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OPTION PRICING UNDER THE KOBOL MODEL

WENTING CHEN[1] and SHA LIN[✉2]

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Abstract

We consider the pricing of European options under a modified Black–Scholes equation having fractional derivatives in the "spatial" (price) variable. To be specific, the underlying price is assumed to follow a geometric Koponen–Boyarchenko–Levendorski process. This pure jump Lévy process could better capture the real behaviour of market data. Despite many difficulties caused by the "globalness" of the fractional derivatives, we derive an explicit closed-form analytical solution by solving the fractional partial differential equation analytically, using the Fourier transform technique. Based on the newly derived formula, we also examine, in theory, many basic properties of the option price under the current model. On the other hand, for practical purposes, we impose a reliable implementation method for the current formula so that it can be easily used in the trading market. With the numerical results, the impact of different parameters on the option price are also investigated.

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1. Introduction

Despite the great success of the Black–Scholes (B–S) model [3] which provides a hedging argument allowing us to price financial derivatives under the risk-neutral measure, many drawbacks of this model caused by some unrealistic assumptions have been mentioned and criticized. One of the major shortcomings is that the B–S model assumes the underlying distribution to be Gaussian, which is in clear contrast to the market observation that the underlying distribution is skewed [21] and leptokurtic [22]. Over the last two decades, numerous efforts [2, 13, 15, 19] have been made to modify the distribution assumption of the B–S model. Notable approaches in this direction include one which assumes that the underlying satisfies a pure jump Lévy process such as the variance gamma process adopted by Madan et al. [17], the finite moment

[1]School of Business, Jiangnan University, Wuxi, Jiangsu Province, China; e-mail: cwtwx@163.com.
[2]School of Mathematics and Applied Statistics, University of Wollongong, NSW 2522, Australia;
e-mail: sl945@uowmail.edu.au.

log stable (FMLS) model by Carr and Wu [8], the Carr–Geman–Madan–Yor (CGMY) model proposed by Carr et al. [7] and the KoBol model introduced by Koponen [16] and Boyarchenko and Levendorski [4].

The stochastic process considered in the current paper is the so-called exponential Koponen–Boyarchenko–Levendorski (KoBol) process, which is proposed based on the fact that the truncated Lévy flights are capable of capturing the empirical probability distribution of the high frequency data, such as the S&P 500 index [18]. With five parameters controlling the essential characteristics, this process could capture the skewness and heavy tails of the underlying return, and also allow the distribution to be either symmetric or asymmetric [21, 22].

Under the KoBol model, it has been shown by Cartea and del-Castillo-Negrete [9] that the option price is governed by a fractional partial differential equation (FPDE) with two fractional derivatives appearing in the spatial direction. Financially, the introduction of the fractional derivatives allows the asset price to jump a finite amount within an infinitesimal time step. Mathematically, in comparison to the derivative with positive integer order, the determination of the so-called fractional derivative requires information of the function in a certain subset of the entire domain of definition, whereas the determination of the former only requires the function value in a small neighbourhood of a certain point. It is this feature that has made the fractional derivative extremely difficult to deal with, both numerically and analytically.

In the literature, fractional derivatives have already been widely used in the option pricing field. For instance, Cartea and del-Castillo-Negrete [9] successfully connected the FMLS, CGMY and KoBol models to FPDEs with spatial-fractional derivatives. Later, several authors considered the pricing of options under the proposed FPDE systems numerically (see [11] and the references therein). Recently, Chen et al. [10, 12] considered the pricing of European options under both the FMLS model and the CGMY model from an analytical point of view by deriving closed-form analytical solutions. They also proposed numerical implementation techniques for their formulae so that those can be easily adopted for trading purposes.

In this paper, we consider the pricing of European options under the KoBol model. It should be noted that our work is a nontrivial extension of [12] because the current FPDE is much more complicated with both left-side and right-side fractional derivatives involved, whereas in the FMLS model, the left-side fractional derivative is the only fractional derivative that appears in the FPDE system. Despite all the difficulties, we have still managed to derive an explicit closed-form analytical formula for European options under the KoBol model. In addition, for trading purposes, a reliable implementation method for the solution is also proposed. Due to the complexity of the newly derived formula, the numerical implementation technique is also quite different and much more complicated than the one proposed by Chen et al. [12]; this is discussed in Section 4.

The rest of the paper is organized as follows. In Section 2, we introduce the FPDE system governing the European option price under the KoBol model. In Section 3, we derive a closed-form solution from the established FPDE system and we examine various basic properties of the option price. In Section 4, some numerical examples and discussions are provided. Concluding remarks are given in the last section.

2. The KoBol model

The KoBol process $\{L_t^{\text{KoBol}}, t \geq 0\}$ introduced recently [4, 16] is a special case of the extended Koponen family of Lévy processes with density defined as

$$\mu(x) = \begin{cases} Dpx^{-1-\alpha}e^{-\lambda x} & x > 0, \\ Dq|x|^{-1-\alpha}e^{-\lambda|x|} & x < 0, \end{cases}$$

where $p + q = 1$, $p > 0$, $q > 0$, $D > 0$, $\lambda > 0$ and $\alpha \in (0, 2]$. As pointed out by Koponen [4], the triplet (α, Dp, Dq) determines the shape of the Lévy density. In particular, α determines whether this process has finite or infinite variation. The parameters p and q determine the overall and relative frequency of the upwind and downwind jumps, respectively. If $p \neq q$, one could expect the fall of the underlying price from the peak to be asymmetric. The parameter λ is the so-called steepness parameter, which determines the exponential decay rate of the tails of the probability density. Usually, the smaller the value of λ, the heavier the tails become. It should be remarked that the KoBol process provides an alternative to the CGMY process for capturing the non-Gaussian characteristic of the underlying return. On the other hand, if $p = q$, the distribution of the KoBol process is identical to the one under the CGMY model with $M = G = \lambda$.

Under a risk-neutral measure Q, the KoBol model assumes the log-price of the underlying asset x_t $(= \ln S_t)$ as

$$d(x_t) = (r - v)\,dt + dL_t^{\text{KoBol}}, \tag{2.1}$$

where r is the risk-free interest rate and v is chosen such that $e^{-rt}S_t$ is a martingale. As pointed out by Koponen [4], under the KoBol model,

$$v = D\Gamma(-\alpha)[p(\lambda - 1)^\alpha + q(\lambda + 1)^\alpha - \lambda^\alpha - \alpha\lambda^{\alpha-1}(q - p)].$$

Now let $V(x, t)$ be the price of European path-independent option with x and t being the current time and the underlying log-price, respectively, both of which satisfy (2.1). Cartea and del-Castillo-Negrete [9] found that $V(x, t)$ is the solution to the FPDE system given by

$$\begin{cases} \dfrac{\partial V(x,t)}{\partial t} + (r - v - \lambda^{\alpha-1}(q-p))\dfrac{\partial V(x,t)}{\partial x} + D\Gamma(-\alpha)\{pe^{\lambda x}{}_xD_\infty^\alpha[e^{-\lambda x}V(x,t)] \\ \quad + qe^{-\lambda x}{}_{-\infty}D_x^\alpha[e^{\lambda x}V(x,t)]\} = [r + D\Gamma(-\alpha)\lambda^\alpha]V(x,t) \\ V(x,T) = \Pi(x), \end{cases} \tag{2.2}$$

where $\Pi(x)$ is the pay-off defined as $\max(e^x - K, 0)$ and $\max(K - e^x, 0)$ for European calls and puts, respectively, with K being the strike price. The one-dimensional left-side and right-side Weyl fractional operators [9], ${}_{-\infty}D_x^Y(\cdot)$ and ${}_xD_\infty^Y(\cdot)$, respectively, are

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defined as

$$
\begin{aligned}
{}_{-\infty}D_x^Y f(x) &= \frac{1}{\Gamma(m-Y)}\frac{d^m}{dx^m}\int_{-\infty}^{x}(x-y)^{m-Y-1}f(y)dy \quad m-1 \le Re(Y) < m,\\
{}_xD_\infty^Y f(x) &= \frac{(-1)^m}{\Gamma(m-Y)}\frac{d^m}{dx^m}\int_{x}^{\infty}(y-x)^{m-Y-1}f(y)dy \quad m-1 \le Re(Y) < m.
\end{aligned}
\tag{2.3}
$$

According to (2.3), it is clear that the fractional derivative is different from the classical derivative with integer order. The determination of the former needs all the information of the target function, whereas the value of the latter at a point is only affected by a small neighbourhood of that particular point.

One should also notice that (2.2) is a generalization of the classical B–S system or even the FPDE system of the FMLS model. With $D = \sigma^2/2\Gamma(-\alpha)$ and $p = q$, (2.2) would transform to the B–S system, whereas (2.2) becomes the FMLS system when

$$
D = \frac{-\sigma^\alpha \sec(\alpha\pi/2)}{2\Gamma(-\alpha)} \quad p = 0 \text{ and } q = 1.
$$

Note that (2.2) appears to be much more complicated in comparison to the B–S system or even the FPDE system for the FMLS model. The current FPDE involves both the left-side and right-side fractional derivatives, with the product of the exponential function and the option price to be differentiated, whereas the one for the FMLS model only has a left-side fractional derivative. Despite the difficulty, we still have managed to derive an explicit closed-form analytical formula for the KoBol model. This issue will be further discussed in the next section.

3. Explicit closed-form analytical solution

In this section, we consider the pricing of European path-independent options under the KoBol model analytically. With the issues to be addressed, this section is further divided into two subsections. First, a closed-form analytical solution is derived by solving the established FPDE system analytically, and then some basic properties of our solution are examined from a theoretical point of view.

3.1. Solution procedure To solve (2.2) analytically, we adopt the Fourier transform technique, because the fractional derivatives usually have simpler behaviour in the Fourier space [20]. Before the Fourier transform can be applied, the following lemma needs to be established.

LEMMA 3.1. *If the Fourier transform is denoted by $F(\cdot)$,*

$$
F\{e^{\lambda x}{}_xD_\infty^\alpha[e^{-\lambda x}f(x)]\} = (\lambda - i\xi)^\alpha F[f(x)](\xi) \tag{3.1}
$$

and

$$
F\{e^{-\lambda x}{}_{-\infty}D_x^\alpha[e^{\lambda x}f(x)]\} = (\lambda + i\xi)^\alpha F[f(x)](\xi), \tag{3.2}
$$

where ξ is the Fourier transform parameter.

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PROOF. According to [20], $F\{{}_xD_\infty^\alpha[f(x)]\} = (i\xi)^\alpha \hat{f}(\xi)$, where $f(\cdot)$ is analytic on $(-\infty, \infty)$. Therefore,

$$\begin{aligned} F\{{}_xD_\infty^\alpha[e^{-\lambda x} f(x)]\}(\theta) &= (i\theta)^\alpha F[e^{-\lambda x} f(x)](\theta) \\ &= (i\theta)^\alpha F[f(x)](\phi)|_{\phi=\theta+i\xi} \\ &= (\lambda + i\xi)^\alpha F[f(x)](\xi). \end{aligned}$$

The second part of this lemma can be proved by using a similar approach, and it is thus omitted. This completes the proof. □

Now applying the Fourier transform with x to (2.2) by using (3.1) and (3.2) gives

$$\begin{cases} \dfrac{\partial \hat{V}}{\partial t} = \{r + i\xi(r - v) - D\Gamma(-\alpha)[p(\lambda + i\xi)^\alpha + q(\lambda - i\xi)^\alpha - \lambda^\alpha + i\xi\alpha\lambda^{\alpha-1}(q - p)]\}\hat{V}, \\ \hat{V}(\xi, T) = \hat{\Pi}(\xi), \end{cases}$$

which can be solved in the Fourier space as

$$\hat{V}(\xi, t) = e^{-(r+i\xi(r-v)-D\Gamma(-\alpha)[p(\lambda+i\xi)^\alpha+q(\lambda-i\xi)^\alpha-\lambda^\alpha+i\xi\alpha\lambda^{\alpha-1}(q-p)])(T-t)}\,\hat{\Pi}(\xi). \tag{3.3}$$

To obtain the option price in the original x space, the Fourier inversion still needs to be carried out, either numerically or analytically: a formidable process that has often hindered the application of this great technique to various problems. In the following, we concentrate on carrying out the inversion of (3.3) analytically.

Applying the inverse Fourier transform to (3.3) gives

$$\begin{aligned} V(x, t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}\hat{V}\,d\xi \\ &= \frac{k_0}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}\{\exp\{-i\xi[k_1 - k_2\alpha\lambda^{\alpha-1}(q - p)]\}\Pi(\xi)\}e^{k_2 p(\lambda+i\xi)^\alpha} e^{k_2 q(\lambda-i\xi)^\alpha}\,d\xi \\ &= k_0 F^{-1}[\exp\{-i\xi[k_1 - k_2\alpha\lambda^{\alpha-1}(q - p)]\}\hat{\Pi}(\xi)\cdot e^{k_2 p(\lambda+i\xi)^\alpha} e^{k_2 q(\lambda-i\xi)^\alpha}] \\ &= k_0\Pi[x + k_1 - k_2\alpha\lambda^{\alpha-1}(q - p)] * \underbrace{F^{-1}[e^{k_2 p(\lambda+i\xi)^\alpha}]}_{I} * \underbrace{F^{-1}[e^{k_2 q(\lambda-i\xi)^\alpha}]}_{II}, \end{aligned} \tag{3.4}$$

where $*$ denotes the convolution of the Fourier transform and

$$k_0 = e^{-r(T-t)-k_2\lambda^\alpha}, \quad k_1 = (r - v)(T - t), \quad k_2 = D\Gamma(-\alpha)(T - t).$$

To determine I and II, we use the fact that the characteristic function of the Lévy stable density is $e^{(i\xi)^\alpha}$ [20]. Then, using the relationship between the characteristic function and the Fourier transform, we obtain

$$F\bigg[\frac{1}{a^{1/\alpha}} f_{\alpha,0}\Big(\frac{|x|}{a}\Big)\bigg] = e^{a(i\xi)^\alpha},$$

where $a > 0$, and $f_{\alpha,0}(x)$ is the Lévy stable density defined as

$$f_{\alpha,0}(x) = \frac{1}{\alpha} H_{2,2}^{1,1}\left[x \,\middle|\, \begin{matrix} (1 - 1/\alpha, 1/\alpha) & (1/2, 1/2) \\ (0, 1) & (1/2, 1/2) \end{matrix}\right]$$

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with H being the Fox function [12]. Then, by using the shift theorem of the Fourier transform [5],

$$F\Big[e^{\lambda x}\frac{1}{a^{1/\alpha}}f_{\alpha,0}\Big(\frac{x}{a^{1/\alpha}}\Big)\Big] = e^{a(\lambda+i\xi)^\alpha},$$
$$F\Big[e^{-\lambda x}\frac{1}{a^{1/\alpha}}f_{\alpha,0}\Big(\frac{x}{a^{1/\alpha}}\Big)\Big] = e^{a(\lambda-i\xi)^\alpha}.$$

Consequently, in the original x-space,

$$I = F^{-1}[e^{k_2 p(\lambda+i\xi)^\alpha}](x) = e^{\lambda x}\frac{1}{(k_2 p)^{1/\alpha}}f_{\alpha,0}\Big(\frac{x}{(k_2 p)^{1/\alpha}}\Big) \tag{3.5}$$

and

$$II = F^{-1}[e^{k_2 q(\lambda-i\xi)^\alpha}](x) = e^{-\lambda x}\frac{1}{(k_2 q)^{1/\alpha}}f_{\alpha,0}\Big(\frac{x}{(k_2 q)^{1/\alpha}}\Big). \tag{3.6}$$

By substituting (3.5) and (3.6) into (3.4), the price of European option with pay-off $\Pi(x)$ is obtained as

$$V(x,t) = k_0\int_{-\infty}^{\infty} L(\eta)\Pi(x-\eta+k_3)\,d\eta,$$

where

$$L(\eta) = \frac{1}{(k_2 p)^{1/\alpha}(k_2 q)^{1/\alpha}}M_1\Big(\frac{|\eta|}{(k_2 p)^{1/\alpha}}\Big) * M_2\Big(\frac{|\eta|}{(k_2 q)^{1/\alpha}}\Big),$$

and $k_3 = k_1 - k_2\alpha\lambda^{\alpha-1}(q-p)$, $M_1(x) = e^{\lambda x}f_{\alpha,0}(x)$, $M_2(x) = e^{-\lambda x}f_{\alpha,0}(x)$.

With the newly derived closed-form formula, it is feasible to further analyse some properties of the European option price. This issue will be discussed in detail in the next subsection.

3.2. Basic properties of the solution In this section, various basic properties of the European option price are analysed. Hereafter, for simplicity, we concentrate on European call options unless otherwise stated. The extension to other path-independent European options is rather straightforward. To avoid confusion, we use $V_c(x,t)$ and $V_p(x,t)$ to represent European call and put option prices, respectively, under the KoBol model.

Proposition 3.2 (Monotonicity). *The European call option price is a monotonically increasing function of the underlying price S.*

Proof. To show the monotonicity of the option price with respect to S, we calculate

$$\begin{aligned}\frac{\partial V_c(x,t)}{\partial x} &= k_0 e^{x+k_3-d_0}L(d_0) - k_0 K L(d_0) + k_0\int_{-\infty}^{d_0} L(\eta)e^{x-\eta+k_3}\,d\eta \\ &= \int_{-\infty}^{d_0} L(\eta)\exp[x-\eta+k_3-r(T-t)-D\Gamma(-\alpha)(T-t)\lambda^\alpha]\,d\eta,\end{aligned} \tag{3.7}$$

where $d_0 = x + k_3 - \ln K$. Since $f_{\alpha,0}(x)$ represents the Lévy stable density, we have $f_{\alpha,0}(x) > 0$ for all x. Consequently, $L(\eta) > 0$ for any η, which indicates that the integrand of (3.7) is always positive. Therefore,

$$\begin{aligned}\frac{\partial V_c(x,t)}{\partial S} &= \frac{1}{S}\frac{\partial V_c(x,t)}{\partial x}\\ &= \frac{1}{S}\int_{-\infty}^{d_0} L(\eta)\exp[x-\eta+k_3-r(T-t)-D\Gamma(-\alpha)(T-t)\lambda^{\alpha}]\,d\eta\\ &> 0,\end{aligned}$$

which implies that V_c is a monotonically increasing function with the underlying S, and this completes the proof. □

In fact, from the above proof, it can also be observed that one of the most important hedging parameters for European calls, that is, Δ_c, is positive, because

$$\Delta_c = \frac{\partial V_c}{\partial S} = \frac{1}{S}\int_{-\infty}^{d_0} L(\eta)\exp\{-\eta - D\Gamma(-\alpha)(T-t)[p(\lambda-1)^{\alpha} + q(1+\lambda)^{\alpha}]\}\,d\eta > 0.$$

A further differentiation of Δ_c with respect to S indicates that

$$\Gamma_c = \frac{\partial^2 V_c(x,t)}{\partial S^2} > 0,$$

and thus the following proposition of the convexity of the option price holds.

Proposition 3.3 (Convexity). *The European call price under the KoBol model is a convex function with the underlying S, that is,*

$$V_c(\theta S_1 + (1-\theta)S_2, t) \le \theta V_c(S_1,t) + (1-\theta)V_c(S_2,t) \quad \textit{where } \theta \in [0,1].$$

In fact, one can easily show that Proposition 3.3 is true for options with convex pay-off functions.

Proposition 3.4 (Asymptotics). *The following are true for the function V_c.*

$$\text{(i)} \lim_{x\to\infty} V_c(x,t) \sim S \quad (\textit{as } S \to \infty), \qquad \text{(ii)} \lim_{x\to-\infty} V_c(x,t) = 0.$$

Proof. (i) By taking $x \to \infty$, the price of the European call option becomes

$$\lim_{x\to\infty} V_c(x,t) = \lim_{x\to\infty} k_0 e^{x+k_3}\int_{-\infty}^{\infty} e^{-\eta}L(\eta)\,d\eta - k_0 K\int_{-\infty}^{\infty} L(\eta)\,d\eta. \tag{3.8}$$

The values of two integrals in the right-hand side (RHS) of (3.8) can be calculated by means of the Fourier transform. To be specific,

$$\int_{-\infty}^{\infty} e^{-\eta}L(\eta)\,d\eta = F[L(\eta)]|_{\xi=i} = \exp[k_2 p(\lambda-1)^{\alpha} + k_2 q(\lambda+1)^{\alpha}] \tag{3.9}$$

and

$$\int_{-\infty}^{\infty} L(\eta)\,d\eta = F[L(\eta)]|_{\xi=0} = e^{k_2\lambda^{\alpha}}. \tag{3.10}$$

Now, by substituting (3.9) and (3.10) into (3.8),

$$\begin{aligned}\lim_{x\to\infty} V_c(x,t) &= \lim_{x\to\infty} k_0 \exp[x + k_3 + k_2 p(\lambda-1)^\alpha + k_2 q(\lambda+1)^\alpha] - k_0 K e^{k_2\lambda^\alpha}\\ &= \lim_{x\to\infty} e^x - e^{-r(T-t)}K,\end{aligned}$$

which is indeed of the same order as S, when S is extremely large. (ii) One can deduce from (3.9) that $\int_{-\infty}^{\infty} e^{-\eta} L(\eta)\, d\eta$ is a bounded integral. Therefore, for any given η, $e^{-\eta}L(\eta)$ is also bounded since $e^{-\eta}L(\eta)$ is continuous and nonnegative. Consequently,

$$\lim_{d_0\to-\infty} \int_{-\infty}^{d_0} e^{-\eta} L(\eta)\, d\eta = 0.$$

Similarly,

$$\lim_{d_0\to-\infty} \int_{-\infty}^{d_0} L(\eta)\, d\eta = 0.$$

Thus, we can reach the conclusion that $\lim_{x\to-\infty} V_c(x,t) = 0$, and the proof is complete. □

Note that the asymptotic behaviour shown in Proposition 3.4 agrees with the far-field boundary conditions appropriate for call options. This indicates the validity of the current solution.

On the other hand, as pointed out in Section 2, our solution reduces to the corresponding B–S price or the FMLS price under certain parameter settings. We examine this limiting behaviour in the following proposition.

Proposition 3.5.

(i) *With $D = \sigma^2/2\Gamma(-\alpha)$ and $p = q$, our pricing formula reduces to the B–S formula with constant volatility σ if $\alpha \to 2$.*
(ii) *When $D = [-\sigma^\alpha \sec(\alpha\pi/2)]/2\Gamma(-\alpha)$, $p = 0$ and $q = 1$, our pricing formula is identical to the FMLS formula [12].*

Proof. (i) By setting $D = \sigma^2/2\Gamma(-\alpha)$ and $p = q$, we obtain

$$\lim_{\alpha\to2} k_0 = e^{-r(T-t)+k_2\lambda^2(T-t)}, \quad \lim_{\alpha\to2} k_1 = k_3 = (r - \tfrac{1}{2}\sigma^2)(T-t), \quad \lim_{\alpha\to2} k_2 = \tfrac{1}{2}\sigma^2(T-t).$$

On the other hand, according to Chen et al. [12], the following identity holds:

$$f_{2,0}(|x|) = \frac{e^{-x^2/4}}{2\sqrt{\pi}},$$

which yields

$$\begin{aligned}\lim_{\alpha\to2} L(x) &= e^{-\lambda x}\frac{1}{4k_2\pi\sqrt{pq}} \int_{-\infty}^{\infty} e^{2\lambda\eta} e^{-[\eta^2/(4k_2p)^{1/2}]-[(x-\eta)^2/(4k_2q)^{1/2}]}\, d\eta\\ &= \frac{1}{4k_2\pi\sqrt{pq}} e^{[((px+4k_2pq\lambda)^2-px^2)/4k_2pq]-\lambda x} \int_{-\infty}^{\infty} e^{-[\eta-(px+4k_2pq\lambda)]^2/4k_2pq}\, d\eta\\ &= \frac{1}{2\sqrt{\pi k_2}} e^{4k_2pq\lambda^2} e^{-x^2/4k_2}.\end{aligned}$$

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Therefore, the European call option price under the current parameter settings can be calculated as

$$\begin{aligned}\lim_{\alpha\to 2} V_c(x,t) &= \frac{k_0}{2\sqrt{\pi k_2}} e^{4k_2pq\lambda^2}\Big[e^{x+k_3}\int_{-\infty}^{\infty} e^{-(\eta^2/4k_2)-\eta}\,d\eta - K\int_{-\infty}^{\infty} e^{-\eta^2/4k_2}\,d\eta\Big]\\ &= k_0e^{4k_2pq\lambda^2}[e^{x+k_3+k_2}N(d_1) - KN(d_2)], \qquad (3.11)\end{aligned}$$

where

$$\begin{aligned} d_1 &= \lim_{\alpha\to 2}\frac{d_0+2k_2}{\sqrt{2k_2}} = \frac{\ln(S/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},\\ d_2 &= \lim_{\alpha\to 2}\frac{d_0}{\sqrt{2k_2}} = \frac{\ln(S/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.\end{aligned}$$

Since $p+q=1$ and $p=q$, we have $p=q=1/2$, and thus

$$k_0e^{4k_2pq\lambda^2} = e^{-r(T-t)} \quad\text{and}\quad k_0e^{4k_2pq\lambda^2}e^{x+k_3+k_2} = e^x. \qquad (3.12)$$

By substituting (3.12) into (3.11), we obtain

$$V_c(x,t) = e^xN(d_1) - Ke^{-r(T-t)}N(d_2) = V^{\text{Call}}_{\text{B–S}}(x,t)$$

with $V^{\text{Call}}_{\text{B–S}}(x,t)$ representing the European call option price under the B–S model. This completes the proof of (i).

(ii) When $D = [-\sigma^\alpha\sec(\alpha\pi/2)]/2\Gamma(-\alpha)$, $p=0$ and $q=1$, it follows that

$$k_0 = e^{-r(T-t)}, \quad k_1 = k_3 = \Big(r+\frac{1}{2}\sigma^\alpha\sec\frac{\alpha\pi}{2}\Big)(T-t), \quad\text{and}\quad k_2 = -\Big(\frac{1}{2}\sigma^\alpha\sec\frac{\alpha\pi}{2}\Big)(T-t).$$

Therefore,

$$\begin{aligned}\lim_{p\to 0} V_c(x,t) &= \lim_{p\to 0} e^{-r(T-t)}\int_{-\infty}^{x+k_1-\ln K}\int_{-\infty}^{\infty}\Big[\frac{f_{\alpha,0}(|s|/(k_2p)^{1/\alpha})f_{\alpha,0}(|\eta-s|/(k_2)^{1/\alpha})}{(k_2p)^{1/\alpha}(k_2)^{1/\alpha}}\\ &\qquad\times (e^{x-\eta+k_1}-K)\Big]\,ds\,d\eta\\ &= e^{-r(T-t)}\int_{-\infty}^{x+k_1-\ln K}\int_{-\infty}^{\infty}\delta(s)\frac{1}{(k_2)^{1/\alpha}}f_{\alpha,0}\Big(\frac{|\eta-s|}{(k_2)^{1/\alpha}}\Big)(e^{x-\eta+k_1}-K)\,ds\,d\eta\\ &= e^{-r(T-t)}\int_{-\infty}^{x+k_1-\ln K}\frac{1}{(k_2)^{1/\alpha}}f_{\alpha,0}\Big(\frac{|\eta|}{(k_2)^{1/\alpha}}\Big)(e^{x-\eta+k_1}-K)\,d\eta,\end{aligned}$$

which is indeed identical to the call option price under the FMLS model. This completes the proof of (ii). □

Finally, we consider the put-call parity under the KoBol model. The so-called put-call parity is a relationship between European vanilla options with the same parameters. From a financial point of view, the introduction of the convexity adjustment v has ensured the existence of the risk-neutral measure, indicating that the "no arbitrage opportunity" assumption still holds for the KoBol model. The put-call parity can thus be achieved by using a similar portfolio analysis as adopted under the B–S model. Mathematically, it can be proved as follows.

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THEOREM 3.6. *For any given $\alpha \in (1,2)$, the put-call parity for European puts and calls with the same parameters can be found as*

$$V_c(x,t) - V_p(x,t) = S - Ke^{-r(T-t)}.$$

PROOF. Since both $V_c(x,t)$ and $V_p(x,t)$ satisfy the governing equation contained in (2.2), $V_c(x,t) - V_p(x,t)$, denoted by V^0 in the following, also satisfies the same governing equation as in (2.2) but with terminal condition

$$V^0(x,T) = \max(e^x - K, 0) - \max(K - e^x, 0) = e^x - K.$$

Following the solution procedure in Section 3.1, we derive

$$\begin{aligned} V^0(x,t) &= k_0 e^{x+k_3} \int_{-\infty}^{\infty} e^{-\eta} L(\eta)\, d\eta - k_0 K \int_{-\infty}^{\infty} L(\eta)\, d\eta \\ &= S - Ke^{-r(T-t)}, \end{aligned}$$

where the last line of the above equation has used (3.9) and (3.10). This completes the proof. □

Note that the put-call parity derived in Theorem 3.6 can greatly facilitate the trading of vanilla options under the KoBol model, in the sense that the price of either a European call or put is easy to reduce from the parity relationship once the price of the "opposite" contract is determined from our formula. In the next section, the implementation details of the current formula will be provided.

4. Numerical examples and discussions

In this section, the implementation details of the current solution are illustrated. Based on the numerical results, some analysis on the impacts of different parameters are provided.

Although our pricing formula is similar to the B–S formula or the one derived under the FMLS model [12], the implementation of our solution is not straightforward because it involves numerical computation of the Fox function and double integrals over infinite domains. For $f_{\alpha,0}(x)$, we adopt the series representation of Chen et al. [12], that is,

$$f_{\alpha,0}(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\Gamma(1 + n/\alpha)}{n!} \sin\Big(\frac{n\pi}{2}\Big)(-x)^{n-1}, \tag{4.1}$$

instead of its usual form in the Mellin space [6] to avoid the extra task of determining the inverse Mellin transform. Since (4.1) converges slowly when x takes on large values, the large asymptotic expression of $f_{\alpha,0}(x)$ is adopted when x exceeds a critical value, that is,

$$f_{\alpha,0}(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\Gamma(1 + n/\alpha)}{n!} \sin\Big(\frac{n\pi\alpha}{2}\Big)|x|^{-1-n\alpha}.$$

Numerical experiments suggest that $x \approx 5$ is an appropriate critical value for all the numerical experiments shown below.

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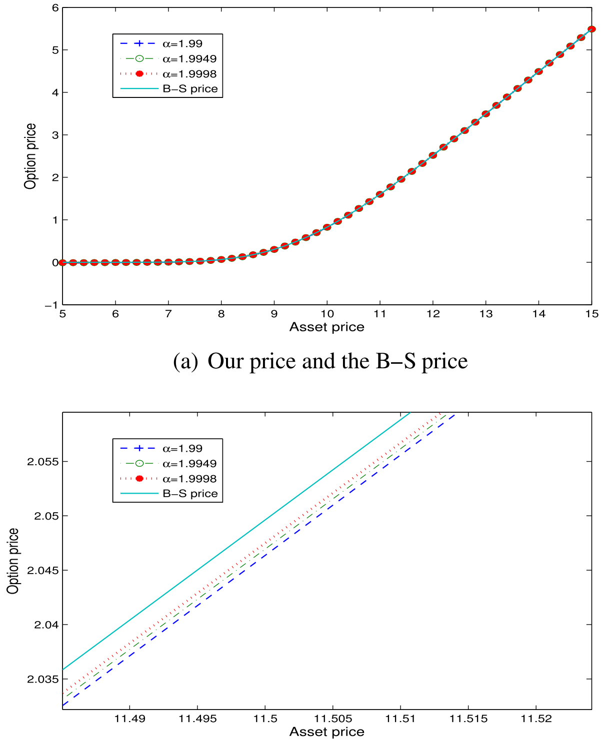

(a) Our price and the B–S price

(b) Enlarged option price difference

FIGURE 1. Parameters are $K = \$10$, $\sigma = 0.2$, $r = 0.1$, $p = q = 0.5$, $\lambda = 1$, $T - t = 0.5$ (year), $D = \sigma^2/2\Gamma(-\alpha)$.

On the other hand, the generalized Laguerre–Gauss (LG) quadrature [14], is used to determine the integrals involved in our pricing formula. Since this quadrature only deals with integrals in semiinfinite domains, we rewrite our solution as

$$\begin{aligned} V_c &= S - Ke^{-r(T-t)} + V_p \\ &= S - Ke^{-r(T-t)} \\ &\quad + \frac{k_0}{(4pq)^{1/\alpha}} \int_{d_0/(k_2/2)^{1/\alpha}}^{\infty} e^{-\lambda(k_2/2)^{1/\alpha} z} \bigg[\int_0^{\infty} e^{2\lambda(k_2/2)^{1/\alpha} y} f_{\alpha,0}\Big(\frac{|y|}{(2p)^{1/\alpha}}\Big) f_{\alpha,0}\Big(\frac{|y-z|}{(2q)^{1/\alpha}}\Big)\, dy \\ &\quad + \int_0^{\infty} e^{-2\lambda(k_2/2)^{1/\alpha} y} f_{\alpha,0}\Big(\frac{|y|}{(2p)^{1/\alpha}}\Big) f_{\alpha,0}\Big(\frac{|y+z|}{(2q)^{1/\alpha}}\Big)\, dy \bigg] [E - e^{x + k_3 - (k_2/2)^{1/\alpha} z}]\, dz, \end{aligned}$$

which can then be determined accurately by the LG quadrature.

One of the most efficient ways to test the reliability of our formula as well as the proposed numerical evaluation technique is to consider the degenerate case as $\alpha \to 2$. Theoretically, as shown in (i) of Proposition 3.5, with $D = \sigma^2/2\Gamma(-\alpha)$ and $p = q$, our pricing formula degenerates to the B–S formula with constant volatility σ as a further limit of $\alpha \to 2$. Figure 1(a) illustrates the comparison between our solution at three

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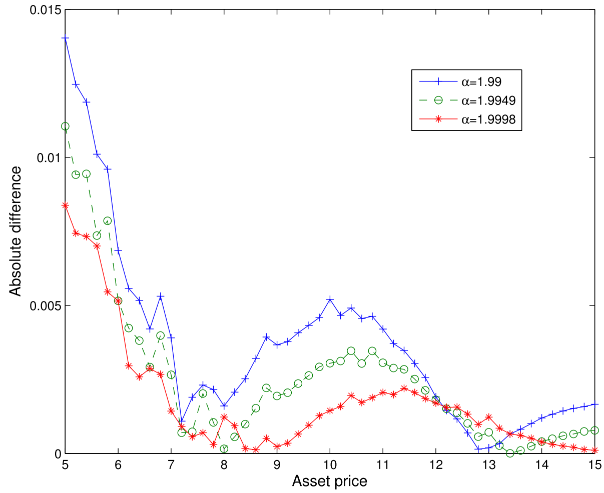


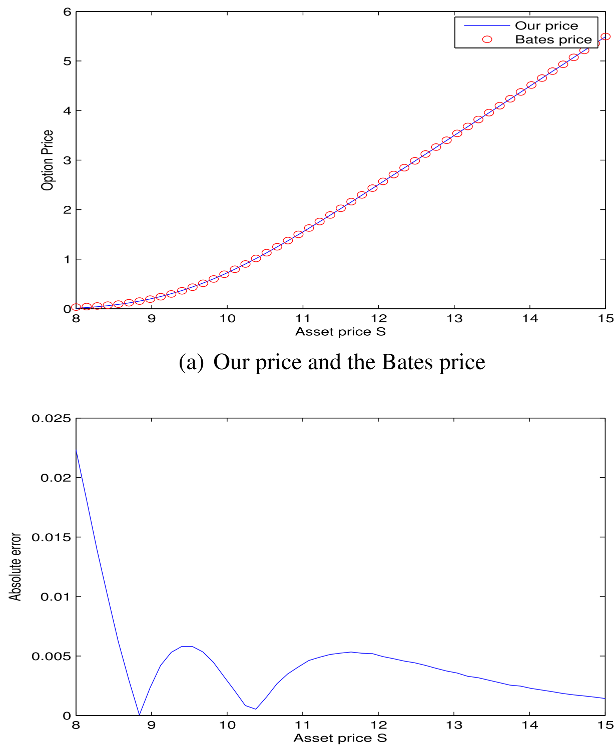
FIGURE 2. Option price difference. Parameters are $K = \$10$, $\sigma = 0.2$, $r = 0.1$, $p = q = 0.5$, $\lambda = 1$, $T - t = 0.5$ (year), $D = \sigma^2/2\Gamma(-\alpha)$.

different α values and the B–S formula, with the pointwise absolute difference further shown in Figure 2. From these two figures, it is clear that under the given parameter settings, our pricing formula approaches the B–S formula as expected. Note that, for the purpose of comparison, we put an enlarged figure in Figure 1(b). Intuitively, the remarkably little difference shown in Figure 2 is the result of the error arising from the implementation technique and the difference between our α- values and two. Since the α are chosen to be very close to two, we believe that the difference shown in Figure 2 is mainly caused by the error of the proposed numerical method.

To further verify the validity of our solution, we compare our results with the Bates formula, as shown in Figure 3. We point out that the Bates formula is actually the option price in the Fourier space without the Fourier inversion being carried out analytically. Theoretically, if there were no errors brought in by the numerical implementation technique, our results should be identical to those calculated from the Bates formula [1]. From Figure 3, it is clear that the maximum pointwise absolute error between our solution and the Bates formula is less than 0.025. This indicates that both our pricing formula and the implementation technique are quite convincing.

With confidence in our solution as well as the proposed implementation technique, we investigate some financial properties of our solution. The hedging parameters of this model are examined first, followed by the impact of some parameters on the option prices.

It is known that with the closed-form analytical solution available, the hedging parameters can also be obtained in closed forms, by differentiating with respect to the corresponding parameters. For example, the delta value can be obtained by differentiating with respect to S, as shown in Section 3.2. One can also get the gamma



(a) Our price and the Bates price

(b) Option price difference

FIGURE 3. Comparison of our price with the Bates formula. Parameters are $\alpha = 1.8$, $K = \$10$, $r = 0.1$, $p = 0.3$, $q = 0.7$, $\lambda = 1$, $T - t = 0.5$ (year), $D = \sigma^2/2\Gamma(-\alpha)$.

value by a further derivative of delta with respect to S. Theoretically, the hedging parameters are not equivalent to the ones under the B–S model, depending on the chosen parameters. For instance, one can observe from Figure 4 that even with the conditions in (i) of Proposition 3.5 being satisfied, the delta values of European calls under the two models are quite different, depending on the α values.

Now we turn to investigating the influences of p, q, λ and α on option prices. The solution produced with $D = 0.08$, $p = q = 0.5$, $r = 0.1$, $T - t = 0.5$ (year), $\alpha = 1.8$ and $\lambda = 1$ is set as the benchmark case. We study the impact of a certain parameter by changing the value of this parameter, but with other parameters fixed as in the benchmark case.

First, we consider the impact of p. As pointed out by Tankov [23], p and q determine the overall and relative frequency of the upward and downward jumps, respectively. As a result, when p increases, more upward jumps would occur, resulting in a higher European call option price, as shown in Figure 5. Similarly, with increasing values of q,

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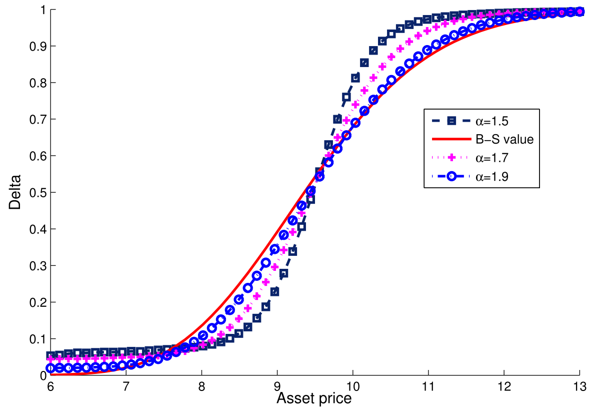

FIGURE 4. Delta values of the European call options. Parameters are $K = \$10$, $r = 0.1$, $p = 0.3$, $q = 0.7$, $\lambda = 1$, $T - t = 0.5$ (year), $D = 0.0063$.

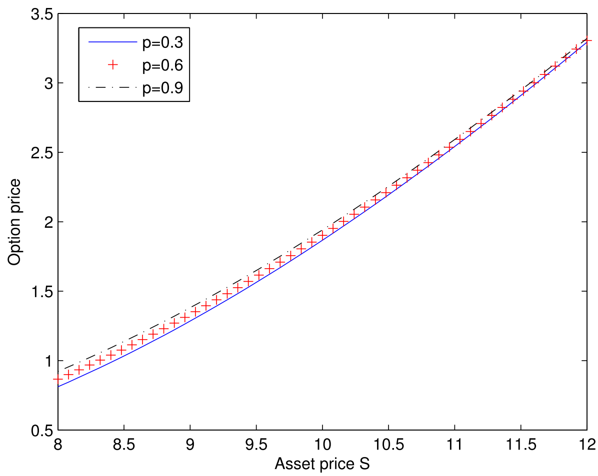

FIGURE 5. Influence of p on our option price.

the call option price would also increase because of the more frequently occurring downward jumps in the underlying price.

Depicted in Figure 6(a) is the variation in the European call option price with respect to λ. From this figure, it is obvious that the call price tends to fall as λ increases. This can be explained by the argument that heavier tails lead to more valuable call options, because the smaller the value of λ, the heavier the tails will be, as mentioned in [4]. A similar argument can also be used to explain the fact that the call option price is a strictly increasing function of α, as shown in Figure 6(b).

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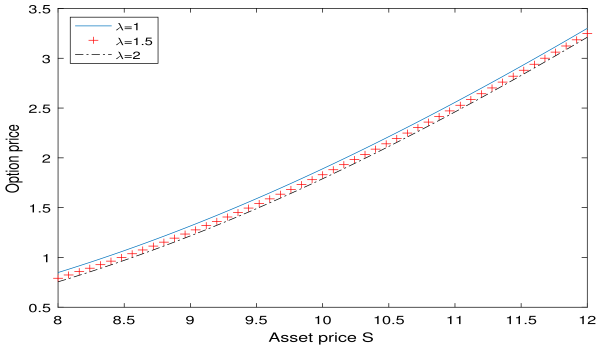
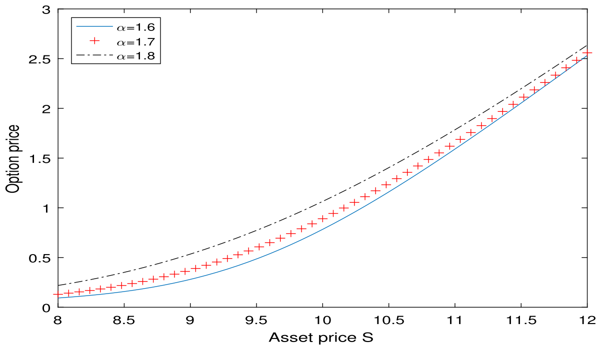


(a) Influence of λ



(b) Influence of α

FIGURE 6. Influences of λ and α on our option price.

5. Conclusion

In this paper, an explicit closed-form analytical solution for European path-independent options under the KoBol model is derived for the first time with the help of the Fourier transform technique. Based on the newly derived formula, many basic properties of the option price under the KoBol model are examined from a theoretical point of view. Numerical experiments suggest the validity of our solution and the proposed numerical implementation technique. We have also proposed a reliable implementation technique for the current solution so that it can be easily used for trading purposes.

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