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Periodic Solutions of Almost Linear Volterra Integro-dynamic Equations on Periodic Time Scales

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Abstract. Using Krasnoselskii's fixed point theorem, we deduce the existence of periodic solutions of nonlinear system of integro-dynamic equations on periodic time scales. These equations are studied under a set of assumptions on the functions involved in the equations. The equations will be called almost linear when these assumptions hold. The results of this paper are new for the continuous and discrete time scales.

1 Introduction and Preliminaries

Consider the nonlinear, infinite delay, Volterra integro-dynamic equation on time scales

$$x^{\Delta}(t) = a(t)h(x(t)) + \int_{-\infty}^{t} C(t,s)g(x(s))\Delta s + p(t), \quad t \in (-\infty, \infty). \tag{1.1}$$

We assume that the functions h, a, p, and g are continuous and that there exist constants H, G and positive constants H^*, G^* such that

$$|h(x) - Hx| \le H^*, \tag{1.2}$$

and

$$|g(x) - Gx| \le G^*. \tag{1.3}$$

Equation (1.1) will be called *almost linear* if (1.2) and (1.3) hold. Existence of periodic solutions of Volterra-type nonlinear integro-differential and summation equations has been intensively investigated in the literature (see [5, 9] and references therein). In recent years, time scales (closed nonempty subset of the real numbers $\mathbb{R}$) and time scale versions of well-known equations have gained much attention (*e.g.*, [3, 4, 10, 11, 14]) since the introduction of the new derivative concept by S. Hilger. This derivative (called Δ-derivative) gives the ordinary derivative if the time scale (denoted $\mathbb{T}$) is the set of reals $\mathbb{R}$, and the forward difference operator if $\mathbb{T} = \mathbb{Z}$. Thus, the need for obtaining separate results for discrete and continuous cases is avoided by unifying them under the umbrella of time scale calculus. For a comprehensive review of this topic we direct the reader to the monograph [4]. Since there are many

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time scales other than $\mathbb{R}$ and $\mathbb{Z}$, the investigation of dynamic equations on time scales yields a general theory. Among time scales, periodic ones deserve special interest, since they enable researchers to develop a theory for the existence of periodic solutions of dynamic equations on time scales (see for example [10, 11]). This paper combines the known continuous and discrete cases with many other time scales that are periodic. Thus the results of this paper are new for the continuous and discrete cases.

In [1], the authors used the notion of degree theory in combination with Lyapunov functionals and showed the existence of periodic solutions of system of integro-dynamic equations on time scales without the requirement of (1.2) and (1.3).

For more on integro-differential equations, we refer to [6–8, 12, 13, 17–19].

For clarity, we restate the following definitions and introductory examples, which can be found in [10]. Also, for the sake of brevity, we assume familiarity with the basic properties of Δ-derivatives and Δ-integrals. For further details consult [4].

Definition 1.1 A time scale $\mathbb{T}$ is said to be *periodic* if there exists a $P > 0$ such that $t \pm P \in \mathbb{T}$ for all $t \in \mathbb{T}$. If $\mathbb{T} \neq \mathbb{R}$, the smallest positive P is called the *period* of the time scale.

Example 1.2 The following time scales are periodic:

(i) $\mathbb{T} = \mathbb{Z}$ has period $P = 1$,
(ii) $\mathbb{T} = h\mathbb{Z}$ has period $P = h$,
(iii) $\mathbb{T} = \mathbb{R}$,
(iv) $\mathbb{T} = \bigcup_{i=-\infty}^{\infty} [(2i-1)h, 2ih]$, $h > 0$ has period $P = 2h$,
(v) $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z}, m \in \mathbb{N}_0\}$, where $0 < q < 1$ has period $P = 1$.

Remark 1.3 All periodic time scales are unbounded above and below.

Definition 1.4 Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period P. We say that the function $f\colon \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = nP$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.

Define the *forward jump operator* σ by

$$\sigma(t) = \inf\{s > t : s \in \mathbb{T}\}$$

and the *graininess function* μ by $\mu(t) = \sigma(t) - t$. A point t of a time scale is called *right scattered* if $\sigma(t) > t$. Hereafter, we denote by x^σ the composite function $x \circ \sigma$.

Remark 1.5 If $\mathbb{T}$ is a periodic time scale with period P, then $\sigma(t \pm nP) = \sigma(t) \pm nP$. Consequently, the graininess function μ satisfies $\mu(t \pm nP) = \sigma(t \pm nP) - (t \pm nP) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period P.

Let $\mathbb{T}$ be a periodic time scale with period P. Let $T > 0$ be fixed, and if $\mathbb{T} \neq \mathbb{R}$, then $T = nP$ for some $n \in \mathbb{N}$. In the following, we present some preliminary material regarding the exponential function on time scales that we will need through the remainder of the paper.

Definition 1.6 A function $h\colon \mathbb{T} \to \mathbb{R}$ is said to be *regressive* provided that $1 + \mu(t)h(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$. The set of all regressive rd-continuous functions $h\colon \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$, while the set $\mathcal{R}^+$ is given by

$$\mathcal{R}^+ = \{h \in \mathcal{R} : 1 + \mu(t)h(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

Let $h \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The *exponential function* on $\mathbb{T}$ is defined by

$$e_h(t,s) = \exp\left(\int_s^t \frac{1}{\mu(z)} \operatorname{Log}(1 + \mu(z)h(z))\,\Delta z\right),$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t,s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y^\Delta = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma.

Lemma 1.7 ([4, Theorem 2.36]) *Let $p, q \in \mathcal{R}$.*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;
(ii) $e_p(\sigma(t),s) = (1 + \mu(t)p(t))e_p(t,s)$;
(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$, *where* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;
(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;
(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(vi) $\left(\frac{1}{e_p(\cdot,s)}\right)^\Delta = -\frac{p(t)}{e_p^\sigma(\cdot,s)}$.

2 Periodic Solutions

Let $\mathbb{T}$ be a periodic time scale with period P. Let $T > 0$ be fixed, and if $\mathbb{T} \neq \mathbb{R}$, then $T = nP$ for some $n \in \mathbb{N}$. In this section we investigate the existence of a periodic solution of (1.1) using Krasnoselskii's fixed point theorem.

We start with a statement of Krasnoselskii's fixed point theorem.

Theorem 2.1 (Krasnoselskii [17]) *Let K be a closed, convex, non-empty subset of a Banach space M. Suppose that A and B map K into M such that*

(i) $x, y \in K$, *implies* $Ax + By \in K$;
(ii) *A is continuous and AK is contained in a compact subset of M;*
(iii) *B is a contraction mapping.*

Then there exists $z \in K$ with $z = Az + Bz$.

The next lemma is essential to our next result. Its proof can be found in [11].

Lemma 2.2 *Let $x \in P_T$. Then $\|x^\sigma\|$ exists and $\|x^\sigma\| = \|x\|$.*

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Lemma 2.3 *Let $\mathbb{T}$ be a periodic time scale with the period P. Suppose that $f\colon \mathbb{T}\times\mathbb{T}^{\kappa}\to\mathbb{R}$ satisfies the assumptions of* [4, Theorem 1.117], *then*

$$\left[\int_{t-T}^{t} f(t,s)\Delta s\right]^{\Delta} = f(\sigma(t),t) - f(\sigma(t),t-T) + \int_{t-T}^{t} f^{\Delta}(t,s)\Delta s,$$

where $T = n_0 P$; $n_0\in\mathbb{N}$ is a positive constant.

Proof

Case 1: If $\sigma(t-T)=t$, then we have $\mu(t)=T=\mu(t-T)$, and therefore from [4, Theorem 1.117], we have

$$\begin{aligned}\int_{t-T}^{t} f^{\Delta}(t,s)\Delta s &= \frac{1}{T}\int_{t-T}^{\sigma(t)-T}\mu(t)f^{\Delta}(t,s)\Delta s\\ &= \frac{1}{T}\int_{t-T}^{\sigma(t)-T}\big[f(\sigma(t),s)-f(t,s)\big]\Delta s\\ &= \frac{\mu(t-T)}{T}\big[f(\sigma(t),t-T)-f(t,t-T)\big]\\ &= f(\sigma(t),t-T)-f(t,t-T).\end{aligned}$$

On the other hand, we find

$$\int_{t-T}^{t} f(t,s)\Delta s = \mu(t-T)f(t,t-T) = Tf(t,t-T),$$

and therefore,

$$\begin{aligned}\left[\int_{t-T}^{t} f(t,s)\Delta s\right]^{\Delta} &= Tf^{\Delta}(t,t-T) = \mu(t)f^{\Delta}(t,t-T)\\ &= f(\sigma(t),\sigma(t-T)) - f(t,t-T)\\ &= f(\sigma(t),t) - f(\sigma(t),t-T) + f(\sigma(t),t-T) - f(t,t-T)\\ &= f(\sigma(t),t) - f(\sigma(t),t-T) + \int_{t-T}^{t} f^{\Delta}(t,s)\Delta s.\end{aligned}$$

Case 2: Let $\sigma(t-T)\neq t$. Then $\sigma(t-T)$ should be less than t, since $T>0$. Hence there exists a number T_0 between $t-T$ and t such that$t-T<T_0<t$. Thus,

$$\int_{t-T}^{t} f(t,s)\Delta s = \int_{t-T}^{T_0} f(t,s)\Delta s + \int_{T_0}^{t} f(t,s)\Delta s.$$

The proof is completed by making use of [4, Theorem 1.117]. ∎

In this section we assume that for all $(t,s)\in\mathbb{T}\times\mathbb{T}$,

$$\sup_{t\in\mathbb{T}}\int_{-\infty}^{t}|C(t,s)|\,\Delta s<\infty. \tag{2.1}$$

We assume $a\in\mathcal{R}^{+}$. This implies that $e_{\ominus(\mathrm{Ha})}(t,t-T)<1$. Suppose there exists a constant $T>0$ such that for $t\in\mathbb{T}$, we have

$$a(t+T)=a(t),\quad p(t+T)=p(t),\quad C(t+T,s+T)=C(t,s). \tag{2.2}$$

Let M be the complete metric space of all continuous T-periodic functions $\phi\colon(-\infty,\infty)\to(-\infty,\infty)$ with the supremum metric. Then for any positive constant m the set

$$P_T=\{f\in M:\|f\|\le m\} \tag{2.3}$$

is a closed convex subset of M.

Lemma 2.4 *If $x\in P_T$, then x is a solution of equation (1.1) if and only if*

$$x(t)=[1-e_{\ominus(\mathrm{Ha})}(t,t-T)]^{-1}\int_{t-T}^{t}\big[\,\mathrm{Ha}(u)x^{\sigma}(u)+a(u)h(x(u))+k(u)\big]\,e_{\ominus(\mathrm{Ha})}(t,u)\Delta u,$$

where

$$k(t)=p(t)+\int_{-\infty}^{t}C(t,s)\big[g(x(s))-Gx(s)\big]\ \Delta s+\int_{-\infty}^{t}C(t,s)Gx(s)\Delta s.$$

Proof For convenience we put (1.1) in the form

$$\begin{aligned}x^{\Delta}(t)+\mathrm{Ha}(t)x^{\sigma}(t)&=\mathrm{Ha}(t)x^{\sigma}(t)+a(t)h(x(t))+p(t)\\&\quad+\int_{-\infty}^{t}C(t,s)\big[g(x(s))-Gx(s)\big]\,\Delta s+\int_{-\infty}^{t}C(t,s)Gx(s)\Delta s.\end{aligned} \tag{2.4}$$

Let

$$k(t)=p(t)+\int_{-\infty}^{t}C(t,s)\big[g(x(s))-Gx(s)\big]\ \Delta s+\int_{-\infty}^{t}C(t,s)Gx(s)\Delta s.$$

Then we may write (2.4) as

$$x^{\Delta}(t)+\mathrm{Ha}(t)x^{\sigma}(t)=\mathrm{Ha}(t)x^{\sigma}(t)+a(t)h(x(t))+k(t). \tag{2.5}$$

Let $x(t)\in P_T$ and assume (2.2). Multiply both sides of (2.5) by $e_{\mathrm{Ha}}(t,0)$ and then integrate both sides from $t-T$ to t to obtain

$$\begin{aligned}&e_{\mathrm{Ha}}(t,0)x(t)-e_{\mathrm{Ha}}(t-T,0)x(t-T)=\\&\qquad\int_{t-T}^{t}\big[\,\mathrm{Ha}(u)x^{\sigma}(u)+a(u)h(x(u))+k(u)\big]\,e_{\mathrm{Ha}}(u,0)\Delta u.\end{aligned}$$

Divide both sides of the above equation by $e_{\mathrm{Ha}}(t,0)$ and use the fact that $x(t-T)=x(t)$ to obtain

$$\begin{aligned}&x(t)[1-e_{\ominus(\mathrm{Ha})}(t,t-T)]=\\&\qquad\int_{t-T}^{t}\big[\,\mathrm{Ha}(u)x^{\sigma}(u)+a(u)h(x(u))+k(u)\big]\,e_{\ominus(\mathrm{Ha})}(t,u)\Delta u,\end{aligned}$$

where we have used Lemma 1.7 to simplify the exponentials.

Since every step is reversible and by using Lemma 2.3, the converse holds. ∎

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Define mappings A and B from P_T into M as follows. For $\phi \in P_T$,

$$(A\phi)(t) = [1 - e_{\ominus(\mathrm{Ha})}(t, t-T)]^{-1}\Big\{ \int_{t-T}^{t} a(u)\big[h(\phi(u)) + H\phi^{\sigma}(u)\big]\, e_{\ominus(\mathrm{Ha})}(t,u)\Delta u + \int_{t-T}^{t}\int_{-\infty}^{u} C(u,s)\big[g(\phi(s)) - G\phi(s)\big]\ \Delta s\, e_{\ominus(\mathrm{Ha})}(t,u)\Delta u \Big\},$$

and for $\psi \in P_T$

$$(B\psi)(t) = [1 - e_{\ominus(\mathrm{Ha})}(t, t-T)]^{-1}\Big\{ \int_{t-T}^{t}\int_{-\infty}^{u} C(u,s)G\psi(s)\,\Delta s\, e_{\ominus(\mathrm{Ha})}(t,u)\Delta u + \int_{t-T}^{t} p(u) e_{\ominus(\mathrm{Ha})}(t,u)\Delta u \Big\}.$$

It can be easily verified that both $(A\phi)(t)$ and $(B\psi)(t)$ are T-periodic and continuous in t. Assume

(2.6)

$$\sup_{t\in\mathbb{T}} \Big| [1 - e_{\ominus(\mathrm{Ha})}(t, t-T)]^{-1} \Big| \int_{t-T}^{t}\int_{-\infty}^{u} |C(u,s)| G^{*} |\psi(s)|\,\Delta s\, e_{\ominus(\mathrm{Ha})}(t,u)\Delta u \le \alpha < 1,$$

and

$$\sup_{t\in\mathbb{T}} \Big| [1 - e_{\ominus(\mathrm{Ha})}(t, t-T)]^{-1} \Big| \Big\{ \int_{t-T}^{t} |a(u)| H^{*} e_{\ominus(\mathrm{Ha})}(t,u)\Delta u + \int_{t-T}^{t}\int_{-\infty}^{u} G^{*}|C(u,s)|\,\Delta s\, e_{\ominus(\mathrm{Ha})}(t,u)\Delta u \Big\} \le \beta < \infty. \tag{2.7}$$

Choose the constant m of (2.3) satisfying

$$\sup_{t\in\mathbb{T}} \Big| [1 - e_{\ominus(\mathrm{Ha})}(t, t-T)]^{-1} \Big| \int_{t-T}^{t} |p(u)| e_{\ominus(\mathrm{Ha})}(t,u)\Delta u + \alpha m + \beta \le m. \tag{2.8}$$

Lemma 2.5 *Assume (2.2), (2.6), and (2.8). Then map B is a contraction from P_T into P_T.*

Proof For $\phi \in P_T$,

$$\begin{aligned} |(B\phi)(t)| &\le m \Big| [1 - e_{\ominus(\mathrm{Ha})}(t, t-T)]^{-1} \Big| \int_{t-T}^{t}\int_{-\infty}^{u} |C(u,s)| G\, ds\ e_{\ominus(\mathrm{Ha})}(t,u)\Delta u \\ &\quad + \Big| 1 - e_{\ominus(\mathrm{Ha})}(t, t-T)]^{-1} \Big| \int_{t-T}^{t} |p(u)| e_{\ominus(\mathrm{Ha})}(t,u)\Delta u \\ &\le \sup_{t\in\mathbb{T}} \Big| 1 - e_{\ominus(\mathrm{Ha})}(t, t-T)]^{-1} \Big| \int_{t-T}^{t} |p(u)| e_{\ominus(\mathrm{Ha})}(t,u)\Delta u + \alpha m < m. \end{aligned}$$

For $\phi, \psi \in P_T$, we obtain

$$|(B\phi)(t) - (B\psi)(t)| \le \alpha \|\phi - \psi\|$$

using (2.6). This proves that B is a contraction mapping from P_T into P_T. ∎

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Lemma 2.6 *Assume (1.1), (1.2), (2.1), (2.2), (2.7), and (2.8). Then map A from P_T into P_T is continuous, and AP_T is contained in a compact subset of M.*

Proof For any $\phi \in P_T$, it follows from (2.7) and (2.8) that

$$|(A\phi)(t)| \le \beta \le m. \tag{2.9}$$

So, A maps P_T into P_T, and the set $\{A\phi\}$ for $\phi \in P_T$ is uniformly bounded. To show that A is a continuous map we let $\{\phi_n\}$ be any sequence of functions in P_T with $\|\phi_n - \phi\| \to 0$ as $n \to \infty$. Then one can easily verify that

$$\|A\phi_n - A\phi\| \to 0 \text{ as } n \to \infty.$$

This proves that A is a continuous mapping. ■

It is trivial to show that $|(A\phi)^{\Delta}(t)|$ is bounded. This would show that the set $\{A\phi\}$ for $\phi \in P_T$ is equicontinuous, by using (2.9). Therefore, by the Arzela–Ascoli Theorem, AP_T is contained in a compact subset of M.

We are now ready to use Krasnoselskii's fixed point theorem to show the existence of a continuous T-periodic solution of (1.1).

Theorem 2.7 *Suppose the assumptions of Lemmas 2.5 and 2.6 hold. Then (1.1) has a continuous T-periodic solution.*

Proof From $\phi, \psi \in P_T$, we get

$$\begin{aligned}
&|(A\phi)(t) + (B\psi)(t)| \\
&\quad \le \sup_{t\ge 0} \big| 1 - e_{\ominus(\mathrm{Ha})}(t, t-T)]^{-1} \big| \int_{t-T}^{t} |p(u)| e_{\ominus(\mathrm{Ha})}(t,u)\Delta u + \alpha m + \beta \\
&\quad \le m,
\end{aligned}$$

which proves that $A\phi + B\psi \in P_T$.

Therefore, by Krasnoselskii's theorem there exists a function $x(t)$ in P_T such that

$$x(t) = Ax(t) + Bx(t).$$

This proves that (1.1) has a continuous T-periodic solution $x(t)$. ■

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