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THE COHOMOLOGICAL DIMENSION OF A DIRECTED SET

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Let R be a ring with identity, and let $\boldsymbol{C}$ be a small, nonempty category. We denote the category of right R-modules by Ab^R and the category of contravariant functors $\boldsymbol{C} \rightarrow Ab^R$ by $Ab^{R\,C^*}$. The limit functor

$$\operatorname{colim}_C : Ab^{R\,C^*} \rightarrow Ab^R$$

is left exact, and its kth right derived functor is denoted by colim^k. The *R-cohomological dimension* of $\boldsymbol{C}$ is defined by

$$\operatorname{cd}_R \boldsymbol{C} = \sup\{k | \operatorname{colim}_C{}^k \neq 0\}.$$

If there is a unitary ring homomorphism $R \rightarrow S$, then it is not difficult to show that $\operatorname{cd}_S C \leqq \operatorname{cd}_R C$.

In this paper we shall obtain the following complete result for the case where $\boldsymbol{C}$ is a directed set. For convenience, we let $\aleph_{-1} = 1$, and we let ∞ denote any infinite ordinal.

THEOREM A. *Let $\aleph_n$ be the smallest cardinal number of a cofinal subset for the directed set $\boldsymbol{C}$ $(-1 \leqq n \leqq \infty)$. Then*

$$\operatorname{cd}_R \boldsymbol{C} = n + 1$$

for all nonzero rings R.

It is perhaps surprising that the result is independent of the ring. This is in contrast to the situation for general partially ordered sets, where the difference $\operatorname{cd}_Z \boldsymbol{C} - \operatorname{cd}_R \boldsymbol{C}$ can be arbitrarily large even if $\boldsymbol{C}$ is required to be finite [**4**, § 34].

The totally ordered case of Theorem A was obtained in [**4**, Corollary 36.9]. The general case for finite n will be obtained from the totally ordered case using Theorem B below. However, the method does not work for infinite n unless $\aleph_n$ is regular, and for the infinite case we revert to a technical lemma of Osofsky used in [**4**].

The following theorem was stated without proof by Roos in [**6**] for the case where U is the inclusion of a cofinal subset in a directed set. A proof for this case was given by Jensen in [**3**]. However, Jensen's proof does not work in the general case. We shall obtain the theorem as a consequence of an appropriate generalization of the "mapping theorem" of Cartan-Eilenberg [**1**, p. 150].

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THEOREM B. *Let $U : \mathbf{C} \to \mathbf{D}$ be a cofinal functor where $\mathbf{C}$ (and hence $\mathbf{D}$) is a filtered category. Then for each $k \geqq 0$ there is a natural isomorphism*

$$\operatorname{colim}_D{}^k N \simeq \operatorname{colim}_C{}^k UN$$

where $N \in Ab^{RD*}$.

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1. The mapping theorem. Notation and terminology will be for the most part as in [**4**]. In particular, the composition fg is to be read as first f and then g. If $\mathfrak{A}$ is a category, then $|\mathfrak{A}|$ is its class of objects and $\mathfrak{A}(A, B)$ is the set of morphisms from A to B. An *additive category* is a category equipped with an abelian group structure on each morphism set such that composition is bilinear. The additive category of abelian groups is denoted by Ab. By a *ringoid* we mean a small, additive category. If $\mathfrak{C}$ is a ringoid, then a *right $\mathfrak{C}$-module* is an additive functor $M : \mathfrak{C} \to Ab$. The category of right $\mathfrak{C}$-modules is denoted $Ab^{\mathfrak{C}}$. However, we shall use the traditional $\operatorname{Hom}_{\mathfrak{C}}(M, N)$ rather than $Ab^{\mathfrak{C}}(M, N)$ for the abelian group of natural transformations from M to N. If M is a right $\mathfrak{C}$-module, $C \in |\mathfrak{C}|$, $x \in M(C)$, and $\lambda \in \mathfrak{C}(C, C')$, then we denote $x\lambda = M(\lambda)(x) \in M(C')$. The category of *left $\mathfrak{C}$-modules* is the category $Ab^{\mathfrak{C}*}$. In this case, we write $\lambda x = M(\lambda)(x) \in M(C)$ for $x \in M(C')$ and $\lambda \in \mathfrak{C}(C, C')$. If M is a right $\mathfrak{C}$-module and N is a left $\mathfrak{C}$-module, then $M \otimes_{\mathfrak{C}} N$ is the abelian group defined by

$$M \otimes_{\mathfrak{C}} N = \bigoplus_{C \in |\mathfrak{C}|} M(C) \otimes_Z N(C)/K$$

where K is the subgroup of the numerator generated by all elements of the form

$$x\lambda \otimes y - x \otimes \lambda y, x \in M(C), y \in N(C'), \lambda \in \mathfrak{C}(C, C').$$

Let $U : \mathfrak{C} \to \mathfrak{D}$ be a map of ringoids, or in other words an additive functor. Then we have the functor $Ab^U : Ab^{\mathfrak{D}*} \to Ab^{\mathfrak{C}*}$ which composes with U, and which has $\mathfrak{D}(\ , U) \otimes_{\mathfrak{C}}$ as its left adjoint. If $Q_{\mathfrak{D}}$ is a left $\mathfrak{D}$-module, then the adjunction

$$\mathfrak{D}(\ , U) \otimes_{\mathfrak{C}} UQ_{\mathfrak{D}} \xrightarrow{\epsilon} Q_{\mathfrak{D}}$$

is given by

$$\epsilon_D(\mu \otimes y) = \mu y, \quad y \in Q_{\mathfrak{D}}(U(C)), \quad \mu \in \mathfrak{D}(D, U(C)).$$

Now if $Q_{\mathfrak{C}}$ is a left $\mathfrak{C}$-module and $\psi : Q_{\mathfrak{C}} \to UQ_{\mathfrak{D}}$ is a map (natural transformation) of $\mathfrak{C}$-modules, then composing $\mathfrak{D}(\ ,U) \otimes_{\mathfrak{C}} \psi$ with ϵ we obtain a map

$$\mathfrak{D}(\ ,U) \otimes_{\mathfrak{C}} Q_{\mathfrak{C}} \xrightarrow{g} Q_{\mathfrak{D}}$$

given explicitly by

$$g_D(\mu \otimes x) = \mu\psi(x), x \in Q_{\mathfrak{C}}(C), \mu \in \mathfrak{D}(D, U(C)).$$

Suppose now that X is a projective resolution for $Q_{\mathfrak{C}}$. Then $\mathfrak{D}(\ ,U) \otimes_{\mathfrak{C}} X$ is a $\mathfrak{D}$-projective left complex over $\mathfrak{D}(\ ,U) \otimes_{\mathfrak{C}} Q_{\mathfrak{C}}$, and so if Y is a projective resolution for $Q_{\mathfrak{D}}$, then we obtain a map of complexes

$$G : \mathfrak{D}(\ ,U) \otimes_{\mathfrak{C}} X \to Y$$

over g. The map G is unique up to homotopy, and so induces well defined maps

$H(UM \otimes_{\mathfrak{C}} X) = H(M \otimes_{\mathfrak{D}} \mathfrak{D}(\ ,U) \otimes_{\mathfrak{C}} X) \to H(M \otimes_{\mathfrak{D}} Y)$
$H(\mathrm{Hom}_{\mathfrak{D}^*}(Y, N)) \to H(\mathrm{Hom}_{\mathfrak{D}^*}(\mathfrak{D}(\ ,U) \otimes_{\mathfrak{C}} X, N) = H(\mathrm{Hom}_{\mathfrak{C}^*}(X, UN)),$

or in other words, maps

$$F^U : \mathrm{Tor}^{\mathfrak{C}}(UM, Q_{\mathfrak{C}}) \to \mathrm{Tor}^{\mathfrak{D}}(M, Q_{\mathfrak{D}})$$
$$F_U : \mathrm{Ext}_{\mathfrak{D}^*}(Q_{\mathfrak{D}}, N) \to \mathrm{Ext}_{\mathfrak{C}^*}(Q_{\mathfrak{C}}, UN)$$

for right $\mathfrak{D}$-modules M and left $\mathfrak{D}$-modules N.

MAPPING THEOREM. *In order that F^U be an isomorphism for all M, it is necessary and sufficient that*

(i) $g : \mathfrak{D}(\ ,U) \otimes_{\mathfrak{C}} Q_{\mathfrak{C}} \simeq Q_{\mathfrak{D}}$
(ii) $\mathrm{Tor}_n^{\mathfrak{C}}(\mathfrak{D}(\ ,U), Q_{\mathfrak{C}}) = 0$ *for* $n > 0$.

If these conditions are satisfied, then F_U is also an isomorphism for all N.

Proof. Assume F^U is an isomorphism. In particular, taking $M = \mathfrak{D}(D,\ )$ for any $D \in |\mathfrak{D}|$, we obtain conditions (i) and (ii).

Conversely, assume that (i) and (ii) hold. If X is a projective resolution of $Q_{\mathfrak{C}}$, then $H_n(\mathfrak{D}(\ ,U) \otimes_{\mathfrak{C}} X) = 0$ for $n > 0$ by (ii), so that $\mathfrak{D}(\ ,U) \otimes_{\mathfrak{C}} X$ is a $\mathfrak{D}$-projective resolution of $\mathfrak{D}(\ ,U) \otimes_{\mathfrak{C}} Q_{\mathfrak{C}}$. Since g is an isomorphism, it follows that G is a homotopy equivalence, and so F^U and F_U are isomorphisms.

The above proof is copied (needless to say) from Cartan-Eilenberg. However, there the theorem is stated only in the "augmented ring" situation, or in other words the case where $\mathfrak{C}$ and $\mathfrak{D}$ are rings, $Q_{\mathfrak{C}}$ and $Q_{\mathfrak{D}}$ are cyclic modules, and ψ takes the generator of $Q_{\mathfrak{C}}$ to the generator of $Q_{\mathfrak{D}}$. All of these conditions are too restrictive for our purposes.

2. Proof of Theorem B. Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be (nonadditive) categories. Recall that a functor $U : \boldsymbol{C} \to \boldsymbol{D}$ is *cofinal* if for each $D \in |\boldsymbol{D}|$ the comma category (D, U) (whose objects are the elements of $\boldsymbol{D}(D, U)$) is nonempty and connected. Recall also that a category $\boldsymbol{C}$ is *filtered* if every pair of objects are

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domains of morphisms with a common target, and if for every pair of morphisms α, α' with common domain and common target there is a morphism β such that $\alpha\beta = \alpha'\beta$.

Now let R be a ring, and let $\boldsymbol{C}$ be a small category. Then we have the ringoid $R\boldsymbol{C}$ whose right modules are the same as the $\boldsymbol{C}$-diagrams of right R-modules [**4**, § 2]. Furthermore, the colimit functor

$$\lim_C : Ab^{RC} \to Ab^R$$

is given by

$$\lim_C M = M \otimes_{RC} \Delta_C R$$

where $\Delta_C R$ is the constant $\boldsymbol{C}^*$-diagram at the left R-module R [**4**, § 16]. Similarly, the limit functor

$$\operatorname{colim}_C : Ab^{R^*C^*} \to Ab^{R^*}$$

is given by

$$\operatorname{colim}_C N = \operatorname{Hom}_{R^*C^*}(\Delta_C R, N).$$

Consider now a cofinal functor $U : \boldsymbol{C} \to \boldsymbol{D}$ where $\boldsymbol{C}$ (and hence $\boldsymbol{D}$) is filtered. Then we have the induced additive functor $R\boldsymbol{C} \to R\boldsymbol{D}$ which we still denote by U. Let us take $\mathfrak{C} = R\boldsymbol{C}$, $\mathfrak{D} = R\boldsymbol{D}$, $Q_{\mathfrak{C}} = \Delta_C R$, and $Q_{\mathfrak{D}} = \Delta_D R$ in the preceding section. Note that $\Delta_C R$ is $\Delta_D R$ composed with U, so that we may take ψ to be the identity. Since $\boldsymbol{C}$ is filtered, colim_C is exact, and so $\Delta_C R$ is flat. Hence, condition (ii) of the mapping theorem is satisfied. To verify condition (i), we consider the map

$$g_D : R\boldsymbol{D}(D, U) \otimes_{RC} \Delta_C R \to R.$$

It is given in this case by

$$g_D(\mu \otimes r) = r, \qquad \mu \in \boldsymbol{D}(D, U(C))$$

where r is considered as an element of the left R-module R sitting at C on the left and at D on the right. We define

$$f : R \to R\boldsymbol{D}(D, U) \otimes_{RC} \Delta_C R$$

as follows. Since the comma category (D, U) is nonempty, there is a $\mu \in \boldsymbol{D}(D, U(C))$ for some C, and so we can define $f(r) = \mu \otimes r$. Then from the fact that (D, U) is connected we see that f is independent of the choice of μ, and it follows easily that $g_D f$ is the identity. Since fg_D is the identity in any case, this establishes that g is an isomorphism. Theorem B is, therefore a special case of the mapping theorem.

3. Proof of Theorem A. If n is an ordinal, we let ω_n denote the first ordinal of cardinal number $\aleph_n$. If X is a directed set, then we define the *cofinality* of X ($\operatorname{cof} X$) to be n where $\aleph_n$ is the smallest cardinal number of a cofinal subset. Thus, $\operatorname{cof} X = -1$ if and only if X has a terminal element, and it is easy to see that $\operatorname{cof} X = 0$ if and only if X contains ω_0 as a (full) cofinal subset. If card $X = \aleph_n$ and each element of X is preceded by only a finite number of elements, then cof $X = n$. In particular, if X is the set of finite subsets (ordered by inclusion) of a set of cardinal number $\aleph_n$, then cof $X = n$. If ω_n is regular, then cof $\omega_n = n$. On the other hand if ω_n is not regular, then n cannot be the cofinality of any totally ordered set.

LEMMA 3.1. *Let X be a directed set of cofinality n. Then there is a cofinal map (functor) $U : X \to \omega_n$.*

Proof. Let $\{x_\alpha | \alpha < \omega_n\}$ be a cofinal subset of X. For $x \in X$, define $U(x)$ to be the first α such that $x \leqq x_\alpha$. Clearly U is order preserving, and if its image were not cofinal in ω_n, then there would be a cofinal subset of X of smaller cardinality.

From the lemma and Theorem B it follows that $\mathrm{cd}_R X \geqq \mathrm{cd}_R \omega_n$ if cof $X = n$. Hence, when ω_n is regular, we obtain from [**4**, Corollary 36.9]

$$\mathrm{cd}_R X \geqq n + 1. \tag{1}$$

Since also $\mathrm{cd}_R X \leqq n + 1$ [**4**, Corollary 16.2], this proves Theorem A in the case where ω_n is regular, and in particular, when n is finite. However, when ω_n is not regular (for example, when $n = \omega_0$), this argument breaks down.

To handle the infinite case, we recall that a *directed set of free generators* for a right $\mathfrak{C}$-module M is a set X of elements of the values $M(C)$ such that

(i) $x\lambda = 0, x \in X, \lambda \in \mathfrak{C} \Rightarrow \lambda = 0$,

(ii) the set X, ordered by $y \leqq x$ if $y = x\lambda$ for some $\lambda \in \mathfrak{C}$, is directed,

(iii) every element of M is of the form $x\lambda$ for some $x \in X$ and $\lambda \in \mathfrak{C}$.

If $Y \subset X, P_{-1}(Y)$ denotes the submodule of M generated by Y. The following lemma is an immediate consequence of a lemma of Osofsky [**5**] which was written down in the required generality in [**4**, Lemma 36.4].

LEMMA 3.2. *Let X be a directed set of free generators for a module M, and suppose that* cof $X > n$ *for some n satisfying $0 \leqq n < \infty$. If* hd $M \leqq k$ *where $0 < k < \infty$, then there is a directed subset $Y \subset X$ such that* cof $Y = n$ *and such that* hd $P_{-1}(Y) \leqq k$.

We need one more easy lemma concerning general preordered sets. A subset U of a preordered set X is *open* if $x \in U, y \leqq x \Rightarrow y \in U$.

LEMMA 3.3. *Let X be a preordered set and let U be an open subset. Let $E : Ab^{RU*} \to Ab^{RX*}$ be the functor which extends a diagram by adding zeros. Then* hd $F =$ hd $E(F)$ *for all $F \in Ab^{RU*}$.*

Proof. E is the left adjoint of the restriction functor T. Since E and T are both exact and ET is the identity on Ab^{RU*}, we see that $E(P)$ is projective if and only if P is projective. Hence, E preserves projective resolutions, and the first projective kernel in a projective resolution for F occurs at exactly the same point where the first projective kernel appears in the corresponding projective resolution for $E(F)$.

Now let X be any directed set, and consider the right RX^*-module $\Delta_X R$ where R is any nonzero ring. Then $\Delta_X R$ has a directed set of free generators which is isomorphic to X as a directed set, namely, the identity elements of R sitting at the various objects of X. We shall identify this directed set of free generators with X. If cof $X = \infty$, we wish to show that $\text{cd}_R X = \infty$, or in other words that hd $\Delta_X R = \infty$. Suppose that hd $\Delta_X R = n < \infty$. By Lemma 3.2, there is a directed set $Y \subset X$ of cofinality n such that hd $P_{-1}(Y) \leqq n$. But $P_{-1}(Y)$ is the diagram which has R at every element of the smallest open set U containing Y, and zeros elsewhere. Hence, by Lemma 3.3 its homological dimension is that of $\Delta_U R$, or in other words $\text{cd}_R U$. But this is contrary to inequality (1), since cof U = cof $Y = n$. This completes the proof of Theorem A.

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