightarrow \mathrm{Ext}^1_K(1, \mathrm{St})$. Hence $\pi(\rho)^K$ is isomorphic to St^K, which is of dimension 1.

(5) We remark that there is also the so-called banal case when l is not congruent to ± 1 modulo p. In this case, there are two situations to consider. In the first one $\pi(\rho) = \mathrm{St} \otimes |\cdot| \circ \det$ and we can take $K = I$, the Iwahori subgroup. In the second one $\pi(\rho)$ is the unique non-split extension of $|\cdot| \circ \det$ by $\mathrm{St} \otimes |\cdot| \circ \det$. Because $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{F}_p)}(1, \mathrm{St}) = 0$ as we are in the banal case, we conclude as above that $K = \mathrm{GL}_2(\mathbb{Z}_p)$ works. $\qquad\square$

6. The fundamental representation

Following the original Deligne approach to the non-abelian Lubin–Tate theory, we define the local fundamental representation. Using it, we refine the Lubin–Tate side of the injections we have considered. Then we recall Emerton's results on the cohomology of the tower of modular curves, yielding by a comparison an information on the local fundamental representation. Our arguments are similar to those given in [Del73].

6.1 Cohomology of the supersingular tube

We have introduced, in § 2.3, the set Δ, spaces $\mathrm{LT}_{\Delta/K_m} = \coprod_{\delta \in \Delta/K_m} \mathrm{LT}_\delta$ and we have obtained a description of the supersingular tube

$$X(Np^m)_{ss} \simeq \mathrm{LT}_{\Delta/K_m} \times_{D^\times(\mathbb{Q})} \mathrm{GL}_2(\mathbb{A}_f^p)/K(N).$$

DEFINITION 6.1. Define the fundamental representation by

$$\widehat{H}^1_{\mathrm{LT},c,\bar{\mathbb{F}}_p} = \varinjlim_m H^1_c(\mathrm{LT}_{\Delta/K_m}, \bar{\mathbb{F}}_p).$$

Similarly we introduce the fundamental representation without support denoting it by $\widehat{H}^1_{\mathrm{LT},\bar{\mathbb{F}}_p}$.

From the description of supersingular points, we have

$$H^1_c(X(Np^m)_{ss}, \bar{\mathbb{F}}_p) = H^1_c(\mathrm{LT}_{\Delta/K_m} \times_{D^\times(\mathbb{Q})} \mathrm{GL}_2(\mathbb{A}_f^p)/K(N), \bar{\mathbb{F}}_p)$$
$$= \{f : D^\times(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_f^p)/K(N) \to H^1_c(\mathrm{LT}_{\Delta/K_m}, \bar{\mathbb{F}}_p)\}.$$

We take the direct limit:

$$\varinjlim_m H^1_c(X(Np^m)_{ss}, \bar{\mathbb{F}}_p) \simeq \{f : D^\times(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_f^p)/K(N) \to \varinjlim_m H^1_c(\mathrm{LT}_{\Delta/K_m}, \bar{\mathbb{F}}_p)\}.$$

Take the limit over N to obtain

$$\widehat{H}^1_{ss,c,\bar{\mathbb{F}}_p} \simeq \{f : D^\times(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_f^p) \to \widehat{H}^1_{\mathrm{LT},\bar{\mathbb{F}}_p}\}$$
$$\simeq \{f : D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) \to \widehat{H}^1_{\mathrm{LT},c,\bar{\mathbb{F}}_p}\}^{D^\times(\mathbb{Q}_p)}$$
$$\simeq (\{f : D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) \to \bar{\mathbb{F}}_p\} \otimes_{\bar{\mathbb{F}}_p} \widehat{H}^1_{\mathrm{LT},c,\bar{\mathbb{F}}_p})^{D^\times(\mathbb{Q}_p)}.$$

Let

$$\mathbf{F} = \{f : D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f) \to \bar{\mathbb{F}}_p\}$$

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where f are locally constant functions, then

$$\widehat{H}^1_{ss,c,\overline{\mathbb{F}}_p} \simeq (\mathbf{F} \otimes_{\overline{\mathbb{F}}_p} \widehat{H}^1_{\mathrm{LT},c,\overline{\mathbb{F}}_p})^{D^\times(\mathbb{Q}_p)}. \tag{1}$$

We get a similar result for the cohomology without support

$$\widehat{H}^1_{ss,\overline{\mathbb{F}}_p} \simeq (\mathbf{F} \otimes_{\overline{\mathbb{F}}_p} \widehat{H}^1_{\mathrm{LT},\overline{\mathbb{F}}_p})^{D^\times(\mathbb{Q}_p)}.$$

6.2 Emerton's results

We recall Emerton's results on the completed cohomology of modular curves. Remark that we are using implicitly the comparison theorem for étale cohomology of a scheme and its analytification which is proved in [Ber95a].

Let us fix a finite set of primes $\Sigma = \Sigma_0 \cup \{p\}$. Let $K^\Sigma = \prod_{l \notin \Sigma} K_l$ where $K_l = \mathrm{GL}_2(\mathbb{Z}_l)$ and choose an open, compact subgroup K_{Σ_0} of $\prod_{l \in \Sigma_0} \mathrm{GL}_2(\mathbb{Z}_l)$. Let $\bar\rho : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an odd, irreducible, continuous representation unramified outside Σ. Remark that by Serre's conjecture (see [Kha06]) $\bar\rho$ is modular. Let us denote by $\mathfrak{m}$ the maximal ideal in the Hecke algebra $\mathbb{T}(K_{\Sigma_0})$ which corresponds to $\bar\rho$. We write also $\bar\rho_{|G_{\mathbb{Q}_p}} = \mathrm{Ind}^{G_\mathbb{Q}}_{G_{\mathbb{Q}_{p^2}}} \alpha$, where α can be considered as a character of $\mathbb{Q}^\times_{p^2}$ by the local class field theory. For the definitions, see [Eme11, §5].

THEOREM 6.2 (Emerton). *We have an isomorphism*

$$\widehat{H}^1_{\overline{\mathbb{F}}_p}[\mathfrak{m}]^{K^\Sigma} \simeq \pi \otimes_{\overline{\mathbb{F}}_p} \pi_{\Sigma_0}(\bar\rho) \otimes_{\overline{\mathbb{F}}_p} \bar\rho$$

where π is a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to $\bar\rho$ by the mod p local Langlands correspondence and $\pi_{\Sigma_0}(\bar\rho)$ is a representation $\mathrm{GL}_2(\mathbb{A}^{\Sigma_0}_f)$ associated to $\bar\rho$ by the modified local Langlands correspondence mod l for $l \in \Sigma_0$ (see [EH]).

Proof. This theorem is proved in [Eme11, Proposition 6.1.20]. In this proposition Emerton assumes that $p > 2$ (which we do as well) and that [Eme11, Conjecture 6.1.9] holds for $\bar\rho$. Observe that in our case $\bar\rho$ satisfies Conjecture 6.1.9, because it is irreducible and hence by [Eme11, Theorem 6.2.13] it satisfies Conjecture 6.1.6 which implies Conjecture 6.1.9 (see [Eme11, Remark 6.1.11]). □

We remark that we can always find $\bar\rho$ which satisfies the theorem above and which at p is isomorphic to our fixed irreducible Galois representation $\bar\rho_p$ (see below).

6.3 Comparison

We will use results of Emerton to describe a part of $\widehat{H}^1_{ss,\overline{\mathbb{F}}_p}$. We start by comparing mod p Hecke algebras for GL_2 and for $D^\times$. On $\mathbf{F}$, after taking K^Σ-invariants, there is an action of a Hecke algebra. For $l \notin \Sigma$, we have a Hecke operator T_l acting on functions of $D^\times(\mathbb{A}_f)$ by

$$T_l(f)(x) = f(xg) + \sum_{i=0}^{l-1} f(xg_i)$$

where $g = \left(\begin{smallmatrix} l & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $g_i = \left(\begin{smallmatrix} 1 & 0 \\ i & l \end{smallmatrix}\right)$ are both considered as elements of $D^\times(\mathbb{A}_f)$ having 1 at places different from l. Let us denote by $\mathbb{T}^D(K_{\Sigma_0})$ the (completed) Hecke algebra, which is the free $\mathcal{O}$-algebra spanned by the operators T_l and S_l for all $l \notin \Sigma$, where $S_l = [K_{\Sigma_0} K^\Sigma \left(\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi \end{smallmatrix}\right) K_{\Sigma_0} K^\Sigma]$. By the results of Serre (see letter to Tate from [Ser96]), systems of eigenvalues for (T_l) of $\mathbb{T}^D(K_{\Sigma_0})$

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on $\mathbf{F}$ are in bijection with systems of eigenvalues for (T_l) of $\mathbb{T}(K_{\Sigma_0})$ coming from mod p modular forms. This allows us to identify maximal ideals of $\mathbb{T}^D(K_{\Sigma_0})$ with those of $\mathbb{T}(K_{\Sigma_0})$, and in what follows we will make no distinction between them.

Let $\bar{\rho}_p$ be the local Galois representation associated to a supersingular representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ by the mod p Langlands correspondence. We assume that there exists a representation $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ which is odd, irreducible, continuous, unramified outside a finite set $\Sigma = \Sigma_0 \cup \{p\}$, and such that $\bar{\rho}_{|G_{\mathbb{Q}_p}} = \bar{\rho}_p$. This is always the case as we may see from the description of the reductions of Galois representations associated to modular forms; see the introduction to [Bre03b] for a discussion (especially Conjecture 1.5) and compare it with the main result of [BG13].

Let us denote by $\mathfrak{m}$ the maximal ideal in the Hecke algebra $\mathbb{T}(K_{\Sigma_0})$ corresponding to $\bar{\rho}$. Results of Emerton apply to $\bar{\rho}$ because we have assumed that $\bar{\rho}_p$ is irreducible. We denote by $K_{\mathfrak{m},\Sigma_0}$ an open compact subgroup of $\prod_{l \in \Sigma_0} \mathrm{GL}_2(\mathbb{Z}_l)$ for which $\pi_{\Sigma_0}(\bar{\rho})^{K_{\mathfrak{m},\Sigma_0}}$ is a one-dimensional vector space (new vectors). We put $K_{\mathfrak{m}} = K_{\mathfrak{m},\Sigma_0} K^{\Sigma}$ and we define

$$\sigma_{\mathfrak{m}} = \mathbf{F}[\mathfrak{m}]^{K_{\mathfrak{m}}}.$$

This is a representation of $D^{\times}(\mathbb{Q}_p)$. We remark that Breuil and Diamond in [BD14] also define a representation of $D^{\times}(\mathbb{Q}_p)$ which serves as a model for a local representation which should appear conjecturally at the place p in the local-global compatibility of the Buzzard–Diamond–Jarvis conjecture (see the next section for a discussion). Their construction is different from our and uses 'types' instead of new vectors.

Let us look again at our cohomology groups. Taking $K_{\mathfrak{m}}$-invariants, which commute with $D^{\times}(\mathbb{Q}_p)$-invariants, we get

$$(\widehat{H}^1_{ss,\bar{\mathbb{F}}_p})^{K_{\mathfrak{m}}} \simeq (\mathbf{F}^{K_{\mathfrak{m}}} \otimes_{\bar{\mathbb{F}}_p} \widehat{H}^1_{\mathrm{LT},\bar{\mathbb{F}}_p})^{D^{\times}(\mathbb{Q}_p)}.$$

Let us define the dual $\sigma_{\mathfrak{m}}^{\vee} = \mathrm{Hom}_{\bar{\mathbb{F}}_p}(\sigma_{\mathfrak{m}}, \bar{\mathbb{F}}_p)$. It is not necessarily a smooth representation. Taking $[\mathfrak{m}]$-part we get

$$(\widehat{H}^1_{ss,\bar{\mathbb{F}}_p}[\mathfrak{m}])^{K_{\mathfrak{m}}} \simeq (\sigma_{\mathfrak{m}} \otimes_{\bar{\mathbb{F}}_p} \widehat{H}^1_{\mathrm{LT},\bar{\mathbb{F}}_p})^{D^{\times}(\mathbb{Q}_p)} =: \widehat{H}^1_{\mathrm{LT},\bar{\mathbb{F}}_p}[\sigma_{\mathfrak{m}}^{\vee}].$$

Thus, by the results proven earlier, we have

$$\pi \otimes_{\bar{\mathbb{F}}_p} \bar{\rho} \simeq (\widehat{H}^1_{\bar{\mathbb{F}}_p}[\mathfrak{m}])^{K_{\mathfrak{m}}}_{(\pi)} \hookrightarrow (\widehat{H}^1_{ss,\bar{\mathbb{F}}_p}[\mathfrak{m}])^{K_{\mathfrak{m}}} \simeq \widehat{H}^1_{\mathrm{LT},\bar{\mathbb{F}}_p}[\sigma_{\mathfrak{m}}^{\vee}],$$

and we arrive at the following theorem.

THEOREM 6.3. *We have a* $\mathrm{GL}_2(\mathbb{Q}_p) \times G_{\mathbb{Q}_p}$-*equivariant injection:*

$$\pi \otimes_{\bar{\mathbb{F}}_p} \bar{\rho} \hookrightarrow \widehat{H}^1_{\mathrm{LT},\bar{\mathbb{F}}_p}[\sigma_{\mathfrak{m}}^{\vee}].$$

We will strengthen this result after proving additional facts about $\sigma_{\mathfrak{m}}$. It is also possible to obtain the analogous result in the p-adic setting. Details will appear elsewhere.

6.4 The mod p Jacquet–Langlands correspondence

We have defined above

$$\sigma_{\mathfrak{m}} = \mathbf{F}[\mathfrak{m}]^{K_{\mathfrak{m}}}.$$

This is a mod p representation of $D^{\times}(\mathbb{Q}_p)$ which is one of our candidates for the mod p Jacquet–Langlands correspondence we search for. We will analyse this representation more carefully in the next section, getting a result about its socle. The question we do not answer here is whether this local representation is independent of the Hecke ideal $\mathfrak{m}$ and, if yes, how to construct it by local means. We make a natural conjecture.

CONJECTURE 6.4. *Let $\mathfrak{m}$ and $\mathfrak{m}'$ be two maximal ideals of the Hecke algebra, which correspond to Galois representations $\bar{\rho}$ and $\bar{\rho}'$ such that $\bar{\rho}_p \simeq \bar{\rho}'_p$. Then we have a $D^\times(\mathbb{Q}_p)$-equivariant isomorphism*

$$\sigma_\mathfrak{m} \simeq \sigma_{\mathfrak{m}'}.$$

This conjecture is natural in view of the fact that $\sigma_\mathfrak{m}$ should play a role of the mod p Jacquet–Langlands correspondence and it should depend only on a local data. In fact, this conjecture follows from the local-global compatibility part of the Buzzard–Diamond–Jarvis conjecture (see [BDJ10, Conjecture 4.7]).

CONJECTURE 6.5. *We have a $D^\times(\mathbb{A})$-equivariant isomorphism*

$$\mathbf{F}[\mathfrak{m}] \simeq \sigma \otimes \pi^p(\bar{\rho})$$

where σ is a $D^\times(\mathbb{Q}_p)$-representation which depends only on $\bar{\rho}_p$, where $\bar{\rho}$ is the Galois representation associated to $\mathfrak{m}$.

The conjecture of Buzzard–Diamond–Jarvis would be proved if one could show the existence of an analogue of the Colmez functor in the context of quaternion algebras. Then, the methods of Emerton from [Eme11] could be applied to give a proof. We come back to the discussion of the mod p Jacquet–Langlands correspondence at the end of the next section.

7. Representations of quaternion algebras: mod p theory

In this section we analyse more carefully mod p representations of quaternion algebras, especially representations $\sigma_\mathfrak{m}$ defined in the preceding section. We also define a naive mod p Jacquet–Langlands correspondence.

7.1 Naive mod p Jacquet–Langlands correspondence

By the work of Vignéras (see [Vig89a]), we know that all irreducible representations of $D^\times$ are of dimension 1 or 2 and are either:

(1) a character of $D^\times(\mathbb{Q}_p)$; or
(2) of the form $\operatorname{Ind}_{\mathcal{O}_D^\times \mathbb{Q}_{p^2}^\times}^{D^\times} \alpha$ where α is a character of $\mathbb{Q}_{p^2}^\times$.

Let $\bar{\rho}_p$ be the mod p two-dimensional irreducible Galois representation which corresponds to the supersingular representation π of $\operatorname{GL}_2(\mathbb{Q}_p)$ by the mod p local Langlands correspondence. As we have mentioned earlier, it is of the form $\operatorname{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} (\omega_2^r \cdot \operatorname{sgn}) \otimes \chi$ where χ is a character and $r \in \{1, \ldots, p\}$.

DEFINITION 7.1. The naive mod p Jacquet–Langlands correspondence is

$$\operatorname{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} (\omega_2^r \cdot \operatorname{sgn}) \otimes \chi \mapsto \operatorname{Ind}_{\mathcal{O}_D^\times \mathbb{Q}_{p^2}}^{D^\times} (\omega_2^r) \otimes \chi$$

where ω_2^r is treated as a character of $\mathbb{Q}_{p^2}$ by the local class field theory and χ is considered both as a character of $G_{\mathbb{Q}_p}$ and $D^\times(\mathbb{Q}_p)$. This gives a bijection between two-dimensional representations of $G_{\mathbb{Q}_p}$ and two-dimensional representations of $D^\times(\mathbb{Q}_p)$. Similar correspondence holds for characters.

We remark that one may also would like to call this the naive mod p Langlands correspondence for $D^\times(\mathbb{Q}_p)$. We get the Jacquet–Langlands correspondence in the usual sense, when we compose it with the mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

Let $\alpha : \mathbb{Q}_{p^2} \to \bar{\mathbb{F}}_p^\times$ be a character. We denote by $\rho(\alpha)$ the representation of $G_{\mathbb{Q}_p}$ obtained by the local class field theory and an induction. We denote by $\sigma(\alpha)$ the $D^\times(\mathbb{Q}_p)$-representation $\mathrm{Ind}_{\mathcal{O}_D^\times \mathbb{Q}_{p^2}}^{D^\times}(\alpha)$. We remark that we also could define the naive mod p Jacquet–Langlands correspondence as

$$\rho(\alpha) \mapsto \sigma(\alpha)$$

but we have chosen our normalisation with a twist by sgn to have the same condition on determinants as for the classical l-adic Jacquet–Langlands correspondence.

7.2 Quaternionic forms

Let D be the quaternion algebra over $\mathbb{Q}$, ramified at p and at ∞. Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and a uniformiser ϖ. Define

$$\mathbf{F} = \varinjlim_K H^0(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f)/K, \bar{\mathbb{F}}_p),$$
$$\mathbf{F}_{\mathcal{O}} = \varinjlim_K H^0(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f)/K, \mathcal{O}).$$

Define also $\mathbf{F}_K = \mathbf{F}_{\mathcal{O}} \otimes_{\mathcal{O}} K$. We can make similar definitions for other $\mathbb{F}_p$ or $\mathbb{Z}_p$-algebras (for example for finite extensions of $\mathbb{F}_p$ or for $\bar{\mathbb{Z}}_p$ in $\mathbf{F}_{\bar{\mathbb{Z}}_p}$ which we will use in the text).

Recall that we have fixed a finite set $\Sigma = \Sigma_0 \cup \{p\}$ and chosen an open, compact subgroup K_{Σ_0} of $\prod_{l \in \Sigma_0} \mathrm{GL}_2(\mathbb{Z}_l)$. On each of the above spaces, after taking K^Σ-invariants, there is an action of the Hecke algebra $\mathbb{T}^D(K_{\Sigma_0})$. Recall also that we have defined $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ an odd, irreducible, continuous representation unramified outside Σ and we have denoted by $\mathfrak{m}$ the maximal ideal in $\mathbb{T}(K_{\Sigma_0})$ (or in $\mathbb{T}^D(K_{\Sigma_0})$) which corresponds to $\bar{\rho}$. We write

$$\bar{\rho}_{|G_{\mathbb{Q}_p}} = \rho(\alpha)$$

where α can be considered as a character of $\mathbb{Q}_{p^2}^\times$ by the local class field theory.

PROPOSITION 7.2. *Take an open, compact subgroup K_p of $D^\times(\mathbb{Q}_p)$ and choose K_{Σ_0} to be an open, compact subgroup of $\prod_{l \in \Sigma_0} \mathrm{GL}_2(\mathbb{Z}_l)$ such that $K_p K_{\Sigma_0} K^\Sigma$ is neat. Then $\mathbf{F}_{\mathfrak{m}}^{K_{\Sigma_0} K^\Sigma}$ is injective as a smooth representation of K_p.*

We do not define the notion of neatness for which we refer to [Pin90, § 0.6]. We only need this condition to ensure that K_p acts freely as in the proof below. Any sufficiently small open compact subgroup is neat.

Proof. Let M be any smooth finitely generated representation of K_p. We have

$$\mathbf{F}^{K_{\Sigma_0} K^\Sigma} = \varinjlim_{K_p'} \mathbf{F}^{K_p' K_{\Sigma_0} K^\Sigma}$$

where $K_p' \subset K_p$ runs over sufficiently small, normal open subgroups of $\mathcal{O}_D^\times$, so that K_p' acts trivially on M. We can associate to M a local system $\mathcal{M}$ on $D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f)/K_{\Sigma_0} K^\Sigma$. Because K_p acts freely on $D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f)/K_{\Sigma_0} K^\Sigma$ by the assumption of neatness, we can descend

this system to each $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K_p'K_{\Sigma_0}K^\Sigma$, where K_p' is as above. Moreover on each $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K_p'K_{\Sigma_0}K^\Sigma$, $\mathcal{M}$ is a constant local system and hence

$$\mathrm{Hom}_{K_p}(M, \mathbf{F}^{K_{\Sigma_0}K^\Sigma}) \simeq \varinjlim_{K_p'} \mathrm{Hom}_{K_p}(M, \mathbf{F}^{K_p'K_{\Sigma_0}K^\Sigma}) \simeq \varinjlim_{K_p'}(\mathbf{F}^{K_p'K_{\Sigma_0}K^\Sigma}(\mathcal{M}^\vee))^{K_p} \simeq \mathbf{F}^{K_p^\times K_{\Sigma_0}K^\Sigma}(\mathcal{M}^\vee)$$

where $\mathbf{F}(\mathcal{M}^\vee) = H^0(D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f), \mathcal{M}^\vee)$. Because $\mathbf{F}^{K_pK_{\Sigma_0}K^\Sigma}(\mathcal{M}^\vee)$ is an exact functor (there is no H^1), we get the result. □

We will now start to analyse socles of quaternionic forms $\mathbf{F}_{\mathfrak{m}}^{K_{\Sigma_0}K^\Sigma}$. Let us start with the following lemma.

LEMMA 7.3. *Let β be a finite-dimensional $\bar{\mathbb{F}}_p$-representation of $\mathcal{O}_D^\times$. We have $\mathrm{Hom}_{\mathcal{O}_D^\times}(\beta^\vee, \mathbf{F}_{\mathfrak{m}}^{K^p}) \simeq \mathbf{F}_{\mathfrak{m}}^{K^p}\{\beta\}$, where $\mathbf{F}_{\mathfrak{m}}^{K^p}\{\beta\}$ is the space of automorphic functions $D(\mathbb{Q})\backslash D(\mathbb{A}_f)/K^p \to \beta$.*

Proof. The isomorphism is given by an explicit map. See [EGH11, Lemma 7.4.3]. □

PROPOSITION 7.4. *The only irreducible $\bar{\mathbb{F}}_p$-representations of $D^\times(\mathbb{Q}_p)$ which appear as submodules in $\mathbf{F}_{\mathfrak{m}}^{K_{\Sigma_0}K^\Sigma}$ are isomorphic to $\sigma^\vee = \sigma(\alpha)^\vee$.*

Proof. Observe that the only irreducible $\bar{\mathbb{F}}_p$-representations of $\mathcal{O}_D^\times$ which can appear in the $\mathcal{O}_D^\times$-socle of $\mathbf{F}_{\mathfrak{m}}^{K^p}$ are duals of the Serre weights of $\bar{\rho}$. This follows from the lemma above and the definition of being modular, i.e. $\bar{\rho}$ is modular of weight β (where β is a representation of $\mathcal{O}_D^\times$) if and only if there exists an open compact subset U of $D^\times(\mathbb{A}_f)$ such that $\mathbf{F}_{\mathfrak{m}}^U\{\beta\} \neq 0$. By the lemma, this is equivalent to $\mathrm{Hom}_{\mathcal{O}_D^\times}(\beta^\vee, \mathbf{F}_{\mathfrak{m}}^U) \neq 0$ which holds if and only if $\beta^\vee \in soc_{\mathcal{O}_D^\times}\mathbf{F}_{\mathfrak{m}}^U$. Now the result follows from [Kha01, Theorem 7], as the only possible weights which can appear in the socle are $\alpha^\vee$ and $(\alpha^p)^\vee$. Hence the $D^\times(\mathbb{Q}_p)$-socle contains only $\sigma(\alpha)^\vee$. □

As a corollary we also get the [$\mathfrak{m}$]-isotypic analogue of the above.

COROLLARY 7.5. *The only irreducible representations which appear as submodules in $\mathbf{F}^{K_{\Sigma_0}K^\Sigma}[\mathfrak{m}]$ are isomorphic to $\sigma^\vee = \sigma(\alpha)^\vee$.*

We are now ready to strengthen the theorem which has appeared before.

THEOREM 7.6. *The representation $\sigma \otimes \pi \otimes \bar{\rho}$ appears as a subquotient in $\widehat{H}_{\mathrm{LT},\bar{\mathbb{F}}_p}^1$.*

Proof. This follows from Theorem 6.3, i.e.

$$\pi \otimes \bar{\rho} \hookrightarrow \widehat{H}_{\mathrm{LT},\bar{\mathbb{F}}_p}^1[\sigma_{\mathfrak{m}}^\vee],$$

and the fact that the only irreducible $D^\times(\mathbb{Q}_p)$-representation which appears as a quotient of $\sigma_{\mathfrak{m}}^\vee$ is σ. □

We remark that if

$$n = \dim_{\bar{\mathbb{F}}_p} \mathrm{Hom}_{D^\times(\mathbb{Q}_p)}(\sigma(\alpha)^\vee, \mathbf{F}^{K_{\Sigma_0}K^\Sigma}[\mathfrak{m}])$$

then one conjectures that $n = 1$ (even in the more general setting, see [Bre14, §8]).

Before moving further, let us recall a (conjectural) structure theorem of Breuil and Diamond for our $D^\times(\mathbb{Q}_p)$-representations, which shows that our candidate for the mod p Jacquet–Langlands correspondence defined above is of entirely different nature than the one with complex coefficients. The proof of the following result is conditional on the local-global compatibility part of the Buzzard–Diamond–Jarvis conjecture. See [BD14] for details.

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PROPOSITION 7.7. *The $D^\times(\mathbb{Q}_p)$-representation $\mathbf{F}^{K_{\Sigma_0}K^\Sigma}[\mathfrak{m}]$ is of infinite length.*

Proof. We give a sketch of the proof, which is contained in [BD14, Corollary 3.2.4]. Firstly observe that it is enough to prove that $\mathbf{F}^{K_{\Sigma_0}K^\Sigma}[\mathfrak{m}]$ is of infinite dimension over $\bar{\mathbb{F}}_p$, because a representation of finite length will be also of finite dimension as $D^\times$ is compact modulo centre. Suppose now that we have an automorphic form π such that the reduction of its associated Galois representation $\bar{\rho}_\pi$ is isomorphic to $\bar{\rho}$ and $\pi^{K_{\Sigma_0}K^\Sigma} \neq 0$. Then there is a lattice $\Lambda_\pi = \mathbf{F}^{K_{\Sigma_0}K^\Sigma}_{\bar{\mathbb{Z}}_p} \cap \pi^{K_{\Sigma_0}K^\Sigma}$ inside $\pi^{K_{\Sigma_0}K^\Sigma}$. Its reduction $\bar{\Lambda}_\pi = \Lambda_\pi \otimes_{\bar{\mathbb{Z}}_p} \bar{\mathbb{F}}_p$ lies in $\mathbf{F}^{K_{\Sigma_0}K^\Sigma}[\mathfrak{m}]$ so it is enough to prove that we can find automorphic representations π as above with $\pi^{K_{\Sigma_0}K^\Sigma}$ of arbitrarily high dimension. This is done by explicit computations of possible lifts in [BD14]. $\square$

This proposition indicates that $\widehat{H}^1_{\mathrm{LT},\bar{\mathbb{F}}_p}$ is a non-admissible smooth representation.

7.3 Non-admissibility

PROPOSITION 7.8. *The $\mathrm{GL}_2(\mathbb{Q}_p)$-representations $\widehat{H}^1_{ss,\bar{\mathbb{F}}_p}$ and $\widehat{H}^2_{X_{ord},\bar{\mathbb{F}}_p}$ are non-admissible smooth $\bar{\mathbb{F}}_p$-representations.*

Proof. If one of them would be admissible, then also the second would because of the exact sequence

$$\widehat{H}^1_{X_{ord},\bar{\mathbb{F}}_p} \to \widehat{H}^1_{\bar{\mathbb{F}}_p} \to \widehat{H}^1_{ss,\bar{\mathbb{F}}_p} \to \widehat{H}^2_{X_{ord},\bar{\mathbb{F}}_p} \to \widehat{H}^2_{\bar{\mathbb{F}}_p}.$$

It is enough to prove that $\widehat{H}^1_{ss,\bar{\mathbb{F}}_p}$ is non-admissible, or even that $\widehat{H}^1_{\mathrm{LT},\bar{\mathbb{F}}_p}$ is non-admissible. Let us look at the Hochschild–Serre spectral sequence for the Iwahori level I of the Lubin–Tate tower

$$H^i(I, \widehat{H}^j_{\mathrm{LT},\bar{\mathbb{F}}_p}) \Rightarrow H^{i+j}_{\mathrm{LT},I,\bar{\mathbb{F}}_p}$$

where we have denoted by $H^{i+j}_{\mathrm{LT},I,\bar{\mathbb{F}}_p}$ the fundamental representation at I-level. Now observe that if $\widehat{H}^1_{\mathrm{LT},\bar{\mathbb{F}}_p}$ were admissible, then $H^0(I, \widehat{H}^1_{\mathrm{LT},\bar{\mathbb{F}}_p})$ would be of finite dimension. Because $H^1(I, \widehat{H}^0_{\mathrm{LT},\bar{\mathbb{F}}_p})$ is of finite dimension (as $\widehat{H}^0_{\mathrm{LT},\bar{\mathbb{F}}_p}$ is), this would mean that $H^1_{\mathrm{LT},I,\bar{\mathbb{F}}_p}$ is finite dimensional. However, geometrically the Lubin–Tate tower at level I is an annulus (this is a standard fact, one can prove it by methods of §8.1), and hence $H^1_{\mathrm{LT},I,\bar{\mathbb{F}}_p}$ has to be of infinite dimension (see [Ber93, Remark 6.4.2]). This contradiction finishes the proof. $\square$

COROLLARY 7.9. *The $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $\widehat{H}^1_{\mathrm{LT},\bar{\mathbb{F}}_p}$ is a non-admissible smooth $\bar{\mathbb{F}}_p$-representation.*

Proof. The corollary follows from the Proposition 7.8. $\square$

7.4 On mod p Jacquet–Langlands correspondence
We come once again to the discussion of the mod p Jacquet–Langlands correspondence. Note that there are three possible candidates for the correspondence which appear in our work:

(1) the two-dimensional irreducible representation σ of $D^\times(\mathbb{Q}_p)$ defined by the naive mod p Jacquet–Langlands correspondence;

(2) the representation $\sigma_\mathfrak{m}$ defined by global means and depending *a priori* on a maximal Hecke ideal $\mathfrak{m}$. It is of infinite length as a representation of $D^\times(\mathbb{Q}_p)$ and contains $\sigma^\vee$ in its socle;

(3) the representation defined via the cohomology

$$\sigma_{\mathrm{LT}} = \mathrm{Hom}_{G_{\mathbb{Q}_p} \times \mathrm{GL}_2(\mathbb{Q}_p)}(\bar{\rho} \otimes_{\bar{\mathbb{F}}_p} \pi, \widehat{H}^1_{\mathrm{LT},\bar{\mathbb{F}}_p}).$$

By the results above, it contains σ as a subquotient.

In the l-adic setting, we can define representations of $D^\times(\mathbb{Q}_p)$ in the similar way and it is known that $\sigma_{\mathrm{LT}} \simeq \sigma_{\mathfrak{m}}^\vee$. Moreover $\sigma_{\mathfrak{m}}$ in the l-adic setting is two-dimensional (at least in the moderately ramified case). This is not the case in the mod p setting as we have shown that representations of candidates (1) and (2) are different (one is two-dimensional, the other is infinite-dimensional). The natural definition of the mod p correspondence seems to be σ_{LT} and it is also natural to ask whether one has $\sigma_{\mathrm{LT}} \simeq \sigma_{\mathfrak{m}}^\vee$ for each appropriate $\mathfrak{m}$ as considered before.

8. Cohomology with compact support

In this section we will discuss what happens when we consider the cohomology with compact support. Our basic result is negative and it states that the first cohomology group with compact support of the fundamental representation $\widehat{H}^1_{\mathrm{LT},c,\bar{\mathbb{F}}_p}$ does not contain any supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ as a subrepresentation. This surprising result, which is very different from the situation known in the l-adic setting where $l \neq p$, leads to a similar exact sequence as we have considered for cohomology without support, but this time, we get that $\pi \otimes \bar{\rho}$ is contained in the H^1 of the ordinary locus.

8.1 Geometry at pro-p Iwahori level

Let $K(1) = \begin{pmatrix} 1+p\mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}$ and let $I(1) = \begin{pmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}$ be the pro-p Iwahori subgroup. We let

$$\mathcal{M}_{\mathrm{LT},K(1)} = \mathrm{Spf}\, R_{K(1)},$$
$$\mathcal{M}_{\mathrm{LT},I(1)} = \mathrm{Spf}\, R_{I(1)}$$

be the formal models for the Lubin–Tate space at levels $K(1)$ and $I(1)$ respectively. We will compute $R_{I(1)}$ explicitly. This is also done in a more general setting in the work of Haines and Rapoport (see [HR12, Corollary 3.4.3]), but here we give a short and elementary argument.

We know that $R_{I(1)} = R_{K(1)}^{I(1)}$ and hence we can use the explicit description of $R_{K(1)}$ by Yoshida to get the result (see [Yos10, Proposition 3.5]). Let $W = W(\bar{\mathbb{F}}_p)$ be the Witt vectors of $\bar{\mathbb{F}}_p$. There is a surjection $W[[\tilde{X}_1, \tilde{X}_2]] \twoheadrightarrow R_{K(1)}$ which maps $\tilde{X}_i$ to X_i where X_i $(i = 1, 2)$ are local parameters for $R_{K(1)}$ which form a $\mathbb{F}_p$-basis of $\mathfrak{m}_{R_{K(1)}}[p] = \{x \in \mathfrak{m}_{R_{K(1)}} | [p](x) = 0\}$, where $[p]$ is explained below. We will find parameters for $R_{I(1)} = R_{K(1)}^{I(1)}$. Observe that for $b \in \mathbb{F}_p$ we have (see [Yos10, ch. 3])

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} X_1 = X_1,$$
$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} X_2 = [b]X_1 +_\Sigma X_2$$

where $+_\Sigma$ is the addition on the universal deformation of the unique formal group over $\bar{\mathbb{F}}_p$ of height two and $[.]$ gives the structure of multiplication by elements of W on the same universal deformation Σ. See [Yos10, ch. 3] for details. We see that X_2 is not invariant under $I(1)$ and hence we define $X_2' = \prod_{b \in \mathbb{F}_p}([b]X_1 +_\Sigma X_2)$ which is. We claim that (X_1, X_2') are local parameters for $R_{I(1)}$. Indeed if z belongs to $R_{I(1)} = R_{K(1)}^{I(1)}$ then we may write it as $z = P(X_1) + X_2 Q(X_1, X_2)$,

where $P \in W[[X_1]]$ and $Q \in W[[X_1, X_2]]$. As $P(X_1)$ is invariant under $I(1)$, we see that also $X_2 Q(X_1, X_2)$ has to be invariant under $I(1)$. Because of the action of $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ on X_2 described above and the fact that $R_{K(1)}$ is a regular local ring hence factorial, we see that X_2' divides $X_2 Q(X_1, X_2)$ (we use here the fact that $[b]X_1 +_\Sigma X_2$ and $[b']X_1 +_\Sigma X_2$ are not associated for $b \neq b'$; this follows from [Str08, Proposition 4.2]). This leads to $z = P(X_1) + X_2' Q'(X_1, X_2)$ for some Q' which is $I(1)$-invariant and hence we conclude by induction and the fact that polynomials are dense in formal series (look at z modulo powers of the maximal ideal $\mathfrak{m}_{R_{I(1)}}$).

Let us observe that for $a \in \mathbb{F}_p^\times$ we have, for $i = 1, 2$, that $[a]X_i = uX_i$, where u is a unit in $R_{K(1)}$. Let us now look at the relation defining $R_{K(1)}$ inside $W[[\tilde{X}_1, \tilde{X}_2]]$ which appears in [Yos10, Proposition 3.5]. We have

$$p = u \prod_{(a_1, a_2) \in \mathbb{F}_p^2 \setminus \{0,0\}} ([a_1]X_1 +_\Sigma [a_2]X_2)$$

where u is some unit in $R_{K(1)}$. Let us write $a \sim b$ whenever $a = ub$ for some unit u in $R_{K(1)}$. Thus we have

$$p \sim \prod_{(a_1, a_2) \in \mathbb{F}_p^2 \setminus \{0,0\}} ([a_1]X_1 +_\Sigma [a_2]X_2) \sim \left(\prod_{a_1 \in \mathbb{F}_p^\times} [a_1]X_1 \right) \left(\prod_{a_1 \in \mathbb{F}_p} \prod_{a_2 \in \mathbb{F}_p^\times} [a_2]([a_1/a_2]X_1 +_\Sigma X_2) \right)$$

$$\sim \left(\prod_{a_1 \in \mathbb{F}_p^\times} [a_1]X_1 \right) \left(\prod_{a_2 \in \mathbb{F}_p^\times} X_2' \right) \sim (X_1 X_2')^{p-1}.$$

Hence we have $p = u'(X_1 X_2')^{p-1}$ for some unit u' in $R_{K(1)}$ *a priori*, but we can see that u' is in fact a unit in $R_{I(1)}$. Because $W[[X, Y]]$ is a complete local ring with an algebraically closed residue field there exists a $(p-1)$th root of u', and hence we can write $p = (X_1' X_2'')^{p-1}$. We want to conclude that this is the only relation in $R_{I(1)}$ which means that there exists a surjection

$$B = W[[\tilde{X}_1', \tilde{X}_2'']] \twoheadrightarrow R_{I(1)}$$

with kernel $f = (\tilde{X}_1' \tilde{X}_2'')^{p-1} - p$. First of all, observe that $R_{I(1)}$ and B/fB are regular local rings of dimension 2 with a surjection $B/fB \twoheadrightarrow R_{I(1)}$. We claim that this map has to be necessarily an injection also. Indeed, this holds for any surjective morphism $A \twoheadrightarrow R$ of regular local rings of the same dimension by using the fact that for a regular local ring we have $\mathrm{gr}_{\mathfrak{m}_A}^\bullet A \simeq \mathrm{Sym}\, \mathfrak{m}_A/\mathfrak{m}_A^2$. This yields an isomorphism at the graded level which lifts to the level of rings. All in all, we conclude that the following proposition is true.

PROPOSITION 8.1. *We have*

$$R_{I(1)} \simeq W[[X, Y]]/((XY)^{p-1} - p).$$

This means that $\mathcal{M}_{\mathrm{LT}, I(1)}$ is made of $p-1$ copies of an open annulus in $\mathbb{P}^1$ after a base change to $W[\sqrt[p-1]{p}][1/p]$:

$$R_{I(1)} \otimes_W W[\sqrt[p-1]{p}][1/p] \simeq \prod_{i=1}^{p-1} W[[X, Y]][1/p]/(XY - \sqrt[p-1]{p} \cdot \zeta_{p-1}^i).$$

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8.2 Cohomology at pro-p Iwahori level

We compute $H^1_c(\mathcal{M}_{\mathrm{LT},I(1)}, \bar{\mathbb{F}}_p)$ (we will omit $\bar{\mathbb{F}}_p$ from the notation in what follows). Let $\mathcal{A}$ be an open annulus in $\mathbb{P}^1$. We can write a long exact sequence

$$0 \to H^0_c(\mathcal{A}) \to H^0(\mathbb{P}^1) \to H^0(\mathbb{P}^1 \backslash \mathcal{A}) \to H^1_c(\mathcal{A}) \to H^1(\mathbb{P}^1).$$

We know that

$$H^1(\mathbb{P}^1) = H^0_c(\mathcal{A}) = 0,$$
$$\dim_{\bar{\mathbb{F}}_p} H^0(\mathbb{P}^1) = 1,$$
$$\dim_{\bar{\mathbb{F}}_p} H^0(\mathbb{P}^1 \backslash \mathcal{A}) = 2,$$

and hence it follows that

$$\dim_{\bar{\mathbb{F}}_p} H^1_c(\mathcal{A}) = 1.$$

Because geometrically $\mathcal{M}_{\mathrm{LT},I(1)}$ is made of $p-1$ copies of $\mathcal{A}$, we have

$$\dim_{\bar{\mathbb{F}}_p} H^1_c(\mathcal{M}_{\mathrm{LT},I(1)}) = p-1.$$

Let $\mathcal{H} = \mathcal{H}_{\mathrm{GL}_2}(I(1)) = \bar{\mathbb{F}}_p[I(1)\backslash \mathrm{GL}_2(\mathbb{Q}_p)/I(1)]$ be the mod p Hecke algebra at the pro-p Iwahori level. Let I be the Iwahori subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$. We look at the action of $\bar{\mathbb{F}}_p[I/I(1)] \simeq \bar{\mathbb{F}}_p[(\mathbb{F}_p^\times)^2]$ on the cohomology. We know by [Str08] that it acts by determinant on connected components of $\mathcal{M}_{\mathrm{LT},K(1)}$ and hence on connected components of $\mathcal{M}_{\mathrm{LT},I(1)}$ so we have a decomposition of $H^1_c(\mathcal{M}_{\mathrm{LT},I(1)})$ into $p-1$ pieces of dimension 1:

$$H^1_c(\mathcal{M}_{\mathrm{LT},I(1)}) = \bigoplus_{\chi:\mathbb{F}_p^\times \to \bar{\mathbb{F}}_p^\times} H^1_c(\mathcal{M}_{\mathrm{LT},I(1)})_\chi$$

where $H^1_c(\mathcal{M}_{\mathrm{LT},I(1)})_\chi$ is the part of $H^1_c(\mathcal{M}_{\mathrm{LT},I(1)})$ on which $\bar{\mathbb{F}}_p[(\mathbb{F}_p^\times)^2]$ acts through $\chi \circ \det$.

8.3 Vanishing result

We will now prove that the supersingular representation π does not appear in $\widehat{H}^1_{\mathrm{LT},c,\bar{\mathbb{F}}_p}$. First of all, remark that it is enough to prove that the $\mathcal{H}$-module $\pi^{I(1)}$ does not appear in $(\widehat{H}^1_{\mathrm{LT},c,\bar{\mathbb{F}}_p})^{I(1)}$, because the functor $\pi \mapsto \pi^{I(1)}$ induces a bijection between supersingular representations and supersingular Hecke modules (see [Vig04]). We have the Hochschild–Serre spectral sequence (see Appendix A)

$$H^i(I(1), \widehat{H}^j_{\mathrm{LT},c,\bar{\mathbb{F}}_p}) \Rightarrow H^{i+j}_{\mathrm{LT},c,I(1),\bar{\mathbb{F}}_p}$$

where we have denoted by $H^{i+j}_{\mathrm{LT},c,I(1),\bar{\mathbb{F}}_p}$ the fundamental representation at $I(1)$-level. This gives a long exact sequence

$$0 \to H^1(I(1), \widehat{H}^0_{\mathrm{LT},c,\bar{\mathbb{F}}_p}) \to H^1_{\mathrm{LT},c,I(1),\bar{\mathbb{F}}_p} \to (\widehat{H}^1_{\mathrm{LT},c,\bar{\mathbb{F}}_p})^{I(1)} \to H^2(I(1), \widehat{H}^0_{\mathrm{LT},c,\bar{\mathbb{F}}_p}).$$

Because $\widehat{H}^0_{\mathrm{LT},c,\bar{\mathbb{F}}_p} = 0$ as $H^0_c(\mathcal{M}_{\mathrm{LT}}, \bar{\mathbb{F}}_p) = 0$ we have an $\mathcal{H}$-equivariant isomorphism

$$H^1_{\mathrm{LT},c,I(1),\bar{\mathbb{F}}_p} \simeq (\widehat{H}^1_{\mathrm{LT},c,\bar{\mathbb{F}}_p})^{I(1)}.$$

This means that if $\pi^{I(1)}$ appears in $(\widehat{H}^1_{\mathrm{LT},c,\bar{\mathbb{F}}_p})^{I(1)}$ then it appears also in $H^1_{\mathrm{LT},c,I(1),\bar{\mathbb{F}}_p}$. However, because $H^1_{\mathrm{LT},c,I(1),\bar{\mathbb{F}}_p}$ consists of multiple copies of $H^1_c(\mathcal{M}_{\mathrm{LT},I(1)}, \bar{\mathbb{F}}_p)$, it is enough to show that

$\pi^{I(1)}$ does not appear in $H_c^1(\mathcal{M}_{\mathrm{LT},I(1)}, \bar{\mathbb{F}}_p)$. To prove it, it suffices to show that no supersingular $\mathcal{H}$-module appear in $H_c^1(\mathcal{M}_{\mathrm{LT},I(1)}, \bar{\mathbb{F}}_p)$. Let M be any supersingular $\mathcal{H}$-module. Then we know that it is two-dimensional and of the form $M_2(0, z, \omega)$ as in [Vig04, § 3.2], where ω is a character of $I/I(1)$. If we write $I/I(1) = \mathbb{F}_p^\times \times \mathbb{F}_p^\times$ and $\omega = \eta_1 \otimes \eta_2$ then $M = (\eta_1 \otimes \eta_2) \oplus (\eta_2 \otimes \eta_1)$ as a $I/I(1)$-module. If M appears in $H_c^1(\mathcal{M}_{\mathrm{LT},I(1)}, \bar{\mathbb{F}}_p)$, then $I/I(1)$ acts on M by determinant and hence $\eta_1 = \eta_2$. This would mean that $H_c^1(\mathcal{M}_{\mathrm{LT},I(1)})_{\eta_1}$ is at least two-dimensional, which is a contradiction. All in all, we conclude that $\pi^{I(1)}$ does not appear in $(\widehat{H}_{\mathrm{LT},c,\bar{\mathbb{F}}_p}^1)^{I(1)}$ and hence we have the following theorem.

THEOREM 8.2. *The supersingular representation π does not appear in $\widehat{H}_{\mathrm{LT},c,\bar{\mathbb{F}}_p}^1$.*

We could rephrase it also as

$$\widehat{H}_{\mathrm{LT},c,\bar{\mathbb{F}}_p,(\pi)}^1 = 0.$$

Remark 8.3. Observe that the above proof does not use in any particular form the fact that we are working with $\mathrm{GL}_2(\mathbb{Q}_p)$, besides the fact that the functor $\pi \mapsto \pi^{I(1)}$ induces a bijection between supersingular representations and supersingular $\mathcal{H}$-modules. Apart from that, the results of Vignéras and Yoshida holds for $\mathrm{GL}_2(F)$ as well, where F is a finite extension $\mathbb{Q}_p$ and show that there are no supersingular modules in the cohomology with compact support of the Lubin–Tate tower at the pro-p Iwahori level. This leads to the conclusion that supersingular representations of $\mathrm{GL}_2(F)$ attached to these supersingular modules by the construction of Paškūnas (see [Pas04]) do not appear in the cohomology with compact support of the Lubin–Tate tower at infinite level. We remark that, contrary to $F = \mathbb{Q}_p$ case, those supersingular representations constructed by Paškūnas do not conjecturally give all the supersingular representations of $\mathrm{GL}_2(F)$.

The above theorem gives us, when combined with the exact sequence for the supersingular locus, an appearance of the mod p local Langlands correspondence in the cohomology of the ordinary locus (in contrast with the mod l situation).

COROLLARY 8.4. *We have an $\mathrm{GL}_2(\mathbb{Q}_p) \times G_{\mathbb{Q}_p}$-equivariant injection*

$$\pi \otimes \bar{\rho} \hookrightarrow \widehat{H}_{ord,\bar{\mathbb{F}}_p}^1.$$

Moreover, this vanishing result can be used in the study of non-admissibility and in the description of the cohomology of certain Shimura curves.

8.4 Non-admissibility

We will now show that our cohomology groups are non-admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. We start with the following proposition.

PROPOSITION 8.5. *The $\mathrm{GL}_2(\mathbb{Q}_p)$-representations $\widehat{H}_{ss,c,\bar{\mathbb{F}}_p}^2$ and $\widehat{H}_{ord,\bar{\mathbb{F}}_p}^1$ are non-admissible smooth $\bar{\mathbb{F}}_p$-representations.*

Proof. If one of them would be admissible, then also the second would be because of the exact sequence

$$\widehat{H}_{ss,c,\bar{\mathbb{F}}_p}^1 \to \widehat{H}_{\bar{\mathbb{F}}_p}^1 \to \widehat{H}_{ord,\bar{\mathbb{F}}_p}^1 \to \widehat{H}_{ss,c,\bar{\mathbb{F}}_p}^2 \to \widehat{H}_{\bar{\mathbb{F}}_p}^2.$$

But we know that $\widehat{H}_{ord,\bar{\mathbb{F}}_p}^1$ is an induced representation

$$\mathrm{Ind}_{B(\infty)}^{\mathrm{GL}_2}\left(\bigoplus_a \widehat{H}_{a,\infty,\bar{\mathbb{F}}_p}^1\right)$$

so if it were admissible, then the localisation of it at π would have to vanish. This is not possible by the corollary above. □

COROLLARY 8.6. *The* $\mathrm{GL}_2(\mathbb{Q}_p)$-*representation* $\widehat{H}^2_{\mathrm{LT},c,\bar{\mathbb{F}}_p}$ *is a non-admissible smooth* $\bar{\mathbb{F}}_p$-*representation.*

Proof. The corollary follows from the Proposition 8.5. □

8.5 Cohomology of Shimura curves

We will briefly sketch another consequence of vanishing of $\widehat{H}^1_{\mathrm{LT},c,\bar{\mathbb{F}}_p,(\pi)}$. First of all, it implies that $H^1_c(\mathcal{M}_{\mathrm{LT}},\bar{\mathbb{F}}_p)_{(\pi)}$ vanishes because $\widehat{H}^1_{\mathrm{LT},c,\bar{\mathbb{F}}_p,(\pi)}$ is just a sum of copies of $H^1_c(\mathcal{M}_{\mathrm{LT}})_{(\pi)}$. Now recall the Faltings isomorphism (see [Far08]) which gives us

$$H^1_c(\mathcal{M}_{\mathrm{LT}},\bar{\mathbb{F}}_p)_{(\pi)} = H^1_c(\mathcal{M}_{Dr},\bar{\mathbb{F}}_p)_{(\pi)} = 0$$

where $\mathcal{M}_{Dr}$ is the Drinfeld tower at infinity (see [Dat12] for details). We have a spectral sequence coming from the p-adic uniformisation of the Shimura curve Sh associated to the algebraic group G'' arising from the quaternion algebra over $\mathbb{Q}$ which is ramified precisely at p and some other prime q:

$$E^{p,q}_2 = \mathrm{Ext}^p_{\mathrm{GL}_2(\mathbb{Q}_p)}(H^{2-q}_c(\mathcal{M}_{Dr,K_p},\bar{\mathbb{F}}_p), C^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}),\bar{\mathbb{F}}_p)^{K^p}) \Rightarrow H^{p+q}_c(Sh^{an}_{K_pK^p},\bar{\mathbb{F}}_p)$$

where we have denoted by G' the algebraic group arising from the quaternion algebra over $\mathbb{Q}$ which is ramified precisely at q and ∞. For this, see [Far04] where it is proven for $\bar{\mathbb{Q}}_l$ but the proof works also for $\bar{\mathbb{F}}_p$ (the proof is also contained in the [Dat06, Appendix B]).

Choose any non-Eisenstein maximal ideal $\mathfrak{n}$ in the Hecke algebra of G'' whose associated Galois representation corresponds at p to the supersingular representation π we have chosen before. Take the direct limit over K_p and localise the above spectral sequence at $\mathfrak{n}$ to get

$$\mathrm{Ext}^p_{\mathrm{GL}_2(\mathbb{Q}_p)}(H^{2-q}_c(\mathcal{M}_{Dr},\bar{\mathbb{F}}_p)_{(\pi)}, C^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}),\bar{\mathbb{F}}_p)^{K^p}_{\mathfrak{n}}) \Rightarrow H^{p+q}_c(Sh^{an}_{K^p},\bar{\mathbb{F}}_p)_{\mathfrak{n}}.$$

The localisation of $H^{2-q}_c(\mathcal{M}_{Dr},\bar{\mathbb{F}}_p)$ at π appears because $C^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}),\bar{\mathbb{F}}_p)^{K^p}_{\mathfrak{n}}$ is π-isotypic. Using our vanishing result we get an interesting isomorphism

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(H^2_c(\mathcal{M}_{Dr},\bar{\mathbb{F}}_p)_{(\pi)}, C^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}),\bar{\mathbb{F}}_p)^{K^p}_{\mathfrak{n}}) \simeq H^1_c(Sh^{an}_{K^p},\bar{\mathbb{F}}_p)_{\mathfrak{n}}.$$

This can be possibly used to study the mod p cohomology of the Shimura curve Sh. We shall treat this issue elsewhere.

9. Concluding remarks

Let us finish by giving some remarks and stating natural questions.

9.1 The *l*-adic case

Observe that our arguments work well also in the mod $l \neq p$ setting and circumvent the use of vanishing cycles. The idea of localisation at a supersingular (supercuspidal) representation appears also in the work of Dat. See especially [Dat12] where the author discusses localisations both for GL_n and quaternion algebras and then uses it to describe the supercuspidal part of the cohomology.

One might want also to see [Shi], which bears some resemblance to certain arguments we use. Shin describes the mod l cohomology of Shimura varieties by using results of Dat about the mod l cohomology of the Lubin–Tate tower. In our work, we start from global results of Emerton to deduce from them statements about local objects.

9.2 Beyond modular curves

The geometric arguments we have given also applies to Shimura curves considered by Carayol in [Car86] and we can consider similar exact sequences relating the ordinary locus and the supersingular locus in this setting. Nevertheless, in this case we cannot go on with arguments as we do not have a definition of the mod p local Langlands correspondence for extensions of $\mathbb{Q}_p$. In fact, such a construction seems a little bit problematic as might be seen from the work of Breuil and Paškūnas [BP12], where the authors show that there are much more automorphic representations than Galois representations. The hope is that by looking at the cohomology of the Lubin–Tate tower, one should be able to tell how the correspondence should look like. We will pursue this subject in our subsequent work.

9.3 Adic spaces

We have chosen to work with Berkovich spaces, but one might as well wonder how the things translate into the setting of adic spaces of Huber [Hub96]. In fact, everything that we have considered can be rewritten in the language of adic spaces and we might consider the same long exact sequences as above (though these exact sequences will be inversed due to the fact that adic spaces behave like formal schemes). The main difference between those two contexts lies in the ordinary locus which in the case of adic spaces will contain additional points which lie in the closure of the ordinary locus from the setting of Berkovich spaces. Nevertheless, the cohomology groups in both settings will be similar and we refer a reader to [Cho14] for details. Let us remark also, that the comparison between mod p étale cohomology of a formal scheme and its (adic) analytification is proved in [Hub96, Theorem 3.7.2].

9.4 Serre's letters

Though it does not appear explicitly in our work (besides the comparison of Hecke algebras), we were influenced by two letters written by Serre (see [Ser96]). It is there that in some sense appears for the first time the modified mod l local Langlands correspondence which goes under the name of the universal unramified representation (see the letter to Kazhdan). Indeed, if we were to suppose that our global lift $\bar{\rho}$ which we have used is actually unramified everywhere outside p, then there is no need to recall either the modified mod l local Langlands correspondence or new vectors, and we could formulate everything in the language of Serre.

Appendix A. The Hochschild–Serre spectral sequence

In the body of the text we have used the unpublished manuscript of Berkovich [Ber95b] where, among others, there appears the Hochschild–Serre spectral sequence for the cohomology with compact support. For the sake of completeness, we will sketch a proof of existence of this spectral sequence here. We thank Vladimir Berkovich for sending us his preprint.

A.1 G-spaces

Recall that an analytic space (in the sense of Berkovich) is a k-analytic space over some non-archimedean field k. Given two analytic spaces X and Y, let $\mathrm{Mor}(X,Y)$ denote the set of morphisms $X \to Y$ and let $\mathcal{G}(X)$ be the group of automorphisms of X. Berkovich defined in [Ber94] a uniform space structure (and in particular, a topology) on $\mathrm{Mor}(X,Y)$. Then, the group $\mathcal{G}(X)$ has the topology induced from $\mathrm{Mor}(X,X)$. We say that the action of a topological group G on an analytic space X is continuous if the induced homomorphism $G \to \mathcal{G}(X)$ is continuous. An analytic space endowed with a continuous action of a topological group G will be called a G-space. A G-equivariant morphism between two G-spaces will be called a G-morphism. The category of analytic spaces is the category of pairs (X, G), where G is a topological group and X is a G-space. We will denote this pair by $X(G)$. A morphism between such spaces $\phi : X'(G') \to X(G)$ is a pair consisting of a continuous homomorphism of topological groups $\nu_\phi : G' \to G$ and a morphism of analytic spaces $\phi : X' \to X$ compatible with the homomorphism ν_ϕ. A G-morphism $\phi : X' \to X$ between G-spaces gives rise to a morphism $\phi : X'(G) \to X(G)$ for which ν_ϕ is the identity map on G. If X is a G-space then the action of G on X extends to a natural action of G on $X(G)$ for which $\nu_g(g') = gg'g^{-1}$, where ν_g is the morphism given by an element $g \in G$. For a G-space X we have a morphism $b : X \to X(G)$ where $X = X(\{1\})$.

A.2 Étale topology

Berkovich has defined the étale topology on analytic spaces and similarly we can define the étale topology on G-analytic spaces. For a G-space X, let $Et(X(G))$ denote the category of étale morphisms $U(G) \to X(G)$. The étale topology on $X(G)$ is the Grothendieck topology on the category $Et(X(G))$ with coverings of $U(G) \to X(G)$ consisting of families $(U_i(G) \to U(G))_{i \in I}$ such that $(U_i \to U)_{i \in I}$ is a covering in the étale topology of X. We denote this site by $X(G)_{et}$ and its corresponding topos by $X(G)_{et}^\sim$. In a similar way, we can also introduce a quasi-étale site $X(G)_{qet}$ and its topos.

We denote by $\Gamma_{X(G)}$ the functor of global sections on $X(G)_{et}$, that is $\Gamma_{X(G)}(F) = F(X(G))$. The higher direct images of $\Gamma_{X(G)}$ on the category of abelian sheaves will be denoted by $F \mapsto H^i(X(G), F)$. Let F be an étale abelian sheaf on $X(G)$. The support of $f \in F(X(G))$ is the (closed) set $\mathrm{Supp}(f) = \{x \in X \mid f_x \neq 0\}$, where f_x is the image of f in F_x. The cohomology groups with compact support are higher direct images of the functor $F \mapsto \Gamma_{c,X(G)}(F) = \{f \in F(X(G)) \mid \mathrm{Supp}(f) \text{ is compact}\}$ and are denoted by $F \mapsto H^i_c(X(G), F)$. We consider also the higher direct functor of $F \mapsto \Gamma_{c,X\{G\}}(F) := \varinjlim \Gamma_{c,X(N)}(F)$ where N runs through open subgroups of G which we will denote by $F \mapsto H^i_c(X\{G\}, F)$. We have $H^i_c(X\{G\}, F) = \varinjlim H^i_c(X(N), F)$. The proposition (that is, [Ber95b, Corollary 1.5.2]) we have used in the main text was the following.

PROPOSITION A.1. *For any étale abelian sheaf F on $X(G)$ there are canonical isomorphisms*

$$H^i_c(X\{G\}, F) \simeq H^i_c(X, b^*F)$$

for $i \geqslant 0$. In particular, there is a spectral sequence

$$E_2^{p,q} = H^p(G, H^q_c(X, b^*F)) \Rightarrow H^{p+q}_c(X(G), F).$$

We have applied it to the Lubin–Tate tower $X = \varprojlim_n \mathcal{M}_{\mathrm{LT},n}$ and $G = I(1)$ the pro-p Iwahori subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$, by applying it to each $\mathcal{M}_{\mathrm{LT},n}$ and taking direct limit in the spectral sequence above (as X is not a Berkovich space we cannot put it directly in the spectral sequence). We are using also the fact that we have an equivalence of topoi $X(G)_{et}^\sim \simeq (G \backslash X)_{et}^\sim$, whenever $G \backslash X$ exists and $X \to G \backslash X$ is étale. This is so in our case, where $G \backslash X = \mathcal{M}_{\mathrm{LT},I(1)}$.

Proof. We sketch the proof of this proposition. The case $i = 0$ follows from the fact that every element of $H_c^0(X, b^*F)$ is fixed by an open subgroup of G. Then the general case follows by constructing the Godement resolution in our context. Namely, for a topological space I denote by $Top(I)$ the site on the category of local homeomorphisms $J \to I$ (with the evident topology). Suppose we have a surjective map $I \to X : i \mapsto x_i$. We endow I with the discrete topology and we fix for each $i \in I$ a geometric point $\bar{x}_i$ over x_i. This gives rise to a morphism of sites $\nu : Top(I) \to X(G)_{et}$. For an étale abelian sheaf F on $X(G)$, its Godement resolution $\mathcal{C}^\bullet(F)$ is constructed as follows.

(i) $\mathcal{C}^0(F) = \nu_* \nu^*(F)$ and let $d^{-1} : F \to \mathcal{C}^0(F)$ be the adjunction map.

(ii) If $m \geqslant 0$, then put $\mathcal{C}^{m+1}(F) = \mathcal{C}^0(\mathrm{coker}\ d^{m-1})$, and let d^m be the canonical map $\mathcal{C}^m(F) \to \mathcal{C}^{m+1}(F)$.

This construction is taken from SGA 4, Exp. XVII, where the following are shown to hold.

(a) $\mathcal{C}^m(F)$ is a flabby sheaf.

(b) The functor $F \mapsto \mathcal{C}^m(F)$ is exact.

(c) The fibre of the complex $\mathcal{C}^\bullet(F)$ at a point $x \in X$ is a canonically split resolution of F_x.

Then Berkovich [Ber95b, Proposition 1.5.1] shows that for any $F \in X(G)_{et}^\sim$ and $m \geqslant 0$, the sheaf $b^*(\mathcal{C}^m(F))$ is soft on X_{et}, where we say that a sheaf $\mathcal{F}$ is soft on X_{et} (after [Ber94, ch. 3]) when it satisfies the following two conditions.

(1) For any $x \in X$, the stalk $\mathcal{F}_x$ is a flabby $\mathrm{Gal}_{\mathcal{H}(x)} = \mathrm{Gal}(\overline{\mathcal{H}(x)}/\mathcal{H}(x))$-module, where $\mathcal{H}(x)$ is the complete field associated to x by the standard procedure.

(2) For any paracompact U étale over X, the restriction of $\mathcal{F}$ to the usual topology $|U|$ of U is a soft sheaf; that is, for any compact subset $T \subset U$, the map $\mathcal{F}(U) \to \mathcal{F}(T)$ is surjective.

Then one shows (see [Ber94, Lemma 3.2]) that we can compute the cohomology with soft sheaves on X_{et}. Remark also that to prove the theorem it is enough to prove that $\mathcal{F} = b^*(\mathcal{C}^0(F))$ is soft, by the inductive definition of the Godement resolution. This is done by checking explicitly the conditions (1) and (2) for such an $\mathcal{F}$. We omit the computations. $\square$

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