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Rational Solutions of Painlevé Equations

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Abstract. Consider the sixth Painlevé equation (P_6) below where α, β, γ and δ are complex parameters. We prove the necessary and sufficient conditions for the existence of rational solutions of equation (P_6) in term of special relations among the parameters. The number of distinct rational solutions in each case is exactly one or two or infinite. And each of them may be generated by means of transformation group found by Okamoto [7] and Bäcklund transformations found by Fokas and Yortsos [4]. A list of rational solutions is included in the appendix. For the sake of completeness, we collected all the corresponding results of other five Painlevé equations (P_1)–(P_5) below, which have been investigated by many authors [1]–[7].

1 Introduction and Main Results

The six Painlevé equations

(P_1)
$$\omega'' = 6\omega^2 + z,$$

(P_2)
$$\omega'' = 2\omega^3 + 3z\omega + \alpha,$$

(P_3)
$$\omega'' = \frac{1}{\omega}(\omega')^2 - \frac{1}{z}\omega' + \frac{1}{z}(\alpha\omega^2 + \beta) + \gamma\omega^3 + \frac{\delta}{\omega},$$

(P_4)
$$\omega'' = \frac{1}{2\omega}(\omega')^2 + \frac{3}{2}\omega^3 + 4z\omega^2 + 2(z^2 - \alpha)\omega + \frac{\beta}{\omega},$$

(P_5)
$$\begin{aligned}\omega'' &= \left(\frac{1}{2\omega} + \frac{1}{\omega - 1}\right)(\omega')^2 - \frac{1}{z}\omega' \\ &\quad + z^{-2}(\omega - 1)^2\left(\alpha\omega + \frac{\beta}{\omega}\right) + \gamma\frac{\omega}{z} + \delta\frac{\omega(\omega + 1)}{\omega - 1},\end{aligned}$$

(P_6)
$$\begin{aligned}\omega'' &= \frac{1}{2}\left(\frac{1}{\omega} + \frac{1}{\omega - 1} + \frac{1}{\omega - z}\right)(\omega')^2 - \left(\frac{1}{z} + \frac{1}{z - 1} + \frac{1}{\omega - z}\right)\omega' \\ &\quad + \frac{\omega(\omega - 1)(\omega - z)}{z^2(z - 1)^2}\left[\alpha + \frac{\beta z}{\omega^2} + \frac{\gamma(z - 1)}{(\omega - 1)^2} + \delta\frac{z(z - 1)}{(\omega - z)^2}\right],\end{aligned}$$

were first derived around the turn of the century in an investigation by Painlevé and his colleagues [2], [5], where α, β, γ, δ are complex parameters. Although first discovered from strictly mathematical considerations, the Painlevé equations have appeared in various of physical applications [1]–[7]. They may also be thought of as

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nonlinear analoques of the classical special functions [6]. Therefore, they have become one of the most important classes of nonlinear differential equations.

Many results [1]–[7] show that the rational solutions of Painlevé equations are related to some partial differential equations in nonlinear wave theory and physical problems. A remarkable example is that the rational solutions are the analogue of multi-solitons for the second Painlevé equation (P_2). Therefore, to characterize the existence of rational solutions and to try to generate all the rational solutions for Painlevé equations (say, Problem*) are significant subjects.

Problem* has been investigated in many articles [1]–[5]. Main results are stated as following theorem where conclusion 5 was proved by Kitaev, Law and McLeod [7].

Theorem A

(1) *There is no rational solution for* (P_1).
(2) *There exists exactly one rational solution for* (P_2) *when the parameter* $\alpha \in \mathcal{Z}$.
(3) (P_3) *with* $\gamma\delta \neq 0$ *has rational solutions if and only if* $\alpha + \beta\varepsilon = 4n$, $n \in \mathcal{Z}$, $\varepsilon^2 = 1$.
(4) *There exist rational solutions for* (P_4) *if and only if the parameters satisfy* $\alpha = n_1$, $\beta = -\frac{2}{9}(6n_2 - 3n_1 + \varepsilon)^2 = -2(1 + 2n_3 - n_1)^2$, $n_1, n_2, n_3 \in \mathcal{Z}$, $\varepsilon^2 = 1$.
(5) (P_5) *with* $\delta \neq 0$ *has a rational solution if and only if for some branch* $\lambda_0 = (-2\delta)^{-\frac{1}{2}}$ *the parameters belong to one of the following cases with* $k, m \in \mathcal{Z}$*:*

   (i) $\{2\alpha = (\lambda_0 + k)^2, -2\beta = m^2\}$ *where* $m \geq 0$, $k + m$ *is odd, and* $\alpha \neq 0$ *where* $k < m$;

   (ii) $\{-2\beta = (\lambda_0 + k)^2, 2\alpha = m^2\}$ *where* $m \geq 0$, $k + m$ *is odd, and* $\beta \neq 0$ *where* $k < m$;

   (iii) $\{-2\beta = (\alpha_1 + m)^2, \lambda_0\gamma = k\}$ *where* $\alpha_1 = 2\alpha$ *so that* $m \geq 0$ *and* $k + m$ *is even;*

   (iv) $\{2\alpha = \frac{1}{4}k^2, -2\beta = \frac{1}{4}m^2, \lambda_0\gamma \notin \mathcal{Z}\}$ *where* $k, m > 0$ *and* k, m *are both odd.*

In other words, only for equation (P_6) the Problem* has not been answered. In this paper, we will solve it making use of the Bäcklund transformation and employing the polynomial Hamiltonian system [5], [7].

Our main results are:

Theorem 1.1 *Equation* (P_6) *has a rational solution if and only if the parameters belong to one of the following cases*

(I) $\{\alpha \in \mathcal{C}, \beta = -\alpha h^2, \gamma = \alpha(h-1)^2, \delta = 0\} \cup \{\alpha = \frac{1}{2}, \beta = -\gamma h^2, \gamma \in \mathcal{C}, \delta = \frac{1}{2} - \gamma(h-1)^2\} \cup \{\alpha \in \mathcal{C}, \beta = \frac{1}{2}, \gamma = \alpha h^2, \delta = \frac{1}{2} - \alpha(h-1)^2\} \cup \{\alpha \in \mathcal{C}, \beta = -\alpha h^2, \gamma = \frac{1}{2}, \delta = \frac{1}{2} - \alpha(h-1)^2\}$ *where* $h \neq 0, 1 \in \mathcal{C}$;
(II) $\{2\alpha = n^2, -2\beta = (\lambda+n)^2, 2\gamma = r^2, 1-2\delta = q^2\} \cup \{-2\beta = n^2, 2\alpha = (\lambda+n)^2, 2\gamma = r^2, 1 - 2\delta = q^2\} \cup \{2\gamma = n^2, 2\alpha = (\lambda + n)^2, -2\beta = r^2, 1 - 2\delta = q^2\} \cup \{1 - 2\delta = n^2, 2\gamma = (\lambda + n)^2, -2\beta = r^2, 2\alpha = q^2\}$ *where* $1 + \lambda + q = r$, $n \in \mathcal{Z}$, r *and* λ *satisfy*

   (1) $r = m + \lambda \notin \mathcal{Z}$, $m \in \mathcal{N}$; *or*

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(2) $\lambda \notin \mathcal{Z}, -r \in \mathcal{N}$; *or*

(3) $r \in \mathcal{Z}, \lambda \in \mathcal{N}$; *or*

(4) $r, \lambda \in \mathcal{Z}, \{\lambda > 1, r \leq 0\} \cup \{\lambda < 0, \lambda < r \leq 0\}$; *or*

(5) $r, \lambda \in \mathcal{Z}, \lambda > 1, r > \lambda$; *or*

(6) $r, \lambda \in \mathcal{Z}, \lambda = 1, r \neq 1$; *or*

(7) $r, \lambda \in \mathcal{Z}, \lambda = 0, r \in \mathcal{Z}$.

Further, we can clarify Theorem 1.1 in the following theorem.

Theorem 1.2 *In Theorem 1.1,*

(1) *if case* $\{\mathcal{I}, \alpha = 0\} \cup \{\mathcal{I}, \gamma = 0\} \cup \{\mathcal{II}$, *(4) or (5) or (6) or (7)*$\}$ *occurs, then equation* (P_6) *has infinite distinct rational solutions;*
(2) *if case* $\{\mathcal{II}, \alpha\beta\gamma(1-2\delta) \neq 0$ *and (1) or (2) or (3)*$\}$ *occurs, then equation* (P_6) *has exactly two distinct rational solutions.*
(3) *if case* $\{\mathcal{I}, \alpha \neq 0\} \cup \{\mathcal{I}, \gamma \neq 0\} \cup \{\mathcal{II}, \alpha\beta\gamma(1-2\delta) = 0$ *and (1) or (2) or (3)*$\}$ *occurs, then equation* (P_6) *has exactly one distinct rational solution.*

In Section 5, we shall give the list of rational solutions for equation (P_6) in each of all above cases after completing the proofs of Theorem 1.1 and Theorem 1.2. Some of them may be generated by means of some Bäcklund transformations and a transformation group. Our proofs depend heavily on the availability the Bäcklund transformations which change a solution to another solution for the same equation with maybe different parameters. In Section 2, we shall introduce these transformations. Another key idea of our proofs is making use of the Hamiltonian system developed by Okamoto [7], which is equivalent to equation (P_6), and will be stated in Section 3. In Section 4, we shall give some preliminary propositions.

For the sake of convenience, we give some definition and notations.

Definition 1.1 A rational function $\frac{P(z)}{Q(z)}$ is said to be a proper if $P(z) \equiv 0$ or $\deg P(z) < \deg Q(z)$, where $P(z)$, $Q(z)$ are polynomials.

Throughout this paper, unless otherwise stated, the expression $\frac{P(z)}{Q(z)}$ denotes a proper, irreducible rational function. $\omega(z) := \omega(z, \alpha, \beta, \gamma, \delta)$ denotes a rational solution of equation (P_6) with parameters α, β, γ, δ. C denotes an arbitrary complex number which may have distinct values in different places. $\{\mathcal{I}.\delta = 0\}$ is a simple notation of the case $\{\mathcal{I}.\alpha \in \mathcal{C}, \beta = -\alpha h^2, \gamma = \alpha(h-1)^2, \delta = 0\}$, and so on.

2 Transformations for Equation (P_6)

As noted before, the availability of transformations is essential in our program. The following theorem is due to Okamoto [8]. It can be checked by direct calculation.

Theorem B *Equation* (P_6) *admits a symmetric group of discrete transformations, generated by the following three transformations:*

$$T_1\big(\omega(z)\big) := \omega_1(z, -\beta, -\alpha, \gamma, \delta) = \omega^{-1}\left(\frac{1}{z}\right);$$

$$T_2\big(\omega(z)\big) := \omega_2(z, -\beta, -\gamma, \alpha, \delta) = 1 - \omega^{-1}\left(\frac{1}{1-z}\right);$$

$$T_3\big(\omega(z)\big) := \omega_3\left(z, -\beta, -\alpha, -\delta + \frac{1}{2}, -\gamma + \frac{1}{2}\right) = \frac{z}{\omega(z)}$$

where $\omega(z, \alpha, \beta, \gamma, \delta)$ *is a solution for equation* (P_6) *with parameters* α, β, γ, δ.

Remark This group has 24 transformations denoted by T_i $(i = \overline{1, 24})$.

For equation (P_6), Fokas and Yortsos [4] were the first to receive the Bäcklund transformations. We shall write them below:

Theorem C *Let* $\omega(z)$ *be a solution of equation* (P_6) *such that*

$$\Phi(\omega) :\equiv 2f' + \frac{g}{z} + \kappa\frac{z+1}{z(z-1)}f \not\equiv 0,$$

where

$$f := z\frac{\omega'}{\omega} + \frac{\tau - \kappa - 1}{2(z-1)}\omega + \frac{\tau + \kappa + 1}{2(z-1)\omega} + \frac{\tau(z+1)}{2(z-1)} - \left(\frac{1}{2} + \frac{\mu}{4}\right),$$

$$g := f^2 + \frac{\mu}{2}f + \upsilon, \quad \kappa := \beta_1 - \alpha_1 - 1 \neq 0,$$

$$\tau := \alpha_1 + \beta_1, \quad (\alpha_1)^2 = 2\alpha, \quad (\beta_1)^2 = -2\beta,$$

$$\mu := -\frac{4}{\kappa}(\gamma + \beta - 1), \quad \upsilon := 2\delta - 1 + \left(\frac{\mu}{4} + \frac{\kappa}{2}\right)^2.$$

Then the function $\widetilde{\omega}(z) := S\big(\omega(z)\big)$ *defined by*

$$\widetilde{\omega}(z) = \omega(z) + \frac{2(z+1)f' - 4\omega f'}{\Phi(\omega)}$$

is a solution of equation (P_6) *with parameters*

$$\widetilde{\alpha} := \frac{1}{2}\big((2\alpha)^{\frac{1}{2}} + 1\big)^2, \quad \widetilde{\beta} := -\frac{1}{2}\big((-2\beta)^{\frac{1}{2}} - 1\big)^2, \quad \widetilde{\gamma} := \gamma, \quad \widetilde{\delta} := \delta.$$

3 Hamiltonian System for Equation (P_6)

In 1986, Okamoto [7] gave some polynomial Hamiltonians $\mathcal{H}_j(z,\omega,\nu)$ associated with each of the six Painlevé equations such that

$$\omega' = \frac{\partial \mathcal{H}_j}{\partial \nu}; \quad \nu' = -\frac{\partial \mathcal{H}_j}{\partial \omega} \tag{3.1}$$

defines a system of first order differential equation (say, Hamiltonian system $\mathcal{H}_j$ for (P_j)) for (ω,ν) where ω solves equation (P_j). He proved that each Hamiltonian system (3.1) is equivalent to the corresponding Painlevé equation. Now we state the result for (P_6).

$$\begin{aligned}\mathcal{H}_6 = \frac{1}{z(z-1)}\Big\{&\omega(\omega-1)(\omega-z)\nu^2\\ &-[-\lambda(\omega-1)(\omega-z)+r\omega(\omega-z)-(q+1)\omega(\omega-1)]\nu\\ &+\frac{1}{4}(C^2-2\alpha)(\omega-z)\Big\}.\end{aligned}$$

Theorem D *Equation* (P_6) *is equivalent to the Hamiltonian system* $\mathcal{H}_6$ *below:*

$$z(z-1)\omega' = \lambda z + [(r-\lambda)z-(C+r)]\omega + C\omega^2 + 2\omega(\omega-1)(\omega-z)\nu, \tag{3.2}$$

$$\begin{aligned}z(z-1)\nu' = \frac{1}{4}(p^2-C^2) &- [(r-\lambda)z-(C+r)]\nu - 2C\omega\nu\\ &-(3\omega^2-2z\omega-2\omega+z)\nu^2,\end{aligned} \tag{3.3}$$

where

$$p^2 = 2\alpha, \quad \lambda^2 = -2\beta, \quad r^2 = 2\gamma, \quad q^2 = 1-2\delta, \quad C = 1+\lambda+q-r. \tag{3.4}$$

4 Preliminary Results for Equation (P_6)

Theorem 4.1 $\omega(z) = h$ $(\neq 0,1)$ *is a constant solution of equation* (P_6) *if and only if*

$$\alpha \in \mathbb{C}, \quad \beta = -\alpha h^2, \quad \gamma = \alpha(h-1)^2, \quad \delta = 0. \tag{4.1}$$

Moreover, if $\alpha = 0$*, then equation* (P_6) *has infinite distinct constant solutions; if* $\alpha \neq 0$*, then equation* (P_6) *has exactly one distinct constant solution.*

Proof Substituting $\omega(z) = h$ into equation (P_6), we have

$$\begin{aligned}&2\big(\beta(h-1)^2+\gamma h^2\big)z^3\\ &\quad+[\alpha h^2(h-1)^2-2\beta h(h-1)^2-\gamma h^2(1+2h)+\delta h^2(h-1)^2]z^2\\ &\quad+[-2\alpha h^3(h-1)^2+\beta h^2(h-1)^2+\gamma h^2(h^2+2h)-\delta h^2(h-1)^2]z\\ &\qquad\qquad+[\alpha h^4(h-1)^2-\gamma h^4]=0.\end{aligned}$$

Thus, the coefficients of z^i, $i = 0, 1, 2, 3$ must be zero, and then (4.1) holds. ■

The following Theorem 4.2 plays an important role in proofs of main results.

Theorem 4.2 *$\omega(z)$ is a nonconstant rational solution of equation* (P_6) *if and only if $\omega(z)$ is a nonconstant rational solution of the Riccati differential equation*

$$z(z-1)\omega' = \lambda z + [(r-\lambda)z - (p+r)]\omega + p\omega^2 \tag{4.2}$$

where $p^2 = 2\alpha$, $\lambda^2 = -2\beta$, $r^2 = 2\gamma$, $q^2 = 1 - 2\delta$, $p = 1 + \lambda + q - r$.

Proof First of all, we prove that $\omega(z)$ is a nonconstant rational solution of equation (4.2) if and only if $v(z) \equiv 0$ if $\big(\omega(z), v(z)\big)$ is a pair nonconstant rational solution of the Hamiltonian system $\mathcal{H}_6$.

The sufficiency is obvious. Now we prove the necessity.

Suppose that $v(z) \not\equiv 0$. Notice that $\omega(z)$ is a rational function, from (3.2) we know that $v(z)$ must be a rational function.

If $z = z_0$ is a pole of $\omega(z)$ with multiplicity τ, then $z = z_0$ is a zero of $v(z)$ with multiplicity at least τ.

From (3.3) we can obtain

$$\omega(z) = -\frac{C}{3v} + \frac{1}{3}(1+z) + \frac{\epsilon}{3}R(z), \tag{4.3}$$

$$R^2(z) = -3z(z-1)v' + a_1 - (a_2 z - a_3)v + (1 - z + z^2)v^2, \tag{4.4}$$

where $\epsilon^2 = 1$, $a_1 := \frac{3}{4}p^2 + \frac{1}{4}C^2$, $a_2 := 3r - 3\lambda + 2C$, $a_3 := C - 3r$, $R(z)$ is a rational function. It is easy to see from (4.3) and (4.4) that $R(z)$ is a polynomial.

If $z = z_1$ is a zero of $v(z)$, from (4.3) we see that $\lim_{z\to z_1} \omega v = -\frac{C}{3}$. Substitute it into (4.4) and (3.3), then we can get $R(z_1) = 0$.

Now we prove that the supposition does not hold according to four different cases. Otherwise:

Case 1: $\deg R(z) > 1$. $v(z)$ is a rational solution of equations (4.4) and (3.3). Then equation (4.4) implies $\lim_{z\to\infty} v(z) = \infty$. Thus by (4.3) we can deduce that $\lim_{z\to\infty}(\omega - z) = \infty$. From equation (3.2), however, we have $\lim_{z\to\infty}(\omega - z) = 0$, a contradiction.

Case 2: $\deg R(z) = 1$. $v(z)$ is a rational solution of equations (4.4) and (3.3). Then we can infer from (4.4) that $\lim_{z\to\infty} v(z) = a\ (\neq 0, \infty)$. If $\lim_{z\to\infty} \omega = \infty$, then we can get from (3.3) that $\lim_{z\to\infty} \frac{z}{\omega} = \frac{3}{2}$. But (3.2) gives $\lim_{z\to\infty} \frac{z}{\omega} = 1$. This contradiction shows that $\lim_{z\to\infty} \omega(z) = b\ (\neq \infty)$. Note that $\omega(z)$ is not a constant, then (4.3) yields $C \neq 0$. Substitute (4.3) into (3.2), then we can obtain that

(4.5)
$$
\begin{aligned}
-Cz(z-1)&\left[\left(\frac{1}{v}\right)' + d'\right] \\
&= Cd - 3dr - 3d^2 + (3\lambda + 3dr - 3d\lambda + 4Cd - 2C)z + 6d(d-1)(d-z)v \\
&\quad - \left[\left(Cr - C\lambda + \frac{2}{3}C^2\right)z - \frac{1}{3}C^2 - Cr\right]\frac{1}{v} + \frac{C^3}{9}\left(\frac{1}{v}\right)^2,
\end{aligned}
$$

where $d := \frac{1}{3}(1+z) + \frac{\epsilon}{3}R(z)$. At present, d is a constant. Thus from (4.5) we know that $v(z)$ only admits simple zeros.

When $d(d-1) \neq 0$, the pole of $v(z)$ must be $z = d$. Hence, $v(z) = a\frac{z-3d+1}{z-d}$, $R(z) = -\epsilon(z - 3d + 1)$. Differentiating equation (4.4) three times in turn, we have

$$
\begin{aligned}
2RR' &= -3(2z-1)v' - 3z(z-1)v'' - a_2v' - (a_2z - a_3)v'' + (2z-1)v^2 \\
&\quad + 2(1 - z + z^2)vv',
\end{aligned} \tag{A.1}
$$

(A.2)
$$
\begin{aligned}
2(R')^2 + 2RR'' &= -6v' - 2(6z - 3 + a_2)v'' - (3z^2 - 3z + a_2z - a_3)v''' + 2v^2 \\
&\quad + 4(2z-1)vv' + 2(1 - z + z^2)\big(vv'' + (v')^2\big),
\end{aligned}
$$

$$
\begin{aligned}
4R'R'' + 2(R'')^2 + 2RR''' &= -18v'' - [3z(z-1) + a_2z - a_3]v^{(4)} + 12vv' \\
&\quad - 3a_2v''' + (2z-1)[6vv'' + 6(v')^2 - 9v'''] \\
&\quad + (1 - z + z^2)(2vv''' + 6v'v'').
\end{aligned} \tag{A.3}
$$

Substitute $v(z) = a\frac{z-3d+1}{z-d}$, $R(z) = -\epsilon(z - 3d + 1)$ into equations (4.4) and (A.1), we can obtain that

(A.4)
$$
\begin{aligned}
(z-d)^2(z-3d+1)^2 &= -3a(2d-1)z(z-1) - a(a_2z - a_3)(z-3d+1)(z-d) \\
&\quad + a_1(z-d)^2 + a^2(1 - z + z^2)(z-3d+1)^2,
\end{aligned}
$$

(A.5)
$$
\begin{aligned}
&2(z-3d+1)(z-d)^3 \\
&= -3a(2d-1)(2z-1)(z-d) + 6a(2d-1)z(z-1) + 2a(a_2z - a_3)(2d-1) \\
&\quad + a^2[(2z-1)(z-d) - a_2a(2d-1)(z-d) + 2(2d-1)(1 - z + z^2)](z-3d+1).
\end{aligned}
$$

In equation (A.4) comparing the coefficient of term z^4, we get $a^2 = 1$. Set $z = d$ we have that

$$
3ad(d-1) = (1 - d + d^2)(2d-1) \tag{A.6}
$$

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Clearly, $2d \neq 1$. Otherwise $a = 0$ implies $v(z) \equiv 0$. In equation (A.5), setting $z = 3d - 1$ and combining equation (A.6), we deduce that $a_3 = a_2 d$. In equation (A.4), setting $z = 3d - 1$ and noting that above results, we obtain that

$$a_1(2d-1) = 3a(3d-1)(3d-2). \tag{A.7}$$

In equation (A.4), comparing the coefficients of terms degree three, two and one of z in turn, we get that

$$12d^2 + 2d + aa_2 - 1 = 0, \tag{A.8}$$

$$13d^2 - 10d - 5aa_2 d + aa_2 - a_1 + 3a(2d-1) + 1 = 0, \tag{A.9}$$

$$-24d^3 + 23d^2 + 10d - 3a(2d-1) + 2da_1 + 7aa_2 d^2 - 1 = 0. \tag{A.10}$$

Equation (A.9) added to equation (A.10) makes that $-24d^3 + 36d^2 + 2aa_2 d^2 + aa_2 + a_1(2d-1) = 0$. Substitute equations (A.7) and (A.8) into above equation, we give that

$$-24d^4 - 28d^3 + 26d^2 - 2d + 1 + a(27d^2 - 27d + 6) = 0. \tag{A.11}$$

We know that d satisfies equations (A.6) and (A.11). On the other hand, equations (A.6) and (A.11) do not have a common solution for either $a = 1$ or $a = -1$. This contradiction shows that $d(d-1) = 0$.

Let $d = 0$. Then $R(z) = -\epsilon(z+1)$, $v(z) = a\frac{z+1}{z+e}$, where e is a constant. Substitute them into equations (4.4) and (4.5), noting that $v(-1) = 0$ gives $3a_1 = C^2$, then eliminate the term containing derivative yields that

(A.12)
$$\left[(3\lambda - 2C)z - \frac{C}{3}(1 - z + z^2)\right] a^2(z+1)^2 + 2Cr(z+1)a(z+e) + \frac{C}{3}(z+1)^2(z+e)^2.$$

Comparing the coefficients of all terms of z we have that $a^2 = 1$,

$$3\lambda - 2C + \frac{C}{3}(1 + 2e) = 0, \tag{A.13}$$

$$6(3\lambda - 2C) + 6rCa + Ce^2 + C + 4Ce = 0, \tag{A.14}$$

$$3(3\lambda - 2C) + C + 6rCa(e+1) + 2Ce = 0, \tag{A.15}$$

$$-1 + 6rae + e^2 = 0. \tag{A.16}$$

From equations (A.13) and (A.15) we can get $2rCa(1+e) = 0$. Hence $r = 0$ or $e = -1$. Noting that the zero of $v(z)$ is $z = -1$, from equation (A.16) we know that $r = 0$ still gives $e = -1$. However, by equation (4.4), we see that $v(z)$ does not have pole $z = 1$ at all, a contradiction.

Let $d = 1$. Then $R(z) = -\epsilon(z-2)$, $v(z) = a\frac{z-2}{z-f}$, where f is a constant. In the manner of the former method we can obtain that

$$\begin{aligned}[2Cz^2 + (2C-9r)z + 2C - 9r - 9]a^2(z-2)^2 \\ + 6Cra(z-2)(z-f) + C(z-2)^2(z-f)^2 = 0.\end{aligned} \tag{A.17}$$

In equation (A.17) comparing coefficients of terms all degree of z, we can get that $a^2 = -\frac{1}{2}$,

$$(2C-9r) + 4Cf = 0, \tag{A.18}$$

$$3(2C-9r) + 9 + 12Cra + 2Cf^2 = 0, \tag{A.19}$$

$$-\frac{5}{2}(2C-9r) - \frac{9}{2} - 6Craf - 12Cra - 4Cf^2 - 4Cf = 0, \tag{A.20}$$

$$-(2C-9r) + 9 + 6Craf + 2Cf^2 = 0. \tag{A.21}$$

From the later three equations we can deduce that $(2C-9r) - 72 + 8Cf = 0$. Then combining equation (A.18) we can obtain that $Cf = 18$, $(2c-9r) = -72$. Set $z = 1$ in equation (A.17). We deduce that $\frac{9}{2} - (2C-9r) - 6Cra(1-f) - Cf^2 = 0$. Subtracting equation (A.21) gives $9+12Cra+6Cf^2 = 0$. Then minus equation (A.19) is $3(2C-9r) - 4Cf^2 = 0$. Thus $f = 3$. Now in equation (4.4) calculating residue at $z = 3$ for $v(z)$ and noting that $\operatorname{Res}_{z=3} v(z) = a$, we have that $a = -\frac{18}{7}$. This contradicts $2a^2 = -1$.

Case 3: $\deg R(z) = 0, R(z) \neq 0$ $\;v(z)$ is a rational solution of equations (4.4) and (3.3). Then $v(z) = \frac{1}{P(z)}$, where $P(z)$ is a polynomial. When $\deg P(z) > 1$, equation (4.3) gives $\lim_{z\to\infty} \omega v = -\frac{C}{3}$. Combining equations (3,3) and (4.4) we can deduce that $R(z) = 0$, a contradiction. Moreover (4.4) can yield $v(z) \not\equiv$ Const. Therefore $\deg P(z) = 1$. Noting that $P(z)$ is a factor of $\omega(\omega-1)(\omega-z)$, we see that the form of $v(z)$ must be as one of the follows: $v(z) = \frac{a}{1+\epsilon R+z}$, $v(z) = \frac{a}{-2+\epsilon R+z}$, $v(z) = \frac{a}{1+\epsilon R-2z}$, where a is a constant.

We will prove a general conclusion which is that there does not exist a solution which is form as $v(z) = \frac{a}{b+z}$ in equation (4.4) under this case, where b is an arbitrary constant. If it is not true, substituting $v(z) = \frac{a}{b+z}$ into equations (4.4), (A.1), (A.2) and (A.3) respectively, we can obtain that

$$\begin{aligned}(b+z)^2R^2 &= 3az(z-1) + a_1(z+1)^2 - (a_2z - a_3)a(b+z) + (1-z+z^2)a^2, \\ 0 &= 3(2z-1)(b+z) - 6z(z-1) + a_2(b+z) - 2(a_2z - a_3) \\ &\quad + a(2z-1)(b+z) - 2a(1-z+z^2), \\ 0 &= 3(b+z)^2 - 2(6z-3+a_2)(b+z) + 3(3z^2 - 3z + a_2z - a_3) + a(b+z)^2 \\ &\quad - 2a(2z-1)(b+z) + 3a(1-z+z^2), \\ 0 &= -6(b+z)^2 + 9(2z-1)(b+z) + 3a_2(b+z) - 4(a_2z - a_3) \\ &\quad + 3a(2z-1)(b+z) - 4a(1-z+z^2) - 12(z^2-z) - 2a(b+z)^2.\end{aligned}$$

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Comparing the coefficients of degree one and constant of z in the above equation, we can obtain that

$$2bR^2 = -3a + 2ba_1 - aa_2b - aa_3 - a^2, \tag{B.1}$$

$$b^2R^2 = b^2a_1 + aba_3 + a^2, \tag{B.2}$$

$$0 = 6b + 3 - a_2 + a + 2ab, \tag{B.3}$$

$$0 = -3b + a_2b + 2a_3 - ab - 2a, \tag{B.4}$$

$$0 = 3b^2 - 2(-3 + a_2)b - 3a_3 + ab^2 + 2ab + 3a, \tag{B.5}$$

$$0 = 6b + 3 + a + 2ba + 3a_2 - 4aa_2, \tag{B.6}$$

$$0 = 6b^2 + 9b + 4a + 2ab^2 + 3ab - 4aa_3 - 3a_2, \tag{B.7}$$

Thus equations (B.3) and (B.6) give $a_2(a-1) = 0$. If $a = 1$, then equation (B.5) yields $a_2 = 8b + 4$. And equation (B.4) implies $2a_2 = 2 + 3b$. Substituting them into equations (B.5) and (B.7), we can infer that $(8b+3)b = 0$ and $4b^2 - 9b - 6 = 0$. It is easy to see that the solution of the former equation $b = 0$ or $b = -\frac{3}{8}$ do not satisfy the later equation. This is impossible. If $a_2 = 0$ and $a \neq 1$, then equations (B.3) and (B.4) can give $4a_3 = 3(a-1)$. Substituting $a_2 = 0$ and $4a_3 = 3(a-1)$ into equations (B.5) and (B.7), we can obtain that $6b - 11a + 9 + 2ab + 6a^2 = 0$. Combining equation (B.3), we can infer that $a = 1$, a contradiction. This contradiction shows that our conclusion holds in this case.

Case 4: $R(z) \equiv 0$ $v(z)$ is a rational solution of equations (4.4) and (3.3). It is easy to verify that $\omega(z) = \frac{1}{3}(1+z)$ is not a solution of equation (P_6) under any parameters. Otherwise, substituting it into (P_6), we get that

$$\begin{aligned}0 = {} & \left(\frac{1}{6} + \frac{3}{4}\beta\right)\frac{1}{z+1} + \left(\frac{1}{6} - \frac{3}{4}\gamma\right)\frac{1}{z-2} + \left(-\frac{5}{6} + 3\delta\right)\frac{1}{1-2z} \\ & + \left(-\frac{2}{27}\alpha + \frac{1}{6}\gamma\right)\frac{1}{z^2} + \left(\frac{1}{6}\beta + \frac{2}{27}\alpha\right)\frac{1}{(z-1)^2} \\ & + \left(-\frac{1}{3} - \frac{1}{18}\alpha - \frac{2}{3}\beta + \frac{1}{12}\gamma + \frac{2}{3}\delta\right)\frac{1}{z} \\ & + \left(-\frac{1}{3} - \frac{1}{18}\alpha + \frac{2}{3}\gamma - \frac{1}{12}\beta + \frac{2}{3}\delta\right)\frac{1}{z-1}.\end{aligned}$$

It is easy to see that each term of above equation must be zero. Hence, by the former four terms, we can obtain that $\alpha = \frac{1}{2}$, $\beta = -\frac{2}{9}$, $\gamma = \frac{2}{9}$, $\delta = \frac{5}{18}$. But these parameters do not make that the last two terms are equivalent zero. This is impossible. Therefore $C \neq 0$. Now equation (4.5) can be written as follows:

$$\begin{aligned}3Cz(z-1)v' = {} & \frac{1}{3}C^3 + [C - r - 1 + (6\lambda + 4r - 2)z + (a_2 + 3C - 1)z^2]v^2 \\ & - C\left(a_2z - \frac{1}{3}C - r\right)v + 2(1+z)(z-2)(1-2z)v^3.\end{aligned}$$

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By the above equation we know that $v(z)$ only admits simple pole $z = -1, 2, \frac{1}{2}$. And whose zero must also be simple. At the same time $\lim_{z\to\infty} v(z) = 0$. Hence equation (4.4) makes us know that if $z = -1, 2, \frac{1}{2}$ is a pole of $v(z)$ then we must have that $\mathrm{Res}_{z=-1}\, v(z) = -2$, $\mathrm{Res}_{z=2}\, v(z) = -2$, $\mathrm{Res}_{z=\frac{1}{2}}\, v(z) = 1$. Set $v_1(z) := -\frac{2}{z+1}$, $v_2(z) := -\frac{2}{z-2}$, $v_3(z) := \frac{2}{2z-1}$. Therefore, $v(z) = v_i(z)$ or $v(z) = v_i(z) + v_j(z)$ or $v(z) = v_1(z) + v_2(z) + v_3(z)$, where $i, j = 1, 2, 3, i \neq j$. On the other hand: the first form of $v(z)$ do not occur in the same proof method of Case 3.

$v(z) = v_1(z) + v_2(z)$ is not a solution of equation (4.4). Otherwise, substituting it into equation (4.4), we get that

$$\begin{aligned} 6z(z-1)(2z^2-2z+5) = a_1(z+1)^2(z-2)^2 + 2(a_2z-a_3)(2z-1)(z+1)(z-2) \\ + 4(1-z+z^2)(2z-1)^2. \end{aligned}$$

Comparing the coefficient of term z^4 and setting $z = 1, 0, -2$ in the above equation, we obtain that $a_1 + 4a_2 - 4 = 0$, $2a_1 - a_2 + a_3 = 0$, $a_1 - a_3 + 1 = 0$, $2a_1 + 20a_2 + 10a_3 + 11 = 0$. It is easy to verify that the system has no any solution, a contradiction.

$v(z) = v_1(z) + v_3(z)$ is not a solution of equation (4.4). Otherwise, substituting it into equation (4.4), we deduce that

$$\begin{aligned} 6z(z-1)(2z^2-6z-1) = a_1(z+1)^2(2z-1)^2 + 2(a_2z-a_3)(2z-1)(z+1)(z-2) \\ + 4(1-z+z^2)(z-2)^2. \end{aligned}$$

Comparing the coefficient of term z^4 and setting $z = 1, 0, 2$ in the above equation, we have that $a_1 + a_2 - 2 = 0$, $a_1 - a_2 + a_3 + 1 = 0$, $a_1 + 4a_3 + 16 = 0$, $3a_1 = -4$. This system is also a contradiction.

$v(z) = v_3(z) + v_2(z)$ is not a solution of equation (4.4). Otherwise, substituting it into equation (4.4), we have that

$$\begin{aligned} 6z(z-1)(2z^2-7) = a_1(z-2)^2(2z-1)^2 + 2(a_2z-a_3)(2z-1)(z+1)(z-2) \\ + 4(1-z+z^2)(z+1)^2. \end{aligned}$$

Comparing the coefficient of term z^4 and setting $z = 1, 0, -1$ in the above equation, we have that $a_1 + a_2 - 2 = 0$, $a_1 - 4a_2 + 4a_3 + 16 = 0$, $a_1 - a_3 - 1 = 0$, $27a_1 + 20 = 0$. This system is still a contradiction.

$v(z) = v_1(z) + v_2(z) + v_3(z)$ is not a solution of equation (4.4). Otherwise, substituting it into equation (4.4), we can obtain that

$$\begin{aligned} &6z(z-1)[(z-2)^2(2z-1)^2 + (z+1)^2(2z-1)^2 - 2(z+1)^2(z-2)^2] \\ &= a_1(z+1)^2(z-2)^2(2z-1)^2 + 36(1-z+z^2)^3 \\ &\quad - 6(a_2z-a_3)(z^2-z+1)(z+1)(z-2)(2z-1). \end{aligned}$$

Comparing the coefficient of term z^6 and setting $z = 1, 0, -2$ in the above equation, we can infer that $a_1 - 3a_2 = 0$, $a_1 + 3a_3 + 9 = 0$, $a_1 + 5a_2 - 5a_3 + 25 = 0$, $210a_2 + 105a_3 + 1373 = 0$. This system is a contradiction, too.

All of these contradictions show that $v(z) \equiv 0$.

Now by Theorem D and the former result we know that this theorem holds. ■

Proposition 4.3 *If $\alpha = 0$, then the Riccati equation (4.2) has the integration as following*

$$\omega(z) = \frac{z^r}{(z-1)^\lambda}\big(\lambda y(z) + C\big)$$

where $y(z)$ is a primitive of $\frac{(z-1)^{\lambda-1}}{z^r}$.

Proof When $\alpha = 0$, (4.2) is a first order linear differential equation which is integrable. Hence this result is trivial. ■

Proposition 4.4 *Either $\int \frac{1}{z^n(z-1)}\,dz$ or $\int \frac{1}{z(z-1)^n}\,dz$ is not a rational function, where $n \in \mathbb{N}$.*

Proof We know that $\int \frac{1}{z(z-1)}\,dz = \ln \frac{z-1}{z} + C$ is not a rational function. Furthermore, the general result follows clearly by induction and from

$$\int \frac{1}{z^n(z-1)}\,dz = \int \frac{1}{z^{n-1}(z-1)}\,dz + \frac{1}{(n-1)z^{n-1}},$$

$$\int \frac{1}{z(z-1)^n}\,dz = -\int \frac{1}{(z-1)^{n-1}z}\,dz - \frac{1}{(n-1)(z-1)^{n-1}}.$$

Theorem 4.5 *Let $\alpha = 0$, $\omega(z)$ be a rational solution of* (P_6). *Then any pole of $\omega(z)$ must be zero or one.*

Moreover

(1) *if $z = 0$ is a pole of ω, then $-r \in \mathbb{N}$;*
(2) *if $z = 1$ is a pole of ω, then $\lambda \in \mathbb{N}$.*

Proof In this case, equation (4.2) becomes

$$z(z-1)\omega' = \lambda z + [(r-\lambda)z - r]\omega.$$

Let $z = z_0$ be a pole of $\omega(z)$. If $z_0 \neq 0, 1$, then $\mathrm{Res}_{z=z_0} \frac{\omega'}{\omega} = 0$ from above formula. This is impossible. Hence $z_0 = 0$ or 1.

If $z = 0$ is a pole of $\omega(z)$ with multiplicity n, then

$$n = -\mathrm{Res}_{z=0}\frac{\omega'}{\omega} = \lim_{z\to 0} z(z-1)\frac{\omega'}{\omega} = \lim_{z\to 0}\left[\frac{\lambda z}{\omega} + (r-\lambda)z - r\right] = -r.$$

If $z = 1$ is a pole of $\omega(z)$ with multiplicity m, then

$$m = -\mathrm{Res}_{z=1}\frac{\omega'}{\omega} = -\lim_{z\to 1} z(z-1)\frac{\omega'}{\omega} = -\lim_{z\to 1}\left[\frac{\lambda z}{\omega} + (r-\lambda)z - r\right] = \lambda.$$

We have proved this theorem. ■

Theorem 4.6 *Let $\alpha = 0$, $\lambda^2 = -2\beta$, $r^2 = 2\gamma$, $q^2 = 1 - 2\delta$, $r = 1 + \lambda + q$. Then*

(I) *if equation* (P_6) *has exactly one nonconstant rational solution* $\omega(z)$ *if and only if the parameters* λ, r *and* q *satisfy*

(1) $r = m + \lambda \notin \mathbb{Z}$, $m \in \mathbb{N}$; *or*

(2) $\lambda \notin \mathbb{Z}$, $-r \in \mathbb{N}$; *or*

(3) $r \notin \mathbb{Z}$, $\lambda \in \mathbb{N}$; *or*

(II) *if equation* (P_6) *has two distinct nonconstant rational solutions if and only if the parameters* $\lambda, r \in \mathbb{Z}$ *and satisfy*

(4) $\{\lambda > 1, r \leq 0\} \cup \{\lambda < 0, \lambda < r \leq 0\}$; *or*

(5) $\lambda > 1$, $r > \lambda$; *or*

(6) $\lambda = 1$, $r \neq 1$; *or*

(7) $\lambda = 0$, $r \in \mathbb{Z}$.

Moreover, equation (P_6) has infinite distinct nonconstant rational solutions.
For each case, the corresponding rational solution is of the form

$$\omega_{(1)}(z) := \omega(z) = \sum_{i=0}^{m-1} (-1)^{m-1-i} \frac{\lambda}{m-1+\lambda-i} \binom{m-1}{i} z^{1+i}(z-1)^{m-1-i};$$

$$\omega_{(2)}(z) := \omega(z) = 1 + \frac{r}{\lambda+1}\left(\frac{z-1}{z}\right) + \frac{r(r+1)}{(\lambda+1)(\lambda+2)}\left(\frac{z-1}{z}\right)^2 + \cdots + \frac{r(r+1)\cdots(-2)(-1)}{(\lambda+1)(\lambda+2)\cdots(\lambda-r)}\left(\frac{z-1}{z}\right)^{-r};$$

$$\omega_{(3)}(z) := \omega(z) = -\frac{\lambda}{r-1}\left(\frac{z}{z-1}\right) - \frac{\lambda(\lambda-1)}{(r-1)(r-2)}\left(\frac{z}{z-1}\right)^2 - \cdots - \frac{\lambda!}{(r-1)(r-2)\cdots(r-\lambda)}\left(\frac{z}{z-1}\right)^{\lambda};$$

$$\omega_{(4)}(z) := \omega(z) = \omega_{(2)}(z) + C\frac{z^r}{(z-1)^{\lambda}};$$

$$\omega_{(5)}(z) := \omega(z) = \omega_{(3)}(z) + C\frac{z^r}{(z-1)^{\lambda}};$$

$$\omega_{(6)}(z) := \omega(z) = \frac{z}{(1-r)(z-1)} + C\frac{z^r}{(z-1)};$$

$$\omega_{(7)}(z) := \omega(z) = Cz^r.$$

Proof (I) By Theorem 4.3 and Theorem 4.5,there exists a constant c_0 such that $\omega(z) = \lambda \frac{z^r}{(z-1)^{\lambda}}\big(y(z) + c_0\big)$. It is clear that $\lambda \neq 0$, $\frac{z^r}{(z-1)^{\lambda}}$ is not a rational. Thus either r or λ is not an integer.

If $\omega(z)$ is a polynomial with degree $m \geq 1$, then $\lim_{z\to\infty} z\frac{\omega'}{\omega} = r - \lambda$. Thus, by Theorem 4.5, we deduce

$$\begin{aligned}
\omega_{(1)}(z) &= \lambda \frac{z^{m+\lambda}}{(z-1)^{\lambda}} \int \frac{(z-1)^{\lambda-1}}{z^{\lambda+m}}\, dz \\
&= \lambda \frac{z^{m+\lambda}}{(z-1)^{\lambda}} \int \left(\frac{z-1}{z}\right)^{\lambda-1} \left(1 - \frac{(z-1)}{z}\right)^{m-1} d\frac{z-1}{z} \\
&= \lambda \frac{z^{m+\lambda}}{(z-1)^{\lambda}} \int \sum_{i=0}^{m-1} \binom{m-1}{i} (-1)^{m-1-i} \left(\frac{z-1}{z}\right)^{m-2+\lambda-i} d\frac{z-1}{z} \\
&= \lambda \frac{z^{m+\lambda}}{(z-1)^{\lambda}} \sum_{i=0}^{m-1} \frac{1}{m-1+\lambda-i} \binom{m-1}{i} (-1)^{n-1-i} \left(\frac{z-1}{z}\right)^{m-1+\lambda-i} \\
&= \lambda z^{m} \sum_{i=0}^{m-1} \frac{1}{m-1+\lambda-i} \binom{m-1}{i} (-1)^{n-1-i} \left(\frac{z-1}{z}\right)^{m-1-i} \\
&= \sum_{i=0}^{m-1} (-1)^{m-1-i} \frac{\lambda}{m-1+\lambda-i} z^{1+i} (z-1)^{m-1-i}.
\end{aligned}$$

If $\omega(z)$ is not a polynomial, then λ, r, q satisfy (2) or (3) corresponding to case (1) or (2) in Theorem 4.5. Moreover, by the partial integration, we can obtain

$$\begin{aligned}
\omega_{(2)}(z) &= \lambda \frac{z^{r}}{(z-1)^{\lambda}} \int \frac{(z-1)^{\lambda-1}}{z^{r}}\, dz \\
&= \frac{z^{r}}{(z-1)^{\lambda}} \int z^{-r}\, d(z-1)^{\lambda} \\
&= \frac{z^{r}}{(z-1)^{\lambda}} \left[z^{-r}(z-1)^{\lambda} - \int (z-1)^{\lambda}\, dz^{-r} \right] \\
&= 1 + \frac{rz^{r}}{(\lambda+1)(z-1)^{\lambda}} \int z^{-r-1}\, d(z-1)^{\lambda+1} \\
&= \cdots \\
&= 1 + \frac{r}{\lambda+1} \left(\frac{z-1}{z}\right) + \frac{r(r+1)}{(\lambda+1)(\lambda+2)} \left(\frac{z-1}{z}\right)^{2} \\
&\qquad + \cdots + \frac{r(r+1)\cdots(-2)(-1)}{(\lambda+1)(\lambda+2)\cdots(\lambda-r)} \left(\frac{z-1}{z}\right)^{-r},
\end{aligned}$$

and

$$\begin{aligned}
\omega_{(3)}(z) &= \lambda \frac{z^{r}}{(z-1)^{\lambda}} \int \frac{(z-1)^{\lambda-1}}{z^{r}}\, dz \\
&= -\frac{\lambda}{r-1} \frac{z^{r}}{(z-1)^{\lambda}} \int (z-1)^{\lambda-1}\, dz^{-r+1}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\lambda}{r-1}\frac{z^r}{(z-1)^\lambda}\left[z^{-r+1}(z-1)^{\lambda-1} - \int z^{-r+1}\,d(z-1)^{\lambda-1}\right]\\
&= -\frac{\lambda}{r-1}\left(\frac{z}{z-1}\right) - \frac{\lambda(\lambda-1)}{(r-1)(r-2)}\frac{z^r}{(z-1)^\lambda}\int (z-1)^{\lambda-2}\,dz^{-r+2}\\
&= \cdots\\
&= -\frac{\lambda}{r-1}\left(\frac{z}{z-1}\right) - \frac{\lambda(\lambda-1)}{(r-1)(r-2)}\left(\frac{z}{z-1}\right)^2\\
&\qquad - \cdots - \frac{\lambda!}{(r-1)(r-2)\cdots(r-\lambda)}\left(\frac{z}{z-1}\right)^\lambda.
\end{aligned}
$$

Conversely, it is easy to verify that for each $i = 1, 2, 3$, $\omega_{(i)}(z)$ is the exactly nonconstant rational solution of equation (P_6) satisfying condition (i), respectively.

(II) Let $\omega_1(z)$ and $\omega_2(z)$ be two distinct nonconstant rational solutions of equation (P_6). By Theorem 4.3 and Theorem 4.5, we see that there exist two distinct constants C_1 and C_2 such that $\omega_1 - \omega_2 := \lambda(C_1 - C_2)\frac{z^r}{(z-1)^\lambda}$ is a rational function. Therefore $r, \lambda \in \mathbb{Z}$ and then $q = r - \lambda - 1 \in \mathbb{Z}$, equation ($P_6$) has infinite distinct rational solutions. At the same time, the primitive $y(z)$ in Theorem 4.3 must be a rational function.

If case $\{\lambda > 1, 0 < r \le \lambda\}$ occurs, then

$$
\begin{aligned}
y(z) &= \int \frac{(z-1)^{\lambda-1}}{z^r}\,dz = \int \sum_{i=0}^{\lambda-1}\binom{\lambda-1}{i} z^{i-r}(-1)^{\lambda-1-i}\,dz\\
&= \sum_{\substack{i=0\\ i\ne r-1}}^{\lambda-1}\frac{(-1)^{\lambda-1-i}}{i-r+1}\binom{\lambda-1}{i} z^{i-r+1} + (-1)^{\lambda+r}\binom{\lambda-1}{r-1}\ln z + C,
\end{aligned}
$$

which is not a rational function.

If case $\{\lambda < 0, r > 0\}$ occurs, then

$$
\begin{aligned}
y(z) &= \int \frac{(z-1)^{\lambda-1}}{z^r}\,dz = \frac{1}{\lambda}\int \frac{1}{z^r}\,d(z-1)^\lambda\\
&= \frac{(z-1)^\lambda}{\lambda z^r} + \frac{r}{\lambda}\int \frac{(z-1)^\lambda}{z^{r+1}}\,dz\\
&= \frac{(z-1)^\lambda}{\lambda z^r} + \frac{r}{\lambda(\lambda+1)}\frac{(z-1)^{\lambda+1}}{z^{r+1}} + \cdots + \frac{r(r+1)\cdots(r-\lambda-2)}{\lambda(\lambda+1)\cdots(-1)}\frac{(z-1)^{-1}}{z^{r-\lambda-1}}\\
&\quad + \frac{r(r+1)\cdots(r-\lambda-1)}{\lambda(\lambda+1)\cdots(-1)}\int \frac{dz}{(z-1)z^{r-\lambda}},
\end{aligned}
$$

which is not a rational function by Theorem 4.4.

If case $\{\lambda < 0, r \le \lambda\}$ occurs, then

$$\begin{aligned}
y(z) &= \int \frac{(z-1)^{\lambda-1}}{z^r}\,dz = \int \frac{(z-1+1)^{-r}}{(z-1)^{1-\lambda}}\,dz \\
&= \int \sum_{i=0}^{-r} \binom{-r}{i} (z-1)^{i-1+\lambda}\,dz \\
&= \sum_{\substack{i=0 \\ i\neq-\lambda}}^{-r} \frac{1}{i+\lambda}\binom{-r}{i}(z-1)^{i+\lambda} + \binom{-r}{-\lambda}\ln(z-1) + C,
\end{aligned}$$

which is not a rational function.

If $\lambda = 1$ and $r = 1$, then

$$y(z) = \int \frac{dz}{z} = \ln z + C,$$

which is not a rational function, too.

All of this infers that r, λ satisfy one of condition (4) to condition (7). Furthermore, we can get the expresses as following in the same manner.

When case (4) occurs, we have

$$\begin{aligned}
\omega_{(4)}(z) &= \lambda \frac{z^r}{(z-1)^\lambda} \int \frac{(z-1)^{\lambda-1}}{z^r}\,dz \\
&= \frac{z^r}{(z-1)^\lambda} \int z^{-r}\,d(z-1)^\lambda \\
&= \frac{z^r}{(z-1)^\lambda}\left[z^{-r}(z-1)^\lambda - \int (z-1)^\lambda\,dz^{-r}\right] \\
&= 1 + \frac{rz^r}{(\lambda+1)(z-1)^\lambda} \int z^{-r-1}\,d(z-1)^{\lambda+1} = \cdots \\
&= 1 + \frac{r}{\lambda+1}\left(\frac{z-1}{z}\right) + \frac{r(r+1)}{(\lambda+1)(\lambda+2)}\left(\frac{z-1}{z}\right)^2 \\
&\qquad + \cdots + \frac{r(r+1)\cdots(-2)(-1)}{(\lambda+1)(\lambda+2)\cdots(\lambda-r)}\left(\frac{z-1}{z}\right)^{-r} + C\frac{z^r}{(z-1)^\lambda} \\
&= \omega_{(2)}(z) + C\frac{z^r}{(z-1)^\lambda}.
\end{aligned}$$

When case (5) occurs, we get

$$\begin{aligned}
\omega_{(5)}(z) &= \lambda \frac{z^r}{(z-1)^\lambda} \int \frac{(z-1)^{\lambda-1}}{z^r}\,dz \\
&= -\frac{\lambda}{r-1}\frac{z^r}{(z-1)^\lambda} \int (z-1)^{\lambda-1}\,dz^{-r+1}
\end{aligned}$$

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$$
\begin{aligned}
&= -\frac{\lambda}{r-1}\frac{z^r}{(z-1)^\lambda}\left[z^{-r+1}(z-1)^{\lambda-1} - \int z^{-r+1}\,d(z-1)^{\lambda-1}\right]\\
&= -\frac{\lambda}{r-1}\left(\frac{z}{z-1}\right) - \frac{\lambda(\lambda-1)}{(r-1)(r-2)}\frac{z^r}{(z-1)^\lambda}\int (z-1)^{\lambda-2}\,dz^{-r+2}\\
&= \cdots\\
&= -\frac{\lambda}{r-1}\left(\frac{z}{z-1}\right) - \frac{\lambda(\lambda-1)}{(r-1)(r-2)}\left(\frac{z}{z-1}\right)^2\\
&\qquad - \cdots - \frac{\lambda!}{(r-1)(r-2)\cdots(r-\lambda)}\left(\frac{z}{z-1}\right)^\lambda + C\frac{z^r}{(z-1)^\lambda}\\
&= \omega_{(3)}(z) + C\frac{z^r}{(z-1)^\lambda}.
\end{aligned}
$$

The rest is trivial.

Theorem 4.7 *Let $\omega(z)$ is a nonconstant rational solution of equation* (P_6). *Then*

(1) *any pole $z = z_0$ of ω is simple and $z_0 \neq 0, 1$, $\mathrm{Res}_{z=z_0}\,\omega = \frac{1}{p}z_0(1-z_0)$, if $\alpha \neq 0$.*
(2) *$\omega(z)$ is one of the forms $\frac{P(z)}{Q(z)}$, $\frac{-\lambda}{r-\lambda} + \frac{P(z)}{Q(z)}$ and $\frac{1+\lambda-r}{p}z + \frac{\lambda p+(r-\lambda-1)(\lambda+q)}{2-(r-\lambda)} + \frac{P(z)}{Q(z)}$ where $\deg P = \deg Q - 1$, if $\alpha \neq 0$.*
(3) *$p \in \mathbb{Z}$ or $\lambda \in \mathbb{Z}$ or $r \in \mathbb{Z}$ or $q \in \mathbb{Z}$.*

Proof By Theorem 4.2, we see that $\omega(z)$ satisfies the equation (4.2). If $\alpha \neq 0$, then we know that from (4.2) any pole $z = z_0$ of $\omega(z)$ must be simple and $z_0 \neq 0, 1$. Moreover we have

$$
\begin{aligned}
\mathrm{Res}_{z=z_0}\,\omega &= \lim_{z\to z_0}(z-z_0)\omega\\
&= \lim_{z\to z_0}(z-z_0)\left[\frac{z(z-1)}{p}\frac{\omega'}{\omega} - \frac{\lambda z}{\omega p} - \frac{(r-\lambda)z+(p+r)}{p}\right]\\
&= \frac{1}{p}z_0(1-z_0).
\end{aligned}
$$

Conclusion 1 follows.

Consider the Laurent expression of $\omega(z)$ near $z = \infty$:

$$\omega(z) = az^n + bz^{n-1} + O(z^{n-2})$$

for some $a, b \in \mathbb{C}$, $n \in \mathbb{N}$, $a \neq 0$. Substitute this into (4.2). Then comparing coefficients of the leading and next terms and noting conclusion 1, we can get conclusion 2.

Now we prove conclusion 3.

If $\alpha = 0$, then $p = 0 \in \mathbb{Z}$.

If $\alpha \neq 0$, then $\omega(z) = \frac{P(z)}{Q(z)}$ implies $\lambda = 0 \in \mathbb{Z}$ by the equation (4.2). When $\omega(z) = \frac{-\lambda}{r-\lambda} + \frac{P(z)}{Q(z)}$ occurs, let $\deg Q = n, z_1, \ldots, z_n$ be zeros of $Q(z)$; from conclusion 1

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and conclusion 2 we obtain

$$\omega(z)=\frac{-\lambda}{r-\lambda}+\frac{1}{p}\sum_{i=1}^{n}\frac{z_i(1-z_i)}{z-z_i}. \tag{4.6}$$

It holds that from the equation (4.2) $\omega(0)=0, 1+\frac{r}{p}$; $\omega(1)=1, \frac{\lambda}{p}$. Therefore $\omega(0)=0$, $\omega(1)=1$ and equation (4.6) gives $p=n$; $\omega(0)=0$, $\omega(1)=\frac{\lambda}{p}$ and equation (4.6) implies $\lambda=n$; $\omega(0)=1+\frac{r}{p}$, $\omega(1)=1$ and equation (4.6) deduces $r=-n$; $\omega(0)=1+\frac{r}{p}$, $\omega(1)=\frac{\lambda}{p}$ and equation (4.6) gives $q=-(n+1)$.

When $\omega(z)=az+b+\frac{P(z)}{Q(z)}$ occurs where $a=\frac{1+\lambda-r}{p}$, let $\deg Q=n$, $z_1,\dots,z_n$ be zeros of $Q(z)$ (in this case, n maybe is zero), from conclusion 1 and conclusion 2 we obtain

$$\omega(z)=\left(1-\frac{q}{p}\right)z+b+\frac{1}{p}\sum_{i=1}^{n}\frac{z_i(1-z_i)}{z-z_i}. \tag{4.7}$$

Hence $\omega(0)=0$, $\omega(1)=1$ and equation (4.7) gives $q=n$; $\omega(0)=0$, $\omega(1)=\frac{\lambda}{p}$ and equation (4.7) implies $r=n+1$; $\omega(0)=1+\frac{r}{p}$, $\omega(1)=1$ and equation (4.7) deduces $\lambda=-(n+1)$; $\omega(0)=1+\frac{r}{p}$, $\omega(1)=\frac{\lambda}{p}$ and the equation (4.2) gives $p=-n$. Note that $\lambda=0$, $r=0$ and $q=-1$ are trivial; we have proved conclusion 3.

The proof of this theorem is complete. ■

Theorem 4.8 *Let $p\in\mathbb{Z}$, $\lambda rq\neq 0$, $p-\lambda\neq 1$ and condition (3.4) hold. Suppose that $\omega(z)$ is a nonconstant rational solution of equation* (P_6)*. Then*

$$S_+\big(\omega(z)\big)=\omega(z)+\frac{2(z+1)f_+'-4\omega f_+'}{\Phi_+(\omega)}$$

and

$$S_-\big(\omega(z)\big)=\omega(z)+\frac{2(z+1)f_-'-4\omega f_-'}{\Phi_-(\omega)}$$

are rational solutions of equation (P_6) *with parameters*

$$p_+=p+1,\quad \lambda_+=-(-\lambda-1),\quad r_+=r,\quad q_+=q$$

and

$$p_-=-(-p+1),\quad \lambda_-=\lambda-1,\quad r_-=r,\quad q_-=q$$

respectively, where S_+ and S_- are two special Bäcklund transformations.

Proof If we can prove $\Phi_+(\omega)\not\equiv 0$ and $\Phi_-(\omega)\not\equiv 0$ then Theorem C will give this theorem.

Suppose that $\Phi_+(\omega) \equiv 0$. Then we get that from the conditions and Theorem C

$$f_+ = \frac{\lambda(z+1)}{(z-1)\omega} + \frac{2p\omega}{z-1} + \frac{(2r-3\lambda-p)z-3p-2r-\lambda}{2(z-1)} - \left(\frac{1}{2}+\frac{\mu}{4}\right),$$

$$z(z-1)f_+' = \frac{1-z}{2}\upsilon - \frac{1}{2}\left[\frac{\mu}{2}(z-1)+\kappa(z+1)\right]f_+ - \frac{z-1}{2}f_+^2.$$

Combining equation (4.2), we have

$$-\frac{\lambda^2(1+z)}{2} + a_1(z)\omega + a_2(z)\omega^2 + a_3(z)\omega^3 + \frac{4p^2}{(z-1)}\omega^4 = 0, \tag{4.8}$$

where

$$\begin{aligned}
a_1 = {} & -\frac{\lambda(z+1)}{z-1}[(r-\lambda)z-(p+r)] - \frac{2\lambda z)}{z-1} + \frac{2(z-1)}{\lambda(z+1)}\left[\frac{\mu}{2}(z-1)+\kappa(z+1)\right]\\
& + \lambda(z+1)\left[\frac{(2r-3\lambda-p)z-3p-2r-\lambda}{2(z-1)} - \left(\frac{1}{2}+\frac{\mu}{4}\right)\right],\\
a_2 = {} & \frac{2(2\lambda p+p+\lambda)}{z-1} + 2p\lambda + 2(p+\lambda) + \frac{z-1}{2}\upsilon\\
& + \frac{1}{2}\left(\frac{\mu}{2}(z-1)+\kappa(z+1)\right)\left[\frac{(2r-3\lambda-p)z-3p-2r-\lambda}{2(z-1)} - \left(\frac{1}{2}+\frac{\mu}{4}\right)\right]\\
& + \frac{z-1}{2}\left[\frac{(2r-3\lambda-p)z-3p-2r-\lambda}{2(z-1)} - \left(\frac{1}{2}+\frac{\mu}{4}\right)\right]^2,\\
a_3 = {} & \frac{2p}{z-1}[(1+r-\lambda)z-p-\lambda] + \frac{p}{z-1}\left[\frac{\mu}{2}(z-1)+\kappa(z+1)\right]\\
& + 2p\left[\frac{(2r-3\lambda-p)z-3p-2r-\lambda}{2(z-1)} - \left(\frac{1}{2}+\frac{\mu}{4}\right)\right]^2.
\end{aligned}$$

If $p \neq 0$, then from (4.8) and Theorem 4.7 we can obtain that $\omega(z)$ is a polynomial. Moreover, still by Theorem 4.7, we get $\omega(z) \equiv az+b$ where $a = \frac{1}{p}(1+\lambda-r)$. Substituting it into equation (4.2) and then comparing the coefficient of the leading term z^2, we have that $q = 0$. This is impossible. Therefore $p = 0$, and (4.8) becomes

$$-\frac{\lambda^2(1+z)}{2} + a_1(z)\omega + a_2(z)\omega^2 = 0 \tag{4.9}$$

Suppose that $a_2(z) \not\equiv 0$ or $a_1(z) \not\equiv 0$. From (4.9), transformation T_3 and Theorem 4.7, we can deduce that $\frac{z}{\omega} := az+b+\frac{c}{z+1}$ and $\omega(0) \neq 0$, where $a, b, c \in \mathbb{C}$. Note that any pole of $\omega(z)$ is either zero or one by Theorem 4.4, we then deduce that $\omega(z) = \frac{z+1}{az}$ or $\frac{z+1}{a(z-1)}$. However, the two forms of $\omega(z)$ are not appeared in Theorem 4.6. This contradiction shows that $a_2(z) \equiv 0$ and $a_1(z) \equiv 0$. Hence $\lambda = 0$, a contradiction. It implies that $\Phi_+(\omega) \not\equiv 0$.

Similarly, we can get $\Phi_-(\omega) \not\equiv 0$. ■

Theorem 4.9 *Let $p = 1+\lambda+q-r \in \mathbb{Z}$, $2\alpha = p^2 \neq 0$, $(\lambda+p)^2 = -2\beta$, $r^2 = 2\gamma$, $q^2 = 1-2\delta$, $\lambda rq \neq 0$. Then*

(I) *if equation* (P_6) *has exactly one nonconstant rational solution* $\omega(z)$ *if and only if* λ, r, q *satisfy (1) or (2) or (3).*

(II) *if equation* (P_6) *has two distinct nonconstant rational solutions if and only if* λ, r, q *satisfy (4) or (5) or (6) or (7).*

Moreover, equation (P_6) *has infinite distinct nonconstant rational solutions. Furthermore,* $\omega(z)$ *are forms of*

$$\omega_{p.i}(z) := \omega(z) = S_p\big(\omega_{(i)}(z)\big), \quad i = 1, 2, \dots, 7 \tag{4.10}$$

respectively, where parameters $\tilde{p} = p \in \mathbb{Z}$, $\tilde{\lambda} = (\lambda + p)$, $\tilde{r} = r$, $\tilde{q} = q$; λ, r, q *satisfy the corresponding conditions (1), (2), ... ,(7).*

Proof In view of Theorem 4.7 and the Bäcklund transformations S_+ and S_- in Theorem 4.8, we can determine the rational solutions below:

When $p = 0$, Theorem 4.6 gives

$$\omega_{0.i}(z) := S_0\big(\omega_{(i)}(z)\big) = \omega_{(i)}(z).$$

When $p = n \in \mathbb{N}$, Theorem 4.8 deduces

$$\begin{aligned}
\omega_{1.i}(z) :&= S_1\big(\omega_{0.i}(z)\big) = S_+\big(\omega_{(i)}(z)\big) \\
&= \omega_{(i)}(z) + \frac{2(z+1)f_1' - 4\omega_{(i)}f_1'}{\Phi_1\big(\omega_{(i)}(z)\big)}, \\
\omega_{n.i}(z) :&= S_n\big(\omega_{(i)}(z)\big) = S_+\big(\omega_{n-1.i}(z)\big) \\
&= \omega_{n-1.i}(z) + \frac{2(z+1)f_n' - 4\omega_{n-1.i}f_n'}{\Phi_n\big(\omega_{n-1.i}(z)\big)}, \\
\omega_{-1.i}(z) :&= S_{-1}\big(\omega_{(i)}(z)\big) = S_-\big(\omega_{(i)}(z)\big) \\
&= \omega_{(i)}(z) + \frac{2(z+1)f_{-1}' - 4\omega_{(i)}f_{-1}'}{\Phi_{-1}\big(\omega_{(i)}(z)\big)}, \\
\omega_{-n.i}(z) :&= S_{-n}\big(\omega_{(i)}(z)\big) = S_-\big(\omega_{-(n-1).i}(z)\big) \\
&= \omega_{-(n-1).i}(z) + \frac{2(z+1)f_{-n}' - 4\omega_{-(n-1).i}f_{-n}'}{\Phi_{-n}\big(\omega_{-(n-1).i}(z)\big)},
\end{aligned}$$

are the rational solutions of equation (P_6), where parameters satisfy

$$\alpha_0 = 0, \quad \alpha_n = \alpha_{n-1} - 1 = -n, \quad \tilde{p}_n = \tilde{p}_{n-1} + 1 = n;$$

$$\beta_0 = -\lambda, \quad \beta_n = \beta_{n-1} - 1 = -\lambda - n, \quad \tilde{\lambda}_n = -(-\tilde{\lambda}_{n-1} - 1) = \lambda + n;$$

$$\alpha_{-n} = \alpha_{-n-1} + 1 = n, \quad \tilde{p}_{-n} = -\tilde{p}_{-(n-1)} + 1 = -n;$$

$$\beta_{-n} = \beta_{-(n-1)} + 1 = \lambda + n, \quad \tilde{\lambda}_{-n} = (\tilde{\lambda}_{-(n-1)} - 1) = \lambda - n.$$

It follows that $\kappa_n = \kappa_{-n} = -\lambda - 1 \neq 0$ (otherwise, $\lambda = -1$ implies $r = 0$ by Theorem 4.6).

The rest is trivial. ∎

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5 Proofs of Main Results

Proof of Theorem 1.1 First, Theorem 4.1 shows that there is a constant solution in equation (P_6) if and only if the parameters of equation (P_6) satisfies case $\{\mathcal{I}.\delta = 0\}$. Moreover, the transformations $T_{20} = T_2 \circ (T_2 \circ T_1 \circ T_3)^2$, $T_{10} = (T_2 \circ T_3)^2$ and $T_4 = T_1 \circ T_3$ change this result to other three cases $\{\mathcal{I}.\alpha = \frac{1}{2}\}$, $\{\mathcal{I}.\beta = -\frac{1}{2}\}$ and $\{\mathcal{I}.\gamma = \frac{1}{2}\}$. The rational solution in these cases are generated as shown in the following figure:

$$\begin{tikzcd}
 & \{\mathcal{I}.\alpha = \frac{1}{2}\} & \\
\{\mathcal{I}.\beta = -\frac{1}{2}\} & \{\mathcal{I}.\delta = 0\} \arrow[l, "T_{10}"'] \arrow[u, "T_{20}"'] \arrow[r, "T_4"] & \{\mathcal{I}.\gamma = \frac{1}{2}\}
\end{tikzcd}$$

Second, Theorem 4.6 tells us that if $\alpha = 0$, then equation (P_6) has a nonconstant rational solution if and only if the parameters belong to one of cases (1)–(7). Starting from the case $\{\mathcal{II}.\alpha = 0\}$, the transformations T_1, $T_{15} = T_1 \circ T_2 \circ T_1$ and $T_8 = (T_2 \circ T_1 \circ T_3)^2$ change it to the cases $\{\mathcal{II}.\beta = 0\}$, $\{\mathcal{II}.\gamma = 0\}$ and $\{\mathcal{II}.\delta = \frac{1}{2}\}$, respectively. And the Bäcklund transformation S_n in Theorem 4.9 changes the case $\{\mathcal{II}.\alpha = 0\}$ to the case $\{\mathcal{II}.\alpha = n^2\}$ for each $n \in \mathbb{Z}$. Furthermore, the transformations T_1, T_{15} and T_8 change $\{\mathcal{II}.\alpha = n^2\}$ to the cases $\{\mathcal{II}. -2\beta = n^2\}$, $\{\mathcal{II}.2\gamma = n^2\}$ and $\{\mathcal{II}.1 - 2\delta = n^2\}$, respectively. The rational solutions in these cases are generated as shown in the figure below:

$$\begin{tikzcd}
 & \{\mathcal{II}.\alpha = 0\} \arrow[d, shift left, "S_n"] & \\
\{\mathcal{II}.2\gamma = n^2\} & \{\mathcal{II}.2\alpha = n^2\} \arrow[l, "T_{15}"'] \arrow[r, "T_8"] \arrow[u, shift left, "S_{-n}"] & \{\mathcal{II}.1 - 2\delta = n^2\} \\
 & \{\mathcal{II}.2\beta = n^2\} \arrow[u, "T_1"'] &
\end{tikzcd}$$

This figure can also be expressed as following:

$$\begin{tikzcd}
 & \{\mathcal{II}.p = 0\} \arrow[d, shift left, "S_n"] & \\
\{\mathcal{II}.r = n\} & \{\mathcal{II}.p = n\} \arrow[l, "T_{15}"'] \arrow[r, "T_8"] \arrow[u, shift left, "S_{-n}"] \arrow[d, <->, "T_1"] & \{\mathcal{II}.q = n\} \\
 & \{\mathcal{II}.\lambda = n\} &
\end{tikzcd}$$

Therefore, at last, it is easy to see that this theorem holds by Theorem 4.6, Theorem 4.7 and above results. ■

Proof of Theorem 1.2 This theorem follows from Theorem 4.1, Theorem 4.6, Theorem 4.8 and the proof of Theorem 1.1. ■

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6 A List of Rational Solutions for $\mathrm{P}_6\ (\alpha, \beta, \gamma, \delta)$

Subcase	$\omega(z)$
$\{\mathfrak{I}.\delta = 0\}$	$h, \quad h(\neq 0,1) \in \mathbb{C}$
$\{\mathfrak{I}.\alpha = \frac{1}{2}\}$	$\frac{hz}{(h-1)z-1}, \quad h(\neq 0,1) \in \mathbb{C}$
$\{\mathfrak{I}.\beta = \frac{1}{2}\}$	$hz + 1 - h, \quad h(\neq 0,1) \in \mathbb{C}$
$\{\mathfrak{I}.\gamma = \frac{1}{2}\}$	$hz, \quad h(\neq 0,1) \in \mathbb{C}$
$\{\mathfrak{II}.p = 0.(1)\}$	$\sum_{i=0}^{m-1} \frac{\lambda(-1)^{m-1-i}}{m-1+\lambda-i} \binom{m-1}{i} z^{1+i}(z-1)^{m-1-i}$
$\{\mathfrak{II}.p = 0.(2)\}$	$\omega_{(2)}(z) := 1 + \frac{r}{\lambda+1}(\frac{z-1}{z}) + \frac{r(r+1)}{(\lambda+1)(\lambda+2)}(\frac{z-1}{z})^2 + \cdots + \frac{r(r+1)\cdots(-2)(-1)}{(\lambda+1)(\lambda+2)\cdots(\lambda-r)}(\frac{z-1}{z})^{-r}$
$\{\mathfrak{II}.p = 0.(3)\}$	$\omega_{(3)}(z) := -\frac{\lambda}{r-1}(\frac{z}{z-1}) - \frac{\lambda(\lambda-1)}{(r-1)(r-2)}(\frac{z}{z-1})^2 - \cdots - \frac{\lambda!}{(r-1)(r-2)\cdots(r-\lambda)}(\frac{z}{z-1})^{\lambda}$
$\{\mathfrak{II}.p = 0.(4)\}$	$\omega_{(2)}(z) + C\frac{z^r}{(z-1)^{\lambda}}$
$\{\mathfrak{II}.p = 0.(5)\}$	$\omega_{(3)}(z) + C\frac{z^r}{(z-1)^{\lambda}}$
$\{\mathfrak{II}.p = 0.(6)\}$	$\frac{z}{(1-r)(z-1)} + C\frac{z^r}{(z-1)}$
$\{\mathfrak{II}.p = 0.(7)\}$	Cz^r
$\{\mathfrak{II}.p = n.(i)\}$	$\omega_{n.i}(z) := S_n\big(\omega_{(i)}(z)\big) = \omega_{n-1.i}(z) + \frac{2(z+1)f_n'-4\omega_{n-1.i}f_n'}{\Phi_n\left(\omega_{n-1.i}(z)\right)}$
$\{\mathfrak{II}.p = -n.(i)\}$	$\omega_{-n.i}(z) := S_{-n}\big(\omega_{(i)}(z)\big) = \omega_{-(n-1).i}(z) + \frac{2(z+1)f_{-n}'-4\omega_{-(n-1).i}f_{-n}'}{\Phi_{-n}\left(\omega_{-(n-1).i}(z)\right)}$
$\{\mathfrak{II}.\lambda = 0.(i)\}$	$T_1\big(\omega_{(i)}(z)\big) = \omega_{(i)}^{-1}(\frac{1}{z})$
$\{\mathfrak{II}.\lambda \in \mathbb{Z}.(i)\}$	$T_1\big(\omega_{\lambda.i}(z)\big) = \omega_{\lambda.i}^{-1}(\frac{1}{z})$
$\{\mathfrak{II}.r = 0.(i)\}$	$T_{15}\big(\omega_{(i)}(z)\big) = [1 - \omega_{(i)}(\frac{z-1}{z})]^{-1}$
$\{\mathfrak{II}.r \in \mathbb{Z}.(i)\}$	$T_{15}\big(\omega_{r.i}(z)\big) = [1 - \omega_{r.i}(\frac{z-1}{z})]^{-1}$
$\{\mathfrak{II}.q = 0.(i)\}$	$T_8\big(\omega_{(i)}(z)\big) = \frac{z[\omega_{(i)}-1]}{\omega_{(i)}(z)-z}$
$\{\mathfrak{II}.q \in \mathbb{Z}.(i)\}$	$T_8\big(\omega_{q.i}(z)\big) = \frac{z[\omega_{q.i}-1]}{\omega_{q.i}(z)-z}$

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