Appendix H
Pion electroproduction

Much can be said about the amplitude for pion electroproduction from the nucleon, N(e, e′ π)N, on general grounds. This is the first inelastic process one encounters in scattering electrons from protons or neutrons. The development in this appendix follows [Fu58, Wa68, Pr69, Wa84]. The pioneering work on the *photoproduction* process was carried out by CGLN [Ch57]. Other important early references on pion electroproduction include [De61, Za66, Vi67, Ad68, Pr70].

The kinematic situation is shown in Fig. 13.1; here particle X is now a pion. The laboratory cross section is given in terms of the covariant matrix elements of the current in Eq. (13.41) by Eq. (13.47). The angular distribution of the pions in the C-M system is given in terms of the helicity amplitudes by Eq. (13.68). With a transition to the L–S basis, and unobserved polarizations, the angular distribution takes the form in Eqs. (13.71, F.13). Here for the nucleon $J^\pi = 1/2^+$ and for pseudoscalar pions $\eta = \eta_1\eta_2\eta_X = -1$.

From Lorentz invariance, the S-matrix for the process N(e, e′ π)N in the one-photon-exchange approximation can be written as

$$\begin{aligned} S_{fi} &= -\frac{(2\pi)^4}{\Omega} i\delta^{(4)}(k_1 + p_1 - k_2 - p_2 - q)\left(\frac{m^2}{2\omega_q E_1 E_2 \Omega^3}\right)^{1/2} T_{fi} \\ T_{fi} &= 4\pi\alpha[i\bar{u}(k_2)\gamma_\mu u(k_1)]\frac{1}{k^2}J_\mu \\ J_\mu &= \left(\frac{2\omega_q E_1 E_2 \Omega^3}{m^2}\right)^{1/2} \langle q p_2^{(-)}|J_\mu|p_1\rangle \end{aligned} \tag{H.1}$$

Assume one has a theory for the pion–nucleon interaction with a set of Feynman diagrams and Feynman rules so that an expression for T_{fi} is at

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hand. Define the Møller potential by

$$\varepsilon_\mu \equiv \bar{u}(k_2)\gamma_\mu u(k_1)\frac{1}{k^2} \tag{H.2}$$

where, as before, $k \equiv k_1 - k_2$. The quantity $\varepsilon_\mu J_\mu$ is then a Lorentz scalar.[1] Conservation of the electromagnetic current states that the amplitude must vanish under the replacement $\varepsilon_\mu \to k_\mu$

$$k_\mu J_\mu = 0 \tag{H.3}$$

With the aid of the Dirac equation and current conservation, the transition amplitude can always be reduced to the following form

$$\varepsilon_\mu J_\mu = \bar{u}(p_2)\left[\sum_{i=1}^{6} a_i(W,\Delta^2,k^2)\,\varepsilon_\mu M_\mu^{(i)}\right]u(p_1) \tag{H.4}$$

The Dirac spinors for the nucleon are now normalized to $\bar{u}u = 1$. The four-momentum transfer to the nucleon used here, and mean four-momentum used below, are defined by

$$\begin{aligned}\Delta &\equiv \frac{1}{2}(k-q)\\ P &\equiv \frac{1}{2}(p_1+p_2)\end{aligned} \tag{H.5}$$

There are six independent kinematic invariants, and they can be taken to be [Fu58]

$$\begin{aligned}M_A &= \frac{1}{2}i\gamma_5\left[\not{\varepsilon}\,\not{k} - \not{k}\,\not{\varepsilon}\right]\\ M_B &= 2i\gamma_5\left[(P\cdot\varepsilon)(q\cdot k) - (P\cdot k)(q\cdot\varepsilon)\right]\\ M_C &= \gamma_5\left[\not{\varepsilon}\,(q\cdot k) - \not{k}\,(q\cdot\varepsilon)\right]\\ M_D &= 2\gamma_5\left[\not{\varepsilon}\,(P\cdot k) - \not{k}\,(P\cdot\varepsilon)\right] - im\gamma_5\left[\not{\varepsilon}\,\not{k} - \not{k}\,\not{\varepsilon}\right]\\ M_E &= i\gamma_5\left[(k\cdot\varepsilon)(q\cdot k) - (q\cdot\varepsilon)\,k^2\right]\\ M_f &= \gamma_5\left[\not{k}\,(k\cdot\varepsilon) - \not{\varepsilon}\,k^2\right]\end{aligned} \tag{H.6}$$

Here the Feynman notation $\not{v} \equiv \gamma_\mu v_\mu$ is employed. Current conservation is evidently satisfied since the replacement $\varepsilon \to k$ causes each invariant to vanish identically.[2] Furthermore, in photoproduction, the last two invariants are absent since $k^2 = k\cdot\varepsilon = 0$ in that case [Ch57].

[1] Strictly speaking one must renormalize the electron wave functions with a factor $(E/m_e)^{1/2}$ so that $\bar{u}u = 1$ for this to be true [Bj65]; however, since all subsequent expressions in this appendix are linear in ε (and we know how to get the correct cross section) the overall normalization of ε here plays no role.

[2] Recall $\not{v}\not{v} = v^2$.

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Without loss of generality, one can reduce the transition amplitude to an expression taken between two-component Pauli spinors by substituting the explicit form of the Dirac spinors introduced previously in Eq. (19.9) and now normalized to $\bar{u}u = 1$

$$u(\mathbf{p}, s) = \left(\frac{E_p + m}{2m}\right)^{1/2} \begin{pmatrix} \eta_s \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E_p + m}\eta_s \end{pmatrix} \tag{H.7}$$

Here $\eta_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\eta_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represent spin up and down along the z-axis, taken to be the direction of the incident nucleon in the C-M system as in Fig. 13.3. Substitution of Eq. (H.7) in Eq. (H.6) and explicit evaluation of the Dirac matrix products leads to the following equivalent, but still exact, expression for the spatial part of the transition matrix element expressed in term of Pauli matrices in the C-M system

$$\begin{aligned}
\hat{\boldsymbol{\varepsilon}} \cdot \mathbf{J} &= \eta_{s_2}^\dagger \left[\sum_{i=1}^{6} G_i(W, \Delta^2, k^2) m_i\right] \eta_{s_1} \\
m_1 &= i\boldsymbol{\sigma} \cdot \hat{\boldsymbol{\varepsilon}} \\
m_2 &= \boldsymbol{\sigma} \cdot \hat{\mathbf{q}} \left[\boldsymbol{\sigma} \cdot (\hat{\mathbf{k}} \times \hat{\boldsymbol{\varepsilon}})\right] \\
m_3 &= i\boldsymbol{\sigma} \cdot \hat{\mathbf{k}}\,(\hat{\mathbf{q}} \cdot \hat{\boldsymbol{\varepsilon}}) \qquad\qquad m_5 = i\boldsymbol{\sigma} \cdot \hat{\mathbf{q}}\,(\hat{\mathbf{k}} \cdot \hat{\boldsymbol{\varepsilon}}) \\
m_4 &= i\boldsymbol{\sigma} \cdot \hat{\mathbf{q}}\,(\hat{\mathbf{q}} \cdot \hat{\boldsymbol{\varepsilon}}) \qquad\qquad m_6 = i\boldsymbol{\sigma} \cdot \hat{\mathbf{k}}\,(\hat{\mathbf{k}} \cdot \hat{\boldsymbol{\varepsilon}})
\end{aligned} \tag{H.8}$$

In this expression $\hat{\mathbf{v}}$ denotes a unit vector. The linear relations between the amplitudes a_i referred to as $\{A, B, \ldots, E\}$ and the G_i is given by [Wa68, Wa84]

$$\begin{aligned}
G_1 &= \left[\frac{(E_1 + m)(E_2 + m)}{4m^2}\right]^{1/2} (W - m) \\
&\quad \times \left[A + (W - m)D - \frac{k \cdot q}{W - m}(C - D) + \frac{k^2}{W - m}F\right] \\
G_2 &= \frac{|\mathbf{q}|\, k^\star (W + m)}{[4m^2(E_1 + m)(E_2 + m)]^{1/2}} \\
&\quad \times \left[-A + (W + m)D - \frac{k \cdot q}{W + m}(C - D) + \frac{k^2}{W + m}F\right] \\
G_3 &= |\mathbf{q}|\, k^\star (W + m) \left[\frac{E_2 + m}{4m^2(E_1 + m)}\right]^{1/2} \\
&\quad \times \left[C - D + (W - m)B - \frac{k^2}{W + m}E\right]
\end{aligned}$$

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$$
\begin{aligned}
G_4 &= \mathbf{q}^2 (W-m) \left[\frac{E_1+m}{4m^2(E_2+m)}\right]^{1/2} \\
&\quad \times \left[C - D - (W+m)B + \frac{k^2}{W-m}E\right] \\
G_5 &= \frac{|\mathbf{q}|\, k^\star}{[4m^2(E_1+m)(E_2+m)]^{1/2}} \\
&\quad \times \{k_0\,[-A + (W+m)(D-F) - k\cdot q(B-E)] \\
&\quad -k\cdot q\,[C - D - (W+m)(B-E)]\} \\
G_6 &= k^{\star 2} \left[\frac{E_2+m}{4m^2(E_1+m)}\right]^{1/2} \\
&\quad \times [-A + k\cdot q(E-B) - (W+m)F - (W-m)D]
\end{aligned} \tag{H.9}
$$

The Coulomb matrix element can be obtained from these results by current conservation

$$
\langle q p_2^{(-)}|\mathbf{J}\cdot\hat{\mathbf{k}}|p_1\rangle = \left(\frac{k_0}{k^\star}\right) \langle q p_2^{(-)}|\rho|p_1\rangle \tag{H.10}
$$

If the Coulomb matrix element is evaluated directly, the result is

$$
\begin{aligned}
\left(\frac{2\omega_q E_1 E_2 \Omega^3}{m^2}\right)^{1/2} \langle q p_2^{(-)}|(-1)J_0\varepsilon_0|p_1\rangle &= \eta_{s_2}^\dagger [m_7 G_7 + m_8 G_8]\eta_{s_1} \\
m_7 &= -i\varepsilon_0\, \boldsymbol{\sigma}\cdot\hat{\mathbf{q}} \\
m_8 &= -i\varepsilon_0\, \boldsymbol{\sigma}\cdot\hat{\mathbf{k}}
\end{aligned} \tag{H.11}
$$

Equation (H.10) allows the identification

$$
\begin{aligned}
G_7 &= \frac{k^\star}{k_0}[G_5 + (\hat{\mathbf{k}}\cdot\hat{\mathbf{q}})\,G_4] \\
G_8 &= \frac{k^\star}{k_0}[G_1 + (\hat{\mathbf{k}}\cdot\hat{\mathbf{q}})\,G_3 + G_6]
\end{aligned} \tag{H.12}
$$

It is convenient to take out the same overall factor as in Eq. (13.41), and one defines new transition amplitudes by

$$
\mathscr{J}_i \equiv \frac{m}{4\pi W} G_i \qquad\qquad i = 1,\dots,8 \tag{H.13}
$$

It then follows from Eq. (H.8) that

$$
\hat{\boldsymbol{\varepsilon}}\cdot\mathscr{J} = \eta_{s_2}^\dagger \left[\sum_{i=1}^{6} \mathscr{J}_i(W,\Delta^2,k^2)m_i\right] \eta_{s_1} \tag{H.14}
$$

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To carry out a multipole analysis of the transition amplitude of the current, the covariant transition matrix element of the current is expanded according to Eq. (13.58)

$$\eta^{\dagger}_{\lambda_2}\left(\sum_{i=1}^{6} m_i \mathscr{I}_i\right)\eta_{s_1} = \tag{H.15}$$
$$\frac{1}{(4k^{\star}q)^{1/2}}\sum_J (2J+1)\mathscr{D}^{J}_{\lambda_1-\lambda_k,\lambda_2}(-\phi_p,-\theta_p,\phi_p)^{\star}\,\langle\lambda_2|T^J(W,k^2)|\lambda_1\,\lambda_k\rangle$$

Here $\lambda_1(\equiv s_1)$ and λ_2 are the initial and final nucleon helicities, and λ_k is the virtual photon helicity. The C-M configuration is shown in Fig. 13.3. A little study shows that the Pauli spinor $\eta^{\star}_{\lambda_2}$ can be expressed in terms of the previous spinor $\eta^{\star}_{s_2}$ (representing spin up or down along the $-\hat{k}^{\star}$ axis) by the rotation

$$\eta^{\dagger}_{\lambda_2} = \sum_{s_2}\mathscr{D}^{1/2}_{\lambda_2,s_2}(-\phi_p,\theta_p,\phi_p)\,\eta^{\dagger}_{s_2} \tag{H.16}$$

Now one has the invariant amplitude expressed in terms of helicity amplitudes. This relation can be inverted using the orthonormality properties of the rotation matrices [Ed74]. Thus, given any invariant amplitude for pion electroproduction, one has all the equivalent helicity amplitudes.

Recall the transformation coefficients to the L–S basis, which provides eigenstates of parity. For the case of the π–N, the transformation in Eq. (F.2) takes the form (again S is suppressed)

$$|JL\rangle = \sum_{\lambda_2}\sqrt{2L+1}(-1)^{1+L+\lambda_2+1/2}\begin{pmatrix} L & 1/2 & J \\ 0 & \lambda_2 & -\lambda_2\end{pmatrix}|J\lambda_2\rangle \tag{H.17}$$

Substitution of this expression in the definition of the transition amplitude in Eq. (F.6) gives

$$c(LJ\tfrac{1}{2}) = \frac{k^{\star}}{\omega^{\star}}\sum_{\lambda_2}\sqrt{2L+1}(-1)^{1+L+\lambda_2+1/2}\begin{pmatrix} L & 1/2 & J \\ 0 & \lambda_2 & -\lambda_2\end{pmatrix}\langle\lambda_2|T^J|\tfrac{1}{2},0\rangle$$
$$t(LJ\lambda_1) = \sum_{\lambda_2}\sqrt{2L+1}(-1)^{1+L+\lambda_2+1/2}\begin{pmatrix} L & 1/2 & J \\ 0 & \lambda_2 & -\lambda_2\end{pmatrix}\langle\lambda_2|T^J|\lambda_1,+1\rangle \tag{H.18}$$

In the second relation $\lambda_1 = \pm 1/2$, and the sum in both relations goes over $\lambda_2 = \pm 1/2$. Thus, once the helicity amplitudes have been obtained, the transition amplitudes into eigenstates of parity follow immediately. The angular correlation coefficients are then given by Eq. (F.13). The transition multipole amplitudes into states of definite parity are sometimes more conventionally defined according to

$$
\begin{aligned}
c(LJ,\frac{1}{2}) &= \pm(4k^\star q)^{1/2}\frac{1}{\sqrt{2}}\frac{k^\star}{\omega^\star}L_{l\pm} \\
t(LJ,\frac{1}{2}) &= \pm(4k^\star q)^{1/2}\frac{1}{\sqrt{2}}\,T^{l\pm}_{1/2} \\
t(LJ,-\frac{1}{2}) &= \pm(4k^\star q)^{1/2}\frac{1}{\sqrt{2}}\,T^{l\pm}_{3/2}
\end{aligned}
\tag{H.19}
$$

Here $J = L \pm 1/2$ with $L \equiv l$.

Although we now have all one needs to obtain the general angular distribution in pion electroproduction, it is useful in comparing with current analyses [Bu94] to derive an equivalent expression directly from Eq. (H.14) by taking simple (two-component) traces. The cross section is given by Eq. (13.47) where the helicity unit vectors are defined in Eq. (13.43) with $\hat{\boldsymbol{\varepsilon}}_0 = \hat{\boldsymbol{\varepsilon}}_{\mathbf{k}3}$. The result is readily shown to be

$$
\overline{|\mathscr{J}_{\mathscr{C}}|^2} = |\mathscr{J}_7|^2 + |\mathscr{J}_8|^2 + 2\mathrm{Re}\,\mathscr{J}_7^\star\mathscr{J}_8\cos\theta_q \tag{H.20}
$$

$$
\begin{aligned}
\overline{|\mathscr{J}^{+1}|^2} + \overline{|\mathscr{J}^{-1}|^2} &= 2(|\mathscr{J}_1|^2 + |\mathscr{J}_2|^2 - 2\mathrm{Re}\,\mathscr{J}_1^\star\mathscr{J}_2\,\cos\theta_q) + \sin^2\theta_q \\
&\quad\times\left(|\mathscr{J}_3|^2 + |\mathscr{J}_4|^2 + 2\mathrm{Re}\,\mathscr{J}_1^\star\mathscr{J}_4 + 2\mathrm{Re}\,\mathscr{J}_2^\star\mathscr{J}_3 + 2\mathrm{Re}\,\mathscr{J}_3^\star\mathscr{J}_4\,\cos\theta_q\right) \\
\mathrm{Im}\,\overline{\mathscr{J}_{\mathscr{C}}^{\;\star}\,[\mathscr{J}^{+1} + \mathscr{J}^{-1})} &= -(1/\sqrt{2})\sin\phi_q\sin\theta_q\;[2\mathrm{Re}\,\mathscr{J}_1^\star\mathscr{J}_7 + 2\mathrm{Re}\,\mathscr{J}_4^\star\mathscr{J}_7 \\
&\quad + 2\mathrm{Re}\,\mathscr{J}_2^\star\mathscr{J}_8 + 2\mathrm{Re}\,\mathscr{J}_3^\star\mathscr{J}_8 \\
&\quad + \cos\theta_q(2\mathrm{Re}\,\mathscr{J}_3^\star\mathscr{J}_7 + 2\mathrm{Re}\,\mathscr{J}_4^\star\mathscr{J}_8)] \\
\mathrm{Re}\,\overline{(\mathscr{J}^{+1})^\star\,(\mathscr{J}^{-1})} &= -(1/2)\cos 2\phi_q\sin^2\theta_q \\
&\quad\times\left(|\mathscr{J}_3|^2 + |\mathscr{J}_4|^2 + 2\mathrm{Re}\,\mathscr{J}_1^\star\mathscr{J}_4 + 2\mathrm{Re}\,\mathscr{J}_2^\star\mathscr{J}_3 + 2\mathrm{Re}\,\mathscr{J}_3^\star\mathscr{J}_4\,\cos\theta_q\right)
\end{aligned}
$$

The amplitudes $\mathscr{J}_i$ for $i = 1,\ldots,4$ are expressed in terms of more familiar multipole amplitudes by

$$
\begin{aligned}
\mathscr{J}_1 &= \sum_l\{[lM_{l+} + E_{l+}]P'_{l+1}(x) + [(l+1)M_{l-} + E_{l-}]P'_{l-1}(x)\} \\
\mathscr{J}_2 &= \sum_l\{[(l+1)M_{l+} + lM_{l-}]P'_l(x)\} \\
\mathscr{J}_3 &= \sum_l\{[E_{l+} - M_{l+}]P''_{l+1}(x) + [E_{l-} + M_{l-}]P''_{l-1}(x)\} \\
\mathscr{J}_4 &= \sum_l\{[M_{l+} - E_{l+} - M_{l-} - E_{l-}]P''_l(x)\}
\end{aligned}
\tag{H.21}
$$

Here $x = \cos\theta_q$ and $P'_l(x) = dP_l(x)/dx$. The notation $l\pm$ indicates that $J = l \pm 1/2$. For $k^2 \to 0$, that is the limit of photoproduction, these four

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equations reduce to those of CGLN [Ch57].[3] In pion electroproduction, the multipole amplitudes are still functions of both the energy in the C-M frame and the four-momentum transfer (W, k^2). In addition in electroproduction, there are the Coulomb multipoles

$$\mathscr{F}_7 = \frac{k^\star}{k_0}(\mathscr{F}_5 + x\mathscr{F}_4) \;=\; \sum_l \{[C_{l-} - C_{l+}]P_l'(x)\} \tag{H.22}$$

$$\mathscr{F}_8 = \frac{k^\star}{k_0}(\mathscr{F}_1 + x\mathscr{F}_3 + \mathscr{F}_6) \;=\; \sum_l \{[C_{l+}P_{l+1}'(x) - C_{l-}P_{l-1}'(x)]\} \tag{H.23}$$

Here $k_0 \equiv \omega^\star$.

These equations can be inverted to solve for the mutlipole amplitudes themselves. Define

$$\mathscr{F}_l^i(W, k^2) = \frac{1}{2}\int_{-1}^{1} P_l(x)\mathscr{F}^i(w, k^2, x)\, dx \tag{H.24}$$

Then use of the properties of the Legendre polynomials [Ed74] and a little algebra lead to

$$\begin{aligned}
lE_{l-} &= \mathscr{F}_l^1 - \mathscr{F}_{l-1}^2 + \frac{l+1}{2l+1}[\mathscr{F}_{l+1}^3 - \mathscr{F}_{l-1}^3] + \frac{l}{2l-1}[\mathscr{F}_l^4 - \mathscr{F}_{l-2}^4] \\
lM_{l-} &= -\mathscr{F}_l^1 + \mathscr{F}_{l-1}^2 - \frac{1}{2l+1}[\mathscr{F}_{l+1}^3 - \mathscr{F}_{l-1}^3] \\
(l+1)E_{l+} &= \mathscr{F}_l^1 - \mathscr{F}_{l+1}^2 - \frac{l}{2l+1}[\mathscr{F}_{l+1}^3 - \mathscr{F}_{l-1}^3] - \frac{l+1}{2l+3}[\mathscr{F}_{l+2}^4 - \mathscr{F}_l^4] \\
(l+1)M_{l+} &= \mathscr{F}_l^1 - \mathscr{F}_{l+1}^2 + \frac{1}{2l+1}[\mathscr{F}_{l+1}^3 - \mathscr{F}_{l-1}^3] \\
C_{l+} &= \mathscr{F}_{l+1}^7 + \mathscr{F}_l^8 \\
C_{l-} &= \mathscr{F}_{l-1}^7 + \mathscr{F}_l^8
\end{aligned} \tag{H.25}$$

Finally, to close the loop, we give the relations between these multipoles and the helicity amplitudes into states of definite parity defined in Eqs. (H.19)

$$\begin{aligned}
(l+1)M_{l+} &= -\frac{i}{\sqrt{2}}[T_{1/2}^{l+} + \left(\frac{l+2}{l}\right)^{1/2} T_{3/2}^{l+}] \\
(l+1)E_{l+} &= -\frac{i}{\sqrt{2}}[T_{1/2}^{l+} - \left(\frac{l}{l+2}\right)^{1/2} T_{3/2}^{l+}] \\
lM_{l-} &= -\frac{i}{\sqrt{2}}[T_{1/2}^{l-} - \left(\frac{l-1}{l+1}\right)^{1/2} T_{3/2}^{l-}] \\
lE_{l-} &= +\frac{i}{\sqrt{2}}[T_{1/2}^{l-} + \left(\frac{l+1}{l-1}\right)^{1/2} T_{3/2}^{l-}]
\end{aligned} \tag{H.26}$$

[3] Recall that $E_{1-} = M_{0+} \equiv 0$.

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The longitudinal multipoles are defined in terms of the Coulomb multipoles with the aid of current conservation

$$C_{l\pm} \equiv \frac{k^\star}{k_0} iL_{l\pm} \equiv \frac{k^\star}{k_0} N_{l\pm} \tag{H.27}$$

If only the electron is detected in an *inclusive* experiment, one must integrate over the final pion direction. Only the terms (A_0, B_0) remain in Eq. (13.71), and the result is written, with the aid of Eqs. (H.19), as

$$\int \frac{d\Omega_q}{4\pi} \overline{|\mathscr{J}_\mathscr{C}|^2} = \sum_{J^\pi} \left(J + \frac{1}{2}\right) |C_{l\pm}|^2 \tag{H.28}$$
$$\int \frac{d\Omega_q}{4\pi} \left(\overline{|\mathscr{J}^{+1}|^2} + \overline{|\mathscr{J}^{-1}|^2}\right) = \sum_{J^\pi} \left(J + \frac{1}{2}\right) \left(|T^{l\pm}_{3/2}|^2 + |T^{l\pm}_{1/2}|^2\right)$$

Consider the role of isospin in pion electroproduction. Let $\alpha = 1, 2, 3$ be the hermitian components of isospin for the produced pion. Recall that the electromagnetic current has the isospin structure

$$J^\gamma_\mu = J^S_\mu + J^{V_3}_\mu \tag{H.29}$$

Isospin invariance of the strong interactions implies that the transition matrix of the current must then have the covariant form

$$T = T^{(+)}\delta_{\alpha 3} + T^{(-)}\frac{1}{2}[\tau_\alpha, \tau_3] + T^{(0)}\tau_\alpha \tag{H.30}$$

The transition amplitudes into states of given total isospin from a proton target then follow as

$$T(\frac{3}{2}, p) = \left(\frac{2}{3}\right)^{1/2} (T^+ - T^-)$$
$$T(\frac{1}{2}, p) = -\left(\frac{1}{3}\right)^{1/2} (T^+ + 2T^- + 3T^0) \tag{H.31}$$

The relations between the multipoles presented in this appendix are all derived in detail in [Wa84]. The reader now has enough background to proceed from any covariant, gauge-invariant expression for the S-matrix in pion electroproduction in the form of Eqs. (H.1) and Eq. (H.13) to individual multipole amplitudes. The coincident angular distribution is then given by Eqs. (13.71, F.13), or by Eqs. (H.20). Simultaneously, one has all the information needed for a general phenomenological analysis of pion electroproduction in terms of contributing multipoles [Bu94].

Finally, for the transition below the two-pion threshold into a π–N state with given (J^π, T), there is a theorem due to Watson that the *phase* of

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the electroproduction amplitude is given by the strong-interaction elastic scattering phase shift in this channel [Wa52]. To understand Watson's theorem, consider a 2-channel process where the first channel $a+b \rightleftharpoons a+b$ is elastic scattering through the strong interaction in a given partial wave, the transition amplitude is weak, say of $O(e)$ as in $\gamma + a \rightleftharpoons a + b$, and the scattering in the second channel $\gamma + a \rightleftharpoons \gamma + a$ is of $O(e^2)$. Time-reversal invariance implies that the S-matrix for this process must be symmetric and unitarity implies that $\mathscr{S}^\dagger \mathscr{S} = 1$ [Ja59]. To $O(e)$, the first condition implies that the S-matrix in this channel must have the form

$$\mathscr{S} = \begin{pmatrix} e^{2i\delta} & 2it \\ 2it & 1 \end{pmatrix} \tag{H.32}$$

Explicit evaluation of the unitarity condition for this 2×2 matrix then leads to the relation

$$t = |t| e^{i\delta} \tag{H.33}$$

Thus the phase of the weak transition amplitude is that of the strong-interaction phase shift. This is Watson's theorem.

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