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AMALGAMATED PRODUCTS AND THE HOWSON PROPERTY

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ABSTRACT. We show that if A is a torsion-free word hyperbolic group which belongs to class (Q), that is all finitely generated subgroups of A are quasiconvex in A, then any maximal cyclic subgroup U of A is a Burns subgroup of A. This, in particular, implies that if B is a Howson group (that is the intersection of any two finitely generated subgroups is finitely generated) then $A *_U B$, $\langle A, t \mid U^t = V\rangle$ are also Howson groups. Finitely generated free groups, fundamental groups of closed hyperbolic surfaces and some interesting 3-manifold groups are known to belong to class (Q) and our theorem applies to them. We also describe a large class of word hyperbolic groups which are not Howson.

0. **Introduction.** Recall that a group G is said to have the *Howson property*, that is G is a *Howson group*, if the intersection of any two finitely generated subgroups of G is finitely generated. In [2] B. Baumslag showed that the class of Howson groups is closed under taking free products. A. Karras and D. Solitar showed (see [12], [13]) that an amalgamated free product of two Howson groups along a finite subgroup is Howson and an HNN-extension of a Howson group over a finite subgroup is Howson. The results of R. Burns [4], [5] and D. Cohen [8] provide a set of sufficient conditions which ensure that an amalgamated free product (or an HNN-extension) of two groups with the Howson property again has the Howson property.

Namely, R. Burns and D. Cohen prove the following (see [8]).

PROPOSITION 0.1. *(a) Let A and B be Howson groups. Then $G = A *_U B$ has the Howson property if U is a Burns subgroup of A (see the definition below) and $H \cap U$ is finitely generated for any finitely generated subgroup H of G.*

(b) Let A be a Howson group and U, V be isomorphic subgroups of A.

Then the HNN-*extension $G = \langle A, t \mid U^t = V\rangle$ has the Howson property if U, V are Burns subgroup of A and $H \cap U$ is finitely generated for any finitely generated subgroup H of G.*

A subgroup U of a group A is called a *Burns subgroup* [8], if it has a left transversal T such that $1 \in T$ and the following conditions are satisfied:

(i) there is a finite subset F of U such that

$$U(T-1) \subset TF;$$

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(ii) for any finitely generated subgroup H of G and any $a \in G$ there is a finite subset F_1 of U such that

$$aH \subset TF_1(H \cap U).$$

In [4] R. Burns showed that maximal cyclic subgroups of finitely generated free groups are Burns. Thus Proposition 0.1 obviously applies if B is a Howson group, A is a finitely generated free group and U, V are maximal cyclic subgroups of A. In particular, this implies that fuchian groups and, more generally, most one-relator groups, which arise as cyclic amalgamations or HNN-extensions of free groups, have the Howson property (see [4], [5], [6] and [8]). Finite subgroups are always Burns and so the class of Howson groups is closed under amalgamations and HNN-extensions along finite subgroups (see [4], [5], [8], [12], [13]).

The purpose of this paper is to clarify the geometric meaning of the notion of a Burns subgroup and to show that there is a large class of groups (which contains all finitely generated free groups and fundamental groups of closed hyperbolic surfaces) such that for any group in this class all maximal cyclic subgroups are Burns. Thus we push further the applicability of Proposition 0.1. Namely, we say that a group G belongs to class (Q) if G is word hyperbolic in the sense of M. Gromov and any finitely generated subgroup of G is quasiconvex in G (see definitions of word hyperbolic groups and quasiconvex subgroups in Section 1). It is not hard to see ([19]) that finitely generated free groups lie in class (Q).

The notion of quasiconvexity corresponds to geometrical finiteness for classical hyperbolic groups (see [21]). The results of G. Swarup [21] and C. Pittet [17] imply that fundamental groups of closed hyperbolic surfaces lie in class (Q). Moreover, it follows from the results of W. Thurston and G. Swarup (see [21]) that if G is a torsion-free geometrically finite Kleinian group without parabolics whose limit set is not the whole sphere S^2 then G lies in (Q). Thus class (Q) is fairly large and it seems that "probabilistically" almost all word hyperbolic groups lie in (Q) since the property of having non-quasiconvex finitely generated subgroups appears to be quite rare and abnormal. As we will see in Section 1 all groups in (Q) have the Howson property.

In this paper we prove the following

THEOREM 0.2. *Let G be a torsion-free group in (Q). Then any maximal cyclic subgroup of G is Burns in G.*

COROLLARY 0.3. *Let A be a torsion-free group in (Q) and U be a maximal cyclic subgroup of A. Let B be any Howson group. Then $G = A *_U B$ has the Howson property.*

COROLLARY 0.4. *Let A be a torsion-free group in (Q) and U, V be maximal cyclic subgroups of A. Then the* HNN-*extension $G = \langle A, t \mid U^t = V \rangle$ has the Howson property.*

Corollaries 0.3 and 0.4 follow immediately from Proposition 0.1 and Theorem 0.2.

Theorem 0.2 combined with Theorem 1 of [8] also imply the following

COROLLARY 0.5. *Let A be a torsion free group from* (Q) *and U be a maximal cyclic subgroup of A and $A \neq U$. Let $G = A *_U B$, $U \neq B$ or $G = \langle A, t \mid U^t = V \rangle$. If a finitely generated subgroup H of G contains an infinite subgroup normal in G then H has finite index in G.*

D. Moldavansky [15] proved that if G is a direct product of a free group of rank 2 and an infinite cyclic group then G is not Howson. In [6] R. Burns and A. Brunner generalized this result to show that any extension of a free group of finite rank by an infinite cyclic group does not have the Howson property. As it follows from the recent results of M. Bestvina and M. Feighn [3], there are word hyperbolic groups which arise in this way and thus are not Howson. However, the first example of a word hyperbolic group without the Howson property was provided by W. Jaco and B. Evans [11, Section V.19] whose results imply that if G is the fundamental group of a closed hyperbolic 3-manifold fibering over a circle then it does not have the Howson property. Later H. Short [19] showed how to obtain word hyperbolic groups without the Howson property using the construction of E. Rips [18] of small cancellation groups with infinite finitely generated normal subgroups of infinite index. Recall that as E. Rips proved in [18], if Q is any finitely presented group then one can construct a $C(7)$-small cancellation group G (which therefore will be word hyperbolic [20] such that there is a finitely generated subgroup K of G with the property that K is normal in G and $G/K = Q$. H. Short [19] observed that if one starts with a group Q without the Howson property then the same effect can be reproduced in G. Indeed, let X and Y be finite subsets of Q such that the group $L = \mathrm{gp}(X) \cap \mathrm{gp}(Y)$ is not finitely generated. Choose finite subsets X' and Y' of G which map onto the sets X and Y respectively. Put $L' = \mathrm{gp}(K, X') \cap \mathrm{gp}(K, Y')$. Then the image of L' in Q is equal to L and therefore L' is not finitely generated. However, K is finitely generated and therefore $\mathrm{gp}(K, X')$ and $\mathrm{gp}(K, Y')$ are also finitely generated. Thus G does not have the Howson property.

The following statement, which will be proved in Section 3, shows that one can drop the requirement that Q be non-Howson and gives a uniform proof of not being Howson for a much larger class of word hyperbolic groups (including the 3-manifold groups mentioned above).

THEOREM 0.6. *Suppose we have a short exact sequence*

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1$$

where G is a subgroup of a torsion-free word hyperbolic group G_1, K is finitely generated and infinite and Q has an element of infinite order. Then G does not have the Howson property

G. A. Swarup [21] conjectured that a finitely presented subgroup K of a word hyperbolic group G is not quasiconvex in G if and only if it has infinite index in its virtual normalizer $N_K = \{g \in G \mid |K{:}\, K \cap gKg^{-1}| < \infty, |gKg^{-1}{:}\, K \cap gKg^{-1}| < \infty\}$ and

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observed that the conjecture holds for Kleinian groups. In light of this conjecture Theorem 0.6 seems to indicate that a torsion free word hyperbolic group is Howson if and only if it belongs to class (Q).

I am grateful to my advisor Gilbert Baumslag for encouragement and to Lisa Carbone for teaching me how to draw pictures using the computer.

1. **Word hyperbolic groups and their quasiconvex subgroups.** This paper contains only a brief discussion about word hyperbolic groups and their quasiconvex subgroups. For details the reader is referred to [1], [10], [7], [9] and [21].

Let G be a group generated by a finite set X. We denote the X-length of an element $g \in G$ by $l_X(g)$. The Cayley graph of G with respect to X is denoted by $\Gamma(G,X)$ and the word metric on $\Gamma(G,X)$ is denoted by d_X. If w is a word in X, we denote the corresponding element of G by $\bar{w}$.

If Δ is a geodesic triangle in a metric space (M,d) with vertices x, y, z and geodesic sides $[x,y]$, $[x,y]$, $[y,z]$ then there are unique points p, q, r on the sides $[x,y]$, $[x,z]$ and $[y,z]$ accordingly such that $d(x,p) = d(x,q)$, $d(z,q) = d(z,r)$ and $d(y,p) = d(y,r)$. The points p,q,r are called the vertices of the *inscribed triangle* in the triangle Δ. A triangle Δ as above is called δ-*slim* if for any points a, b on $[x,y]$ and $[x,z]$ such that $d(x,a) = d(x,b) \leq d(x,p) = d(x,q)$ we have $d(a,b) \leq \delta$ and the symmetric condition holds for y and z.

A group G is called *word hyperbolic* if for some (for any) finite generating set X of G there is δ such that all geodesic triangles in $\big(\Gamma(G,X), d_X\big)$ are δ-slim.

Recall (see [1] for details) that if G is a word hyperbolic group and A is a subgroup of G, A is called *quasiconvex* in G if any of the following equivalent conditions is satisfied:

(a) for some (any) finite generating set X of G there is an $\epsilon > 0$ such that any geodesic $[a_1,a_2]$ in the Cayley graph $\Gamma(G,X)$ joining points $a_1, a_2 \in A$ is contained in the ϵ-neighborhood of A;

(b) A is finitely generated and for some (any) finite generating set Y of A and some (any) finite generating set X of G there is a linear function $F(y) = By$ such that

$$d_Y(a_1,a_2) \leq F\Big(d_X(a_1,a_2)\Big), \quad a_1, a_2 \in A.$$

We will list some of the important properties of quasiconvex subgroups of word hyperbolic groups.

PROPOSITION 1.1 (SEE [1]). *Let G be a word hyperbolic group.*

(a) If A is a quasiconvex subgroup of G then A is word hyperbolic (in particular A is finitely generated and finitely presented);

(b) if A and B are quasiconvex in G then $A \cap B$ is quasiconvex in G;

(c) if C is a cyclic or finite subgroup of G then C is quasiconvex in G;

(d) if $A \leq B \leq G$ and $|B{:}A| < \infty$ then A is quasiconvex in G if and only if B is quasiconvex in G.

(e) if $A \leq G$, $\phi \in \mathrm{Aut}(G)$ then A is quasiconvex in G if and only if $\phi(A)$ is quasiconvex in G;

(f) *if A is a quasiconvex subgroup of G, X and Y are finite generating sets of G and A then there is $\epsilon > 0$ such that for any d_Y-geodesic word W, for any d_X-geodesic word w with $\overline{w} = \overline{W}$ and for each initial segment u of w there is an initial segment U of W such that $d_X(\overline{u}, \overline{U}) \leq \epsilon$.*

A word hyperbolic group G is said to *belong* to class (Q) if all its finitely generated subgroups are quasiconvex. It is now clear from Proposition 1.1(a)(b) that such a group has the Howson property.

PROOF OF THEOREM 0.2. Amalgams of word hyperbolic groups along their quasiconvex subgroups were studied in [10], [7], [16], [3], [14] and other works. Our main reference here is the paper of G. Baumslag, S. Gersten, M. Shapiro and H. Short [7]. One of the particular advantages of their approach is that, given a word hyperbolic group G with a finite generating set X and a quasiconvex subgroup C, they construct a good left transversal for C in G.

LEMMA 2.1. *Let G be a word hyperbolic group and C be a malnormal quasiconvex subgroup of G that is for every $g \in G - C$ we have $gCg^{-1} \cap C = \{1\}$. Fix a finite generating set Z for C and a finite generating set X, containing Z, for G.*

Then there is a constant $K > 1$ such that if $g \in G - C$ is a shortest (with respect to the word metric d_X) element in the coset gC then for any $c \in C$ the element cg is at most K away (in the word metric d_X) from any shortest element h in the coset cgC.

PROOF. Let $\delta > 1$ be an integer such that all geodesic triangles in the Cayley graph $\Gamma(G, X)$ are δ-slim. Since C is quasiconvex in G, by Proposition 1.1(f) there is an integer $\epsilon > 1$ such that for any d_Z-geodesic word W over Z, for any d_X-geodesic word w representing $\overline{W}$ and for any initial segment w_1 of w there is an initial segment W_1 of W such that $d_X(\overline{w_1}, \overline{W_1}) \leq \epsilon$.

Let N be the total number of d_X-geodesic words of length at most $2\epsilon + 2\delta + 4$. Put $K = 200(N+1)(\epsilon + \delta + 100)$. Suppose g, h and c are as in Lemma 2.1 and $c_1 \in C$ is such that $cg = hc_1$. We claim that c_1 is short, namely $l_X(c_1) \leq K$.

Suppose, on the contrary, $l_X(c_1) > K$. Let u, v, y, y_1 and w be d_X-geodesic representatives of g, h, c, c_1 and cg and let Y, Y_1 be d_Z-geodesic representatives of c and c_1. Consider a geodesic quadrilateral in $\Gamma(G, X)$ with sides y, u, v and y_1 which corresponds to the relation $cg = hc_1$ (see Figure 1). The geodesic triangles yuw^{-1} and vy_1w^{-1} are δ-slim. Let the points n, t, s and p, r, q be the vertices of the their inscribed triangles as it is shown in Figure 1.

Notice that $d_X(h, q) = d_X(h, r) \leq \delta + \epsilon + 2$. Indeed, if $d_X(h, q) = d_X(h, r) > \delta + \epsilon + 2$ then there is a vertex z_r on Y_1 such that $d_X(r, z_r) \leq \epsilon + 1$. But in this case $d_X(1, z_r) \leq d_X(1, q) + \delta + \epsilon + 1 < d_X(1, q) + d_X(q, h) = d_X(1, h)$ what contradicts our assumption that h is shortest in hC.

We claim that $d_X(1, p) < d_X(1, s)$. Indeed, suppose $d_X(1, p) \geq d_X(1, s)$ as it is shown in Figure 1. But in this case $d_X(p, cg) = d_X(r, cg) = l(y_1) - d_X(r, h) > K - \delta - \epsilon - 2 > 2\delta + \epsilon + 1$ since we already know that $d_X(r, h) \leq \delta + \epsilon + 2$ and we assumed $l(y_1) > K$. There exist

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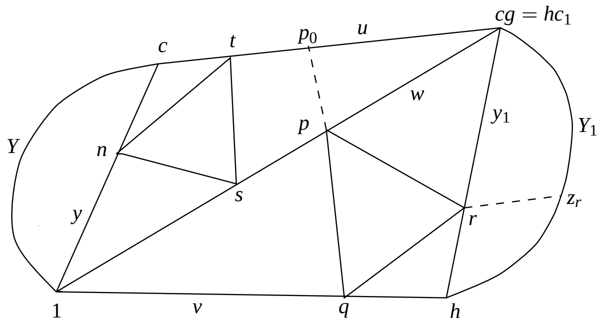

FIGURE 1

a point p_0 on u and a vertex z_r on Y_1 such that $d_X(p_0, p) \leq \delta$ and $d_X(r, z_r) \leq \epsilon + 1$ (see Figure 2). Therefore $d_X(c, z_r) \leq d_X(c, p_0) + 2\delta + \epsilon + 1 < d_X(c, p_0) + d_X(p_0, cg) = d_X(c, cg)$ what contradicts our assumptions that g is shortest in gC.

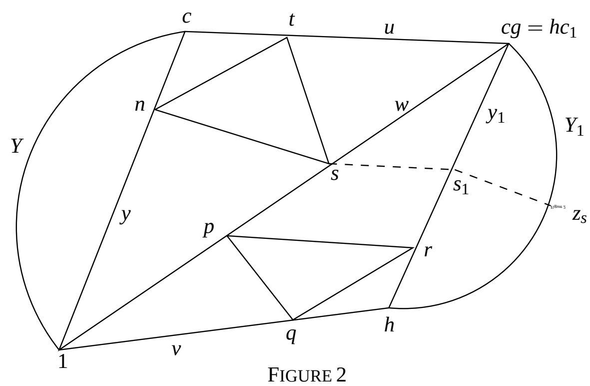

FIGURE 2

It is now easy to see that $d_X(s, cg) = d_X(t, cg) \leq 2\delta + \epsilon + 1$. Indeed, suppose that $d_X(s, cg) > 2\delta + \epsilon + 1$ and that we have a situation as shown in Figure 2. Then there is a point s_1 on y_1 and a vertex z_s on Y_1 such that $d_X(s, s_1) \leq \delta$ and $d_X(s_1, z_s) \leq \epsilon + 1$. Therefore $d_X(c, z_s) \leq d_X(c, t) + 2\delta + \epsilon + 1 < d_X(c, t) + d_X(t, cg) = d_X(c, cg)$ what contradicts the fact that g is shortest in gC.

Thus we have established that $d_X(h, q) = d_X(h, r) \leq \delta + \epsilon + 2$, $d_X(1, p) < d_X(1, s)$ and $d_X(s, cg) = d_X(t, cg) \leq 2\delta + \epsilon + 1$. This implies that $d_X(p, s) > l(y_1) - d_X(h, r) - d_X(s, cg) - 2\delta > 100(N + 1)(\epsilon + \delta + 100)$. Therefore there is a sequence of vertices $f_1, f_2, \ldots, f_{N+1}$ on w between p and s such that $d_X(f_i, f_j) = 10(\delta + \epsilon + 2)|i - j|$. Then for any $i = 1, 2, \ldots, N+1$ there is a vertex z_i on Y and a vertex d_i on Y_1 such that $d_X(f_i, z_i) \leq \delta + \epsilon + 1$ and $d_X(f_i, d_i) \leq \delta + \epsilon + 1$. Moreover, for $i \neq j$ we have $z_i \neq z_j$ and $z_i \neq d_j$ since $d_X(z_i, z_j) \geq d(f_i, f_j) - \delta - \epsilon - 1 > 0$ and $d_X(d_i, d_j) \geq d(f_i, f_j) - \delta - \epsilon - 1 > 0$. Thus

$d_X(z_i, d_i) \le 2(\delta + \epsilon + 2)$ for each i. By the choice of N there are two distinct indices $i < j$ such that $z_i^{-1}d_i = z_j^{-1}d_j = x \in G$ as it is shown in Figure 3. Then $x(d_i)^{-1}d_jx^{-1} = z_i^{-1}z_j$ and so $x \in C$ since C is malnormal in G. But in this case $g \in C$ which contradicts our assumptions. Lemma 2.1 is proved.

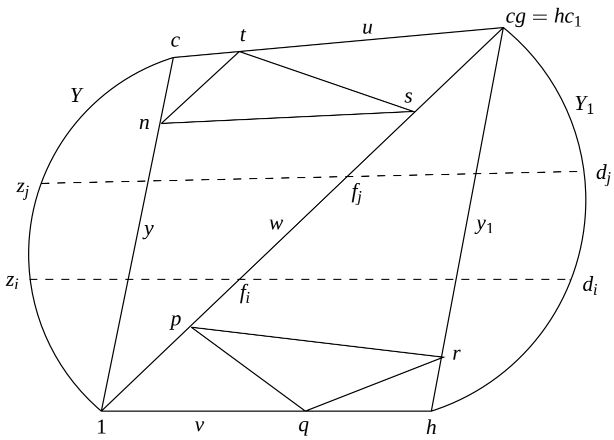

FIGURE 3

LEMMA 2.2. *Let G be a torsion-free word hyperbolic group and $C = \langle c \rangle$ be a maximal cyclic subgroup of G. Let H be a quasiconvex subgroup of G and $a \in G$. Suppose further that $H \cap C = \{1\}$. Fix a finite generating set B of H and a finite generating set X, containing c, of G. Put $Z = \{c\}$.*

There is an integer $M > 1$ such that for any $h \in H$ if $ah = gc^k$, where g is a shortest (with respect to the word metric d_X) element in ahC, then $|k| \le M$.

PROOF. We fix a d_X-geodesic representative for each element of B. This allows us to think about any word W over B as a path in the Cayley graph $\Gamma(G, X)$. For the remainder of the proof the phrase "a vertex on W" will always refer to a vertex representing an initial segment of W as a B-word. Likewise, the phrase "an initial segment of W" will always refer to an initial segment of W as a B-word.

First, observe that C is malnormal and quasiconvex in G since G is torsion-free and so the normalizer of C is cyclic (see [1], [14]). Let $\delta > 1$ be an integer such that all geodesic triangles in the Cayley graph $\Gamma(G, X)$ are δ-slim. Since C is quasiconvex in G, by Proposition 1.1(f) there is an integer $\epsilon > 1$ such that for any d_Z-geodesic word W over Z, for any d_X-geodesic word w representing $\bar{W}$ and for any initial segment w_1 of W there is an initial segment W_1 of W such that $d_X(\overline{w_1}, \overline{W_1}) \le \epsilon$. Let u_a be a d_X-geodesic representative of a. Since geodesic triangles in $\Gamma(G, X)$ are δ-slim, for any d_X-geodesic word u and a d_X-geodesic representative w of $a\bar{u}$ for any initial segment w_1 of w there is an initial segment u_1 of u such that $d_X(\overline{w_1}, a\overline{u_1}) \le 2l(u_a) + \delta$.

Also, because H is quasiconvex in G, Proposition 1.1(f) implies that there is an integer $\epsilon_1 > 1$ such that for any d_B-geodesic word W over B, for any d_X-geodesic word w

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representing $\overline{W}$ and for any initial segment w_1 of w there is an initial segment W_1 of W such that $d_X(\overline{w_1}, \overline{W_1}) \le \epsilon_1$.

Put N to be the total number of d_X-geodesic words of length at most $2\delta+\epsilon_1+2l(u_a)+\epsilon+2$. The subgroup C is quasiconvex in G (Proposition 1.1(c)) and so there is an integer $K_1 > 1$ such that for any integer n

$$|n| \le K_1 l_X(c^n).$$

Put $M = 1000(N+1)K_1\big(\delta + \epsilon + \epsilon_1 + l(u_a) + 10\big)$. Suppose $h \in H$ and $ah = gc^k$ where $|k| > M$ and g is shortest in ahC. We assume that $k > 0$ and it will be seen from the proof that the case $k < 0$ is completely analogous. Let u, v, y, w be d_X-geodesic representatives of h, g, c^k and $ah = gc^k$. Let $Y = c^k$ be a d_Z-geodesic representative of c^k and let U be a d_B-geodesic representative of h. Consider now a geodesic quadrilateral with the sides u_a, u, v and y as shown in Figure 4. Denote by p, r, q the vertices of the inscribed triangle in the geodesic triangle with the sides w, y and v. As in the proof of Lemma 2.1 we conclude that $d_X(q,g) = d_X(r,g) \le \delta + \epsilon + 2$ because otherwise g is not shortest in gC. Since $|k| > M$, $d_X(r, gc) > 50(N+1)(\delta+\epsilon+\epsilon_1+l(u_a)+10)$, and so there is a sequence of vertices $f_1, \ldots, f_{N+1}$ on y between r and gc such that

$$d_X(f_i, f_j) = 10(\delta + \epsilon + \epsilon_1 + l(u_a) + 10)|i - j|.$$

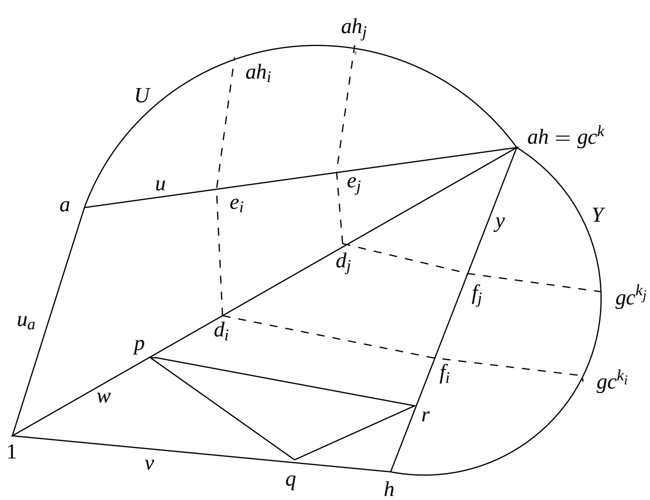

FIGURE 4

Then for each $i = 1, 2, \ldots, N+1$ there is a point d_i on w with $d_X(f_i, d_i) \le \delta$. Moreover, as we noticed before, for each $i = 1, 2, \ldots, N+1$ there is a point e_i on u such that $d_X(d_i, e_i) \le 2l(u_a) + \delta$. Furthermore, for each $i = 1, 2, \ldots, N+1$ there is a vertex ah_i on U, $h_i \in H$, such that $d_X(e_i, ah_i) \le \epsilon_1 + 1$. Thus $d_X(f_i, ah_i) \le 2\delta + \epsilon_1 + 2l(u_a) + 1$. So for $i \ne j$ $h_i \ne h_j$ because in this case

$$d_X(ah_i, ah_j) \ge 10\big(\delta + \epsilon + \epsilon_1 + l(u_a) + 10\big)|i-j| - 2\big(2\delta + \epsilon_1 + 2l(u_a) + 1\big) > 0.$$

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On the other hand for each $i = 1, 2, \ldots, N+1$ there is k_i, $0 \leq k_i \leq k$ such that $d_X(f_i, gc^{k_i}) \leq \epsilon + 1$. In particular if $i \neq j$ then $d_X(gc^{k_i}, gc^{k_j}) \geq 10(\delta + \epsilon + \epsilon_1 + l(u_a) + 10)|i - j| - 2 - 2\epsilon > 0$ and so $k_i \neq k_j$. Therefore for each $i = 1, 2, \ldots, N+1$

$$d_X(ah_i, gc^{k_i}) \leq 2\delta + \epsilon_1 + 2l(u_a) + 1 + \epsilon + 1.$$

By the choice of N there are some distinct i, j, $i < j$ such that $h_i^{-1}a^{-1}gc^{k_i} = h_j^{-1}a^{-1}gc^{k_j} = x \in G$. This implies that $xc^{k_i - k_j}x^{-1} = b$ where $b = h_i^{-1}h_j \in H$, $b \neq 1$. Put $b_1 = h_j^{-1}h \in H$. Then $x = b_1 c^{k_j - k}$. We conclude that $b_1 c^{k_j - k} c^{k_i - k_j} c^{k - k_j} b_1^{-1} = b$ and so

$$c^{k_i - k_j} = b_1^{-1} b b_1.$$

This contradicts our assumption that $H \cap C = \{1\}$. Lemma 2.2 is proved.

COROLLARY 2.3. *Let G be a torsion-free word hyperbolic group and $C = \langle c \rangle$ be a maximal cyclic subgroup of G. Let a be an element of G and H be a quasiconvex subgroup of G. Let B be a finite generating set for H and X be a finite generating set for G containing c. Fix a lexicographic order on X.*

Put T to be the set of all d_X-geodesic words u such that if v is another d_X-geodesic word with the property $\overline{u}C = \overline{v}C$ then either u is shorter than v or $l(u) = l(v)$ and u is lexicographically smaller than v. (Notice that T contains the empty word e representing 1).

Then there is a finite subset C_1 of C such that

$$aH \subset \overline{T}C_1(H \cap C).$$

PROOF. If $H \cap C = \{1\}$ then put $C_1 = \{c^k \mid -M \leq k \leq M\}$ where M is a constant provided by Lemma 2.2. Then by Lemma 2.2 $aH \subset \overline{T}C_1$.

Suppose now that $H \cap C \neq \{1\}$. Then $H \cap C$ has finite index n in C. Put $C_1 = \{c^k \mid -n \leq k \leq n\}$. Thus, obviously,

$$aH \subset \overline{T}C_1(C \cap H).$$

This completes the proof of Corollary 2.3.

COROLLARY 2.4 (THEOREM 0.2). *Let G be a torsion free group from class (Q) and $C = \langle c \rangle$ be a maximal cyclic subgroup of G. Then C is a Burns subgroup of G.*

PROOF. This follows immediately from Corollary 2.3 and Lemma 2.1.

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2. **Groups which do not have the Howson property.**

THEOREM 3.1 (THEOREM 0.6). *Suppose we have a short exact sequence*

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1$$

where G is a subgroup of a torsion-free word hyperbolic group G_1, K is finitely generated and infinite, and Q has an element of infinite order. Then G does not have the Howson property.

PROOF. We may assume that K is a subgroup of G. Let s be an element of G which projects to an element of infinite order in Q.

Since K is an infinite subgroup of a torsion-free word hyperbolic group, it follows from [9, Chapter 8, Theorem 37] that K has an element f of infinite order. Clearly f does not commute with s since otherwise some power of s would lie in a cyclic subgroup generated by f (see [1], [14]) which contradicts our choice of s. Thus by [10, Theorem 5.3.E] there is some power $t = s^n$ of s, $n \neq 0$ and some power $k = f^m$ of f, $m \neq 0$ such that t and k generate a free subgroup of rank 2 in G. Denote it by $H = s\,\mathrm{gp}(t, k)$.

We claim that $H_0 = H \cap K$ is not finitely generated. Indeed, if $k_i = t^{-i}kt^i$, $i \neq 0$ and $k_0 = k$ then $s\,\mathrm{gp}(\{k_i \mid i \in \mathbb{Z}\} \subset H_0$. On the other hand if $w(t, k) \in H_0$ then the exponent sum of t in w is equal to 0 since no nonzero power of t lies in K and K is normal in G. Thus $w(t, k)$ can be rewritten as a word over $\{k_i \mid i \in \mathbb{Z}\}$. Therefore $H_0 = s\,\mathrm{gp}(\{k_i \mid i \in \mathbb{Z}\}$.

It is not hard to see that H_0 is actually free on the generators $\{k_i \mid i \in \mathbb{Z}\}$. Indeed, if $v(k_{-N}, \ldots, k_N)$ is a nontrivial relation between these generators then it translates into a nontrivial relation between t and k. This is impossible since H is free on t, k. Thus $H_0 = H \cap K$ is not finitely generated and therefore G does not have the Howson property.

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