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Nonstandard Ideals from Nonstandard Dual Pairs for $L^1(\omega)$ and $l^1(\omega)$

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Abstract. The Banach convolution algebras $l^1(\omega)$ and their continuous counterparts $L^1(\mathbb{R}^+, \omega)$ are much studied, because (when the submultiplicative weight function ω is radical) they are pretty much the prototypic examples of commutative radical Banach algebras. In cases of "nice" weights ω, the only closed ideals they have are the obvious, or "standard", ideals. But in the general case, a brilliant but very difficult paper of Marc Thomas shows that nonstandard ideals exist in $l^1(\omega)$. His proof was successfully exported to the continuous case $L^1(\mathbb{R}^+, \omega)$ by Dales and McClure, but remained difficult. In this paper we first present a small improvement: a new and easier proof of the existence of nonstandard ideals in $l^1(\omega)$ and $L^1(\mathbb{R}^+, \omega)$. The new proof is based on the idea of a "nonstandard dual pair" which we introduce. We are then able to make a much larger improvement: we find nonstandard ideals in $L^1(\mathbb{R}^+, \omega)$ containing functions whose supports extend all the way down to zero in $\mathbb{R}^+$, thereby solving what has become a notorious problem in the area.

1 Preliminary Definitions

Our notion of a *radical weight* $\omega\colon \mathbb{R}^+ \to \mathbb{R}^+$ will be the usual one; ω must be measurable and submultiplicative, and one must have $\omega(s)^{1/s} \to 0$ as $s \to \infty$.

Given a radical weight ω, we define the radical convolution algebras $l^1(\omega)$ and $L^1(\mathbb{R}^+, \omega)$ in the usual way. If the Banach algebra $\mathcal{A}$ is either $l^1(\omega)$ (the *discrete case*) or $L^1(\mathbb{R}^+, \omega)$ (the *continuous case*), we identify the dual space $\mathcal{A}^*$ with $l^\infty(1/\omega)$ or $L^\infty(\mathbb{R}^+, 1/\omega)$, so $\mathcal{A} \cap \mathcal{A}^*$ contains all the functions with compact support (bounded measurable functions in the continuous case) from $\mathbb{N} \to \mathbb{C}$ (discrete case) or $\mathbb{R}^+ \to \mathbb{C}$ (continuous case). Many of our proofs are equally valid in the discrete case or the continuous case. In particular, the proof that nonstandard ideals I exist with $\alpha(I) > 0$ is the same in both cases. Only when we want $\alpha(I) = 0$ do we have to specialise to the continuous case.

If $s \in \mathbb{N}_0$ (discrete case) or $\mathbb{R}^+ \cup \{0\}$ (continuous case) the *standard ideal* $I_s \subset \mathcal{A}$ is the collection of all functions $f \in \mathcal{A}$ that are supported in the interval $[s, \infty)$. Any nonzero closed ideal that is not one of the ideals I_s is called a *nonstandard ideal*. If $\mathcal{A}$ has no nonstandard ideals it is called *unicellular*. Also the right shift operator $R^s\colon \mathcal{A} \to \mathcal{A}$ is as usual the map with $R^s f(t) = f(t-s)$ (if $t > s$) or zero otherwise, and the left shift $L^s\colon \mathcal{A}^* \to \mathcal{A}^*$ is the dual map with $L^s f(t) = f(t+s)$. We know that $\|R^s\|_{\mathcal{A}} = \|L^s\|_{\mathcal{A}^*} \le \omega(s)$; and because ω is a *radical* weight we know that $L^s\colon \mathcal{A} \to \mathcal{A}$ and $R^s\colon \mathcal{A}^* \to \mathcal{A}^*$ are densely defined but *unbounded* operators. A finite linear combination of the maps L^s (or R^s) will be called a "generalised polynomial" in $L =$

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L^1 (or $R = R^1$), generalised because in the continuous case non-integer values of s are allowed.

2 A Brief History of the Problem

The main example in this area is the paper [6] by Marc Thomas, which solved a problem of Shilov dating back to 1940. The extension by Dales and McClure [2] to the continuous case leaves unsolved the question of whether there are nonstandard ideals with $\alpha(I) = 0$, something which Dales recognised as significant at the time, and that question is solved affirmatively in this paper.

In the positive direction, the main theorem is the result of Domar [3] which asserts that if the weight function ω is star-shaped, *i.e.*, $-\frac{1}{n}\log\omega(n)$ is an increasing function of $n \in \mathbb{N}$, then $l^1(\omega)$ has no nonstandard ideals. Domar also proved a continuous version of this result: if $\eta(t) = -\log\omega(t)$ is a convex function of t, and if for some $\varepsilon > 0$ one has $\eta(t)/t^{1+\varepsilon} \to 0$, then $L^1(\mathbb{R}^+, \omega)$ has no nonstandard ideals. There is also an account of this work in Dales [1].

The question of whether one can have nonstandard weak-$*$ closed ideals (see definition and discussion below) was raised by S. Grabiner (private communication; his interests in the weak-$*$ topology on $L^1(\mathbb{R}^+, \omega)$ and in nonstandard ideals are seen in [4, 5]). The question of whether one can have compact multiplication on the algebras $L^1(\mathbb{R}^+, \omega)$ and $l^1(\omega)$ and yet still have nonstandard ideals is also discussed and answered affirmatively by Thomas in the discrete case, and by Dales and McClure in the continuous case. In this paper we give examples of weak-$*$ closed ideals where the weight function can be chosen so as to give compact multiplication and yet nonstandard ideals still exist on $L^1(\mathbb{R}^+, \omega)$, with $\alpha(I) = 0$.

3 Further Definitions

The definitions of Section 1 are standard practise. Not quite so standard, but to us very useful, are the following:

Definition 3.1

(i) Let $\mathcal{A}$ denote the algebra $l^1(\omega)$ or $L^1(\mathbb{R}^+, \omega)$. Let $f \in \mathcal{A}$ and $\phi \in \mathcal{A}^*$. We define the function $[f:\phi]\colon \mathbb{N}_0 \to \mathbb{C}$ (respectively, $\mathbb{R}^+ \to \mathbb{C}$) by

$$[f:\phi](s) = \phi(R^s f). \tag{3.1}$$

We will refer to this function as the *interaction* between f and ϕ.

(ii) If $f, g\colon \mathbb{N} \to \mathbb{C}$ (discrete case) or $\mathbb{R}^+ \to \mathbb{C}$ (continuous case) and $s \in \mathbb{N}$ (discrete case) or $\mathbb{R}^+$, we say $f \ll s$ if f is supported on $[0, s]$, and $g \gg s$ if g is supported (essentially supported in the continuous case) on $[s, \infty)$. We say $g \gg f$ ("g lies beyond f") if $f \ll s$ and $g \gg s$ for some s (continuous case), but in the discrete case we require $f \ll s - 1$ and $g \gg s$ for some $s \in \mathbb{N}$. Either way, the condition given ensures $[g:f] = 0$ when $g \in \mathcal{A}$ and $f \in \mathcal{A}^*$ are such that g lies beyond f.

(iii) For nonzero f, we also define $\alpha(f)$ to be the minimum of the support of f (essential support in the continuous case), and $\beta(f)$ its maximum (or infinity

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if the support is not compact). If $I \subset \mathcal{A}$ is a nonzero closed ideal, we define $\alpha(I)$ to be the minimum of $\alpha(f)$ for $f \in I$ (and this really is a minimum, not an infimum, because if $\|f_i\| = 1$ and $\alpha(f_i)$ is a decreasing sequence tending to t, one has $\alpha(\sum_{n=1}^{\infty} \pm 2^{-n} f_n) = t$ for a suitable choice of signs).

(iv) If p is a nonzero generalised polynomial, we likewise define $\alpha(p)$ to be the largest power of R that divides $p(R)$, so that one has $\alpha(p(R)f) = \alpha(p) + \alpha(f)$ for all such p and f. We define $\beta(p)$ to be the degree of p.

(v) If $f \gg \phi$ with $f \in \mathcal{A}$ and $\phi \in \mathcal{A}^*$, then obviously $[f:\phi] = 0$. We say the elements $f \in \mathcal{A}$ and $\phi \in \mathcal{A}^*$ are a *nonstandard dual pair* if they are both nonzero, we do *not* have $f \gg \phi$, yet we find that the function $[f:\phi]$ is identically zero.

Obviously it is the last part of this definition that gives us a method—so to speak, a style of proof—when looking for nonstandard ideals. The algebra $\mathcal{A}$ as defined above will have nonstandard ideals if and only if it has nonstandard dual pairs; for $[f:\phi] = 0$ if and only if ϕ annihilates the closed principal ideal generated by f (which is easily seen to be the closed linear span of all the translates $R^s f$, $s \geq 0$). If that ideal is standard, *i.e.,* if it is the ideal I_α, $\alpha = \alpha(f)$, then the only functionals ϕ annihilating it will be those with $\phi \ll \alpha$ and hence, $\phi \ll f$. So the existence of a nonstandard dual pair (f, ϕ) definitely implies the existence of a nonstandard ideal (with $\alpha(I) = \alpha(f)$). Conversely, a simple application of the Hahn–Banach theorem shows that if there is any nonstandard ideal I, then there is a nonstandard dual pair (f, ϕ) with $\alpha(f) = \alpha(I)$. Thus, the hunt for nonstandard ideals becomes a hunt for nonstandard dual pairs.

One further subtlety, which can also be attacked using nonstandard dual pairs, is the issue of the weak-$*$ topology and the question of whether one can have weak-$*$ closed nonstandard ideals. In the discrete case $\mathcal{A} = l^1(\omega)$, the weak-$*$ topology is the topology $\sigma(l^1(\omega), c_0(1/\omega))$; in the continuous case it is defined to be the topology $\sigma(L^1(\mathbb{R}^+, \omega), C_0(\mathbb{R}^+, 1/\omega))$, where $C_0(\mathbb{R}^+, 1/\omega))$ is the Banach space of continuous functions $\phi\colon \mathbb{R}^+ \to \mathbb{C}$ such that $\phi(t)/\omega(t) \to 0$ as $t \to \infty$.

The Hahn–Banach theorem is again our friend when converting statements about weak-$*$ closed ideals into statements about dual pairs; one may check that there is a weak-$*$ closed, nonstandard ideal $I \subset \mathcal{A}$ with $\alpha(I) = t$ if and only if there is a nonstandard dual pair (f, ϕ) with $\phi \in c_0(1/\omega)$, (respectively, $C_0(\mathbb{R}^+, 1/\omega)$) and $\alpha(f) = t$.

4 Geometric Progressions and Cancellation in $[f:\phi]$

This paper is mainly about explicit construction of nonstandard dual pairs; let us assume that $\mathcal{A}$ is $l^1(\omega)$ or $L^1(\mathbb{R}^+, \omega)$ as above, with $f \in \mathcal{A}$ and $\phi \in \mathcal{A}^*$ elements of compact support such that we *do not* have $f \gg \phi$ but instead $\phi \gg f$. In the continuous case, the Titchmarsh convolution theorem tells us that the interaction $[f:\phi]$ is nonzero with

$$\alpha([f:\phi]) = \alpha(\phi) - \beta(f). \tag{4.1}$$

Equation (4.1) is, of course, obvious in the discrete case.

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Now choose any $a > \beta(\phi) - \alpha(f)$ so that $R^a f \gg \phi$ and $[R^a f : \phi] = 0$. Choose also a nonzero complex constant A. Since $[R^a f : R^a g] = [f : g]$ for any compactly supported f and g, there is some cancellation in the expression $[f + AR^a f : \phi - A^{-1}R^a\phi]$. In fact, two terms $[f : \phi] + [AR^a f : -A^{-1}R^a\phi]$ cancel out, the term $[R^a f : \phi] = 0$ so $[f + AR^a f : \phi - A^{-1}R^a\phi] = [f : A^{-1}R^a\phi]$, and $\alpha([f + AR^a f : \phi - A^{-1}R^a\phi]) = a + \alpha(\phi) - \beta(f) = a + \alpha([f : \phi])$. Indeed, we may take a finite "geometric progression"

$$\phi' = \sum_{k=0}^{n} (-A^{-1}R^a)^k \phi \tag{4.2}$$

and obtain even more cancellation: $[f + AR^a f : \phi'] = [f : (-A^{-1}R^a)^n \phi]$ with $\alpha([f + AR^a f : \phi']) = na + \alpha(\phi) - \beta(f) = na + \alpha([f : \phi])$. Of course, any hope that we might achieve $[f + AR^a f : \phi'] = 0$ by letting $n \to \infty$ is immediately dashed because ω has to be a radical weight; the infinite series $\sum_{k=0}^{\infty} (-A^{-1}R^a)^k \phi$ will certainly not converge, whether in $l^\infty(1/\omega)$ or in $L^\infty(\mathbb{R}^+, 1/\omega)$.

Nonetheless, "geometric progressions" can be used to perturb a pair (f, ϕ) in such a way that the new elements $f' = f + AR^a f$ and ϕ' as in (4.2) have an interaction that starts much further along the real line than $[f : \phi]$. We shall now use this idea to construct our first nonstandard dual pairs (f, ϕ).

5 Some Dual Pairs (f, ϕ) Which Are "Nonstandard if They Converge"

In this section we shall prove a lemma which can be used to exhibit nonstandard dual pairs in "real life" cases. We begin with a recursive definition that takes two increasing sequences of positive integers as its input, and gives an elaborate "progression of geometric progressions" as its output. This definition, and the lemma that follows it, are not quite complicated enough to give nonstandard ideals I with $\alpha(I) = 0$; but they form the basis of a very nice short proof that nonstandard ideals exist with $\alpha(I) > 0$. This proof is given in Section 7, after a suitable weight function is defined in Section 6. Then in Sections 8–9 we add the necessary extra complexity to get ideals with $\alpha(I) = 0$.

Definition 5.1 Let $(a_n)_{n=0}^{\infty}$ and $(A_n)_{n=0}^{\infty}$ be strictly increasing sequences of positive integers with $a_0 = 1$. Define $q_0(R) = R^2$, $\rho_0(R) = A_0 R$ (where R is, as usual, the right shift operator), and for $n \in \mathbb{N}_0$ define

$$p_n(R) = \sum_{k=0}^{n} A_k R^{a_k}. \tag{5.1}$$

Then recursively define polynomials $\rho_n(R), q_n(R), h_n(R), \delta_n(R)$ $(n \in \mathbb{N})$ as follows.

$$h_n(R) = -A_n^{-1} p_{n-1}(L) R^{a_n}, \tag{5.2}$$

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(N.B. The operator (5.2) is indeed a polynomial in R for all n, because L is a left inverse for R and the degree of p_{n-1} is $a_{n-1} < a_n$.)

$$\delta_n(R) = -A_n^{-1} R^{a_n} \rho_{n-1}(R), \tag{5.3}$$

$$q_n(R) = \sum_{k=0}^{a_n^2-1} h_n(R)^k \delta_n(R), \tag{5.4}$$

and finally

$$\rho_n(R) = h_n(R)^{a_n^2} \rho_{n-1}(R). \tag{5.5}$$

These definitions allow us to state our key lemma.

Lemma 5.2 *Let sequences be given as in Definition* 5.1*. Let $\mathcal{A} = l^1(\omega)$ or $L^1(\mathbb{R}^+, \omega)$ as usual, and let $f_0 \in \mathcal{A}$, $\phi_0 \in \mathcal{A}^*$ with $f_0 \ll \phi_0 \ll a_1/2 - 2$. Let us also assume the a_n increase sufficiently rapidly that $a_{n+1} > 2a_n$ for all $n \geq 0$. Noting that the degree $\beta(q_n)$ is a function of $a_0, a_1, \ldots a_n$, let us assume, as a further condition of rapid increase, that $\beta(q_n) < a_{n+1} - a_1/2$ for all $n \in \mathbb{N}_0$. In the continuous case, let us assume that ϕ_0 is a continuous function.*

Suppose that the weight ω is chosen in such a way that the sums

$$f = \sum_{n=0}^{\infty} A_n R^{a_n} f_0 = \lim_{n\to\infty} p_n(R) f_0 \tag{5.6}$$

and

$$\phi = \sum_{n=0}^{\infty} q_n(R)\phi_0 \tag{5.7}$$

are norm convergent in $\mathcal{A}$ and $\mathcal{A}^*$, respectively. Then (f, ϕ) is a nonstandard dual pair (with $\alpha(f) = 1 + \alpha(f_0)$, and ϕ weak-$*$ continuous).

In the course of proving this lemma, we shall also give some more explanation of Definition 5.1. Note that this lemma is not dependent on a specific choice of weight function ω. We just need to find one such that the sums (5.6) and (5.7) do indeed converge.

Proof The method of proof is to establish the following fact: for all $n \in \mathbb{N}_0$,

$$\Big[\, p_n(R) f_0 : \sum_{k=0}^{n} q_k(R)\phi_0 \Big] = [\, f_0 : \rho_n(R)\phi_0]. \tag{5.8}$$

This is enough to prove the lemma, for the following reason:

For $n > 0$, note that the polynomial $h_n(R) = -A_n^{-1} p_{n-1}(L) R^{a_n}$ has $\alpha(h_n) = a_n - \beta(p_{n-1}) = a_n - a_{n-1} \geq a_n/2$, by our rapid increase condition. We can

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then use (5.5) to estimate $\alpha(\rho_n) = a_n^2\alpha(h_n) + \alpha(\rho_{n-1}) \geq a_n^3/2$. And then, since $\phi_0 \gg f_0$, $\alpha([f_0:\rho_n(R)\phi_0]) \geq \alpha(\rho_n) + \alpha(\phi_0) - \beta(f_0) \geq \alpha(\rho_n) \geq a_n^3/2$. As $n \to \infty$, this expression tends to infinity. Given formula (5.8), we therefore know that $\alpha([p_n(R)f_0:\sum_{k=0}^n q_k(R)\phi_0])$ tends to infinity. But if the sums (5.6) and (5.7) are norm convergent, certainly the function $[f:\phi]$ is the pointwise limit, as n tends to infinity, of the functions $[p_n(R)f_0:\sum_{k=0}^n q_k(R)\phi_0]$. Therefore $[f:\phi] = 0$. But f is a sum of A_0Rf_0 and higher terms which translate f_0 further to the right; so $\alpha(f) = 1+\alpha(f_0)$. And likewise ϕ is a sum of $R^2\phi_0$ and higher terms, of which the n-th term translates ϕ_0 at least $\alpha(\delta_n)$ to the right ((5.4) giving $q_n(R)$ as a sum of terms $h_n(R)^k\delta_n(R)$). But $\alpha(\delta_n) \geq a_n$ by (5.3) so the higher terms involve translating by at least $a_1 > 2$. We therefore have $\alpha(\phi) = 2 + \alpha(\phi_0)$. Plainly, then, f and ϕ are both nonzero and we *do not* have $f \gg \phi$. So (f, ϕ) is a nonstandard dual pair. Now we have chosen ϕ_0 weak-$*$ continuous; this is automatic in the discrete case, and in the continuous case, it follows because we have chosen ϕ_0 to be continuous. Therefore, ϕ will be a norm-convergent sum of weak-$*$ continuous functionals, so it will be weak-$*$ continuous. Thus the lemma will be proved.

All that remains is to prove equation (5.8). The proof is by induction on n. When $n = 0$, (5.8) is the assertion $[A_0Rf_0:R^2\phi_0] = [f_0:A_0R\phi_0]$, which is true for any compactly supported f_0 and ϕ_0 ($[A_0Rf_0:R^2\phi_0](t) = A_0\langle R^{1+t}f_0, R^2\phi_0\rangle = A_0\langle R^t f_0, R\phi_0\rangle = [f_0:A_0R\phi_0](t)$ for all $t \geq 0$).

For $n > 0$, let us first consider the polynomial q_n. We are committed to making $q_n(R)$ a "geometric progression" as in (5.4). It has been chosen such that much of the expression $[p_n(R)f_0:\sum_{k=0}^n q_k(R)\phi_0]$ will cancel for reasons similar to our elementary argument in the previous section. The convenient way to see this cancellation is to note that $p_n(R) = A_nR^{a_n} + p_{n-1}(R)$. We have arranged for a term of form $[p_{n-1}(R)f_0:h_n(R)^k\psi]$ to cancel with a term $[A_nR^{a_n}f_0:h_n(R)^{k+1}\psi]$, where $h_n(R)$ is the (operator) ratio for our "geometric progression" $q_n(R)$. For $[g:\psi]$ is a bilinear function of g and ψ; $[g:\psi] = [R^{a_n}g:R^{a_n}\psi]$ for all compactly supported g and ψ; and $L\colon \mathcal{A}^* \to \mathcal{A}^*$ is the dual map to $R\colon \mathcal{A} \to \mathcal{A}$. Therefore for any compactly supported ψ, one has

$$\begin{aligned}
[p_{n-1}(R)f_0:h_n(R)^k\psi] &= [p_{n-1}(R)R^{a_n}f_0:R^{a_n}h_n(R)^k\psi] \\
&= [R^{a_n}f_0:p_{n-1}(L)R^{a_n}h_n(R)^k\psi] \\
&= [A_nR^{a_n}f_0:A_n^{-1}p_{n-1}(L)R^{a_n}h_n(R)^k\psi] \\
&= -[A_nR^{a_n}f_0:h_n(R)^{k+1}\psi].
\end{aligned}$$

Using this formula we can simplify the expression $[p_n(R)f_0:q_n(R)\phi_0]$. For

$$\begin{aligned}
(5.9)\quad [p_n(R)f_0:q_n(R)\phi_0] &= \sum_{k=0}^{a_n^2-1}[(A_nR^{a_n} + p_{n-1}(R))f_0:h_n(R)^k\delta_n(R)\phi_0] \\
&= [A_nR^{a_n}f_0:\delta_n(R)\phi_0] + [p_{n-1}(R)f_0:h_n(R)^{a_n^2-1}\delta_n(R)\phi_0]
\end{aligned}$$

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because all the other terms cancel. Also

$$(5.10)\qquad \begin{aligned}[A_nR^{a_n} f_0 : \delta_n(R)\phi_0] &= -[A_nR^{a_n} f_0 : A_n^{-1}R^{a_n}\rho_{n-1}(R)\phi_0] \\ &= -[f_0 : \rho_{n-1}(R)\phi_0] \qquad \text{by (5.3)} \\ &= -\Big[p_{n-1}(R) f_0 : \sum_{k=0}^{n-1} q_k(R)\phi_0\Big]\end{aligned}$$

by induction hypothesis. Then(5.10) and (5.9) give us

$$(5.11)\qquad \begin{aligned}[p_n(R) f_0 : q_n(R)\phi_0] + \Big[p_{n-1}(R) f_0 : \sum_{k=0}^{n-1} q_k(R)\phi_0\Big] \\ = [p_{n-1}(R) f_0 : h_n(R)^{a_n^2-1}\delta_n(R)\phi_0] \\ = [f_0 : p_{n-1}(L) h_n(R)^{a_n^2-1}\delta_n(R)\phi_0] \\ = [f_0 : h_n(R)^{a_n^2}\rho_{n-1}(R)\phi_0] = [f_0 : \rho_n(R)\phi_0]\end{aligned}$$

by (5.5). For (5.1), (5.3) and (5.2) give us $p_{n-1}(L)\delta_n(R) = h_n(R)\rho_{n-1}(R)$. Hence, $[p_n(R) f_0 : \sum_{k=0}^{n} q_k(R)\phi_0]$ is equal to

$$(5.12)\qquad \begin{aligned}[p_n(R) f_0 : q_n(R)\phi_0] + \Big[p_{n-1}(R) f_0 : \sum_{k=0}^{n-1} q_k(R)\phi_0\Big] + \Big[A_nR^{a_n} f_0 : \sum_{k=0}^{n-1} q_k(R)\phi_0\Big] \\ = [f_0 : \rho_n(R)\phi_0] + \Big[A_nR^{a_n} f_0 : \sum_{k=0}^{n-1} q_k(R)\phi_0\Big].\end{aligned}$$

Now, provided

$$(5.13)\qquad R^{a_n} f_0 \gg \sum_{k=0}^{n-1} q_k(R)\phi_0,$$

the term $[A_nR^{a_n} f_0 : \sum_{k=0}^{n-1} q_k(R)\phi_0]$ in (5.12) is zero, which gives us (5.8) and completes our proof by induction. But for all $k < n$ we have $\beta(q_k) < a_{k+1} - a_1/2 \le a_n - a_1/2$ by the more complicated of our two "rapid increase" conditions. So

$$\beta\Big(\sum_{k=0}^{n-1} q_k(R)\phi_0\Big) < a_n - a_1/2 + \beta(\phi_0) < a_n - 2,$$

because $\phi_0 \ll a_1/2 - 2$. Therefore the vector $R^{a_n} f_0$ does indeed "lie beyond" $\sum_{k=0}^{n-1} q_k(R)\phi_0$, so (5.13) is established and the Lemma is proved. ∎

6 The Weight Function

Given an increasing sequence $(a_n) \subset \mathbb{N}$, with $a_0 = 1$, we use it to define a weight function (and also to define the other sequence (A_n)). We give the simplest definition which will work; if one wants examples with compact multiplication, something slightly more complicated is needed, which we discuss in Section 11.

Definition 6.1 Given the underlying sequence (a_n), define $A_n = a_{n+2}$ $(n \geq 0)$, and let $\omega\colon \mathbb{R}^+ \to \mathbb{R}^+$ be the greatest submultiplicative weight such that $\omega(t) \leq 1$ for all t, and

$$\omega(a_n) \leq (2^n A_n)^{-1} \tag{6.1}$$

for all $n \geq 0$.

Note that one can write down an explicit formula for ω:

$$\omega(t) = \min\{1\} \cup \Big\{ \prod_{i=1}^{m} (2^{n_i} A_{n_i})^{-1} : \sum_{i=1}^{m} a_{n_i} \leq t \Big\}. \tag{6.2}$$

Then ω is continuous on $\mathbb{R}^+ \setminus \mathbb{N}_0$, and right continuous everywhere.

Lemma 6.2 *Let*

$$\mu_k = \max\{(2^i A_i)^{1/a_i} : 0 \leq i \leq k\} = \max\{(2^i a_{i+2})^{1/a_i} : 0 \leq i \leq k\}. \tag{6.3}$$

Suppose the rapid increase condition $\mu_{n+1} > 2\mu_n^2$ holds for $n \in \mathbb{N}_0$. Then

$$\omega(t + a_n) = (2^n A_n)^{-1}\omega(t) = \omega(a_n)\omega(t) \tag{6.4}$$

for every $t < a_{n+1} - a_n$, and ω is a radical weight.

Proof For (6.4) note that $\omega(t + a_n) \leq (2^n A_n)^{-1}\omega(t)$ follows from submultiplicativity; if strict inequality were to hold, pick the minimal sequence $(n_i)_{i=1}^m$ to use in (6.2) and give $\omega(t + a_n)$. Then $\sum_{i=1}^m a_{n_i} \leq t + a_n$ yet

$$\prod_{i=1}^{m} (2^{n_i} A_{n_i})^{-1} < (2^n A_n)^{-1}\omega(t). \tag{6.5}$$

No n_i can exceed n because $t + a_n < a_{n+1}$ by hypothesis. If, say, $n_m = n$, we can use the sequence $(n_i)_{i=1}^{m-1}$ to estimate $\omega(t) \leq \prod_{i=1}^{m-1}(2^{n_i} A_{n_i})^{-1}$; yet that, together with (6.5), will give us the contradiction $\omega(t) < \omega(t)$. So in fact no n_i can exceed $n - 1$. Therefore $\omega(t + a_n) = \omega_{n-1}(t + a_n)$ where

$$\omega_r(t) = \min\{1\} \cup \Big\{ \prod_{i=1}^{m} (2^{n_i} A_{n_i})^{-1} : n_i \leq r, \sum_{i=1}^{m} a_{n_i} \leq t \Big\}. \tag{6.6}$$

But the weight ω_{n-1} plainly satisfies $\omega_{n-1}(s) \geq \mu_{n-1}^{-s}$ where μ_k is as above. Let $t+a_n = ra_n + t_0$ with $r > 0$, $0 \leq t_0 < a_n$. Certainly,

$$\omega(t + a_n) \leq (2^n A_n)^{-r} \leq (2^n A_n)^{-(t+a_n)/2a_n} = \mu_n^{-(t+a_n)/2}, \tag{6.7}$$

provided the sequence μ_n is increasing. If we even have $\mu_n > \mu_{n-1}^2$, then it is impossible for $\omega(t + a_n)$ to equal $\omega_{n-1}(t + a_n)$ for any $t \geq 0$, because (6.7) gives us a strictly smaller estimate than $\mu_{n-1}^{-(t+a_n)}$. This contradiction proves that (6.4) does indeed hold. Now the condition $\mu_n > 2\mu_{n-1}$ also gives us $\mu_k \to \infty$, or simply $(2^k A_k)^{1/a_k} = \omega(a_k)^{-1/a_k} \to \infty$; so ω is indeed a radical weight. ■

We conclude this section by stating our main theorem:

Theorem 6.3 *Let ω be the radical weight of Definition 6.1. Provided the underlying sequence $(a_n)_{n=0}^{\infty}$ increases sufficiently rapidly, the Banach algebras $l^1(\omega)$ and $L^1(\mathbb{R}^+, \omega)$ both have weak-$*$ closed nonstandard ideals. In addition, $L^1(\mathbb{R}^+, \omega)$ has weak-$*$ closed nonstandard ideals I with $\alpha(I) = 0$.*

7 Proof of Theorem 6.3: Ideals with $\alpha(I) > 0$.

In this section we prove all of Theorem 6.3 except the part about achieving ideals with $\alpha(I) = 0$. Thus we reproduce the main results of both [6, 2] in a rather shorter way. So let ω be the weight of Definition 6.1, and let us establish some "rapid increase" conditions on the underlying sequence (a_n) which ensure that the conditions of Lemma 5.2 are satisfied by any nonzero f_0 and ϕ_0 with $f_0 \ll \phi_0 \ll 5$ (say). This will give us nonstandard dual pairs, and hence nonstandard ideals, in both $l^1(\omega)$ and $L^1(\mathbb{R}^+, \omega)$; though in the $L^1(\mathbb{R}^+, \omega)$ case, plainly one will have $\alpha(I) \geq 1$, so $\alpha(I) = 0$ cannot be achieved by this method. We will assume the (a_n) at least increase fast enough that the conclusions of Lemma 6.2 hold.

In that case, $A_n = 2^{-n}\omega(a_n)^{-1}$, so $\|A_n R^{a_n} f_0\| \leq 2^{-n}\|f_0\|$ for any $f_0 \in \mathcal{A}$ (for $\|R^{a_n}\|_{\mathcal{A}} \leq \omega(a_n)$); so it is obvious at least that the sum (5.6) converges. The "rapid increase" conditions of Lemma 5.2 can also be assumed, and we also have $f_0 \ll \phi_0 \ll a_1/2 - 2$ whenever $f_0 \ll \phi_0 \ll 5$ provided (say) $a_1 > 15$. So our one and only problem is to show that the sum (5.7) converges in $\mathcal{A}^*$. This is the heart of the matter, and this we now proceed to do. We do most of the combinatorics in the next three lemmas; then we complete the proof at the end of this section.

Lemma 7.1 *Provided the sequence (a_n) increases sufficiently rapidly, the following is true. For all $n \in \mathbb{N}$, $0 \leq k < a_n^2$ and $\phi_0 \in \mathcal{A}^*$ with $\phi_0 \ll 5$, we have*

$$\|h_n(R)^k \delta_n(R)\phi_0\|_{\mathcal{A}^*} \leq 2^{n(k+1)}\|p_{n-1}(L)^k\|_{\mathcal{A}^*} \cdot \|\rho_{n-1}(R)\phi_0\|_{\mathcal{A}^*}. \tag{7.1}$$

(Here the norm $\|p_{n-1}(L)^k\|_{\mathcal{A}^*}$ denotes its norm as a multiplier on $\mathcal{A}^*$.) Furthermore,

$$\|\rho_n(R)\phi_0\|_{\mathcal{A}^*} \leq 2^{na_n^2}\|p_{n-1}(L)^{a_n^2}\|_{\mathcal{A}^*} \cdot \|\rho_{n-1}(R)\phi_0\|_{\mathcal{A}^*}. \tag{7.2}$$

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Proof Let $n \in \mathbb{N}$. The degree $\beta(h_n) < a_n$, hence by (5.5) $\beta(\rho_n) < a_n^3 + \beta(\rho_{n-1})$ and given $a_k > 2a_{k-1}$ for all k that tells us $\beta(\rho_n) < 2a_n^3 + \beta(\rho_0) = 1 + 2a_n^3$. Now with $\phi_0 \ll 5$ and $k < a_n^2$, we find

$$h_n(R)^k \delta_n(R)\phi_0 = p_{n-1}(L)^k(-A_n^{-1}R^{a_n})^{k+1}\rho_{n-1}(R)\phi_0 \\ = p_{n-1}(L)^k(-A_n^{-1}R^{a_n})^{k+1}\gamma \tag{7.3}$$

where the vector $\gamma = \rho_{n-1}(R)\phi_0 \ll 5 + 1 + 2a_{n-1}^3 = 6 + 2a_{n-1}^3$. Assume as a condition of rapid increase that $a_n^3 + 6 + 2a_{n-1}^3 < a_{n+1}$ for all $n > 0$. By (6.4) we will have $\omega(t + (k+1)a_n) = \omega(a_n)^{t+1}\omega(t)$ for all t in the support of γ. Hence, $\|R^{(k+1)a_n}\gamma\|_{\mathcal{A}^*} = \omega(a_n)^{-k-1}\|\gamma\|_{\mathcal{A}^*}$. Substituting $\omega(a_n) = 2^{-n}/A_n$ we obtain

$$\|(-A_n^{-1}R^{a_n})^{k+1}\gamma\|_{\mathcal{A}^*} = 2^{n(k+1)}\|\gamma\|_{\mathcal{A}^*} \tag{7.4}$$

and substituting this in (7.3) we obtain (7.1). For the other inequality we note $\rho_n(R)\phi_0 = p_{n-1}(L)^{a_n^2}(-A_n^{-1}R^{a_n})^{a_n^2}\gamma$, and the identity (7.4), with $k = a_n^2 - 1$, gives us (7.2) as required. ■

Lemma 7.2 *Provided the sequence (a_n) increases sufficiently rapidly, the following is true. For all $n \in \mathbb{N}$ we have*

$$\|p_{n-1}(L)\|_{\mathcal{A}^*} \le 2. \tag{7.5}$$

Furthermore for $n > 1$ we have

$$\|p_{n-1}(L)^{a_n}\|_{\mathcal{A}^*} \le 1/\sqrt{A_n}. \tag{7.6}$$

Proof For the first inequality, we have

$$\|p_{n-1}(L)\|_{\mathcal{A}^*} = \|p_{n-1}(R)\|_{\mathcal{A}} \le \sum_{k=0}^{n-1} A_k\omega(a_k) \le \sum_{k=0}^{n-1} 2^{-k} < 2.$$

For the second inequality note that since $L|p_{n-1}(L)$, $L^{a_n}|p_{n-1}(L)^{a_n}$. Since $\omega(a_n) < A_n^{-1}$, we have $\|p_{n-1}(L)^{a_n}\|_{\mathcal{A}^*} = \|p_{n-1}(R)^{a_n}\|_{\mathcal{A}} \le A_n^{-1} \cdot |p_{n-1}|^{a_n}$, where $|p|$ denotes the sum of the absolute values of the coefficients of p. But $|p_{n-1}|$ depends on elements of our underlying sequence only up to $A_{n-1} = a_{n+1}$, so we are perfectly entitled to assume $A_n^{-1} \cdot |p_{n-1}|^{a_n} < 1/\sqrt{A_n}$, for all n, as a condition of rapid increase, which of course gives us (7.6). ■

Lemma 7.3 *Provided the sequence (a_n) increases sufficiently rapidly, the following is true. For all $n \in \mathbb{N}$ we have*

$$\|\rho_n(R)\phi_0\|_{\mathcal{A}^*} \le A_n^{-a_n/3} \cdot \|\rho_0(R)\phi_0\|_{\mathcal{A}^*} \tag{7.7}$$

and for $n > 1$ we have

$$\|q_n(R)\phi_0\|_{\mathcal{A}^*} \le 2^{-n}\|\rho_0(R)\phi_0\|_{\mathcal{A}^*}. \tag{7.8}$$

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Proof By (7.2) and (7.6) we have $\|\rho_n(R)\phi_0\|_{\mathcal{A}^*} \le A_n^{-a_n/2} \cdot 2^{na_n^2}\|\rho_{n-1}(R)\phi_0\|_{\mathcal{A}^*}$. Assume as a condition of rapid increase that $A_n^{-a_n/2} \cdot 2^{na_n^2} < A_n^{-a_n/3}$ for all $n \in \mathbb{N}$. This gives (7.7) immediately when $n = 1$; for larger values we get $\|\rho_n(R)\phi_0\|_{\mathcal{A}^*} \le \|\rho_{n-1}(R)\phi_0\|_{\mathcal{A}^*} \cdot \prod_{k=1}^n A_k^{-a_k/2}$ which implies (7.7). Likewise if $k < a_n^2$ and $n > 1$, the previous lemma gives $\|p_{n-1}(L)^k\|_{\mathcal{A}^*} \le 2^k$. By (7.1) and (7.7) this gives $\|h_n(R)^k\delta_n(R)\phi_0\| \le 2^{n(k+1)+k}\|\rho_{n-1}(R)\ \phi_0\|_{\mathcal{A}^*} \le 2^{nk+n+k}\ A_{n-1}^{-a_{n-1}/2}\|\rho_0(R)\phi_0\|_{\mathcal{A}^*}$. But since $A_{n-1} = a_{n+1}$ we can assume as a condition of rapid increase that

$$2^{na_n^2+n+a_n^2} < A_{n-1}^{a_{n-1}/4} \tag{7.9}$$

for every $n > 0$ (note this is the place where we really use the fact that $A_n = a_{n+2}$ rather than the more natural a_{n+1}). Then

$$\|h_n(R)^k\delta_n(R)\phi_0\|_{\mathcal{A}^*} \le A_{n-1}^{-a_{n-1}/4}\|\rho_0(R)\phi_0\|_{\mathcal{A}^*}.$$

Summing from $k = 0$ to $a_n^2 - 1$ we find using (5.4) that $\|q_n(R)\phi_0\|_{\mathcal{A}^*} \le a_n^2 \cdot A_{n-1}^{-a_{n-1}/4}\|\rho_0(R)\phi_0\|_{\mathcal{A}^*} \le A_{n-1}^{2-a_{n-1}/4}\|\rho_0(R)\phi_0\|_{\mathcal{A}^*}$. We may assume that $A_{n-1}^{2-a_{n-1}/4} < 2^{-n}$ for all n. This establishes (7.8) and proves the lemma. ■

Proof that (5.7) *converges when* $\phi_0 \ll 5$*:* We can assume Lemmas 7.1–7.3. Equation (7.8) plainly implies that the sum (5.7) converges. Thus all conditions of lemma 5.2 are satisfied, for any $f \ll \phi \ll 5$, provided the underlying sequence (a_n) increases sufficiently rapidly. Thus $l^1(\omega)$ and $L^1(\mathbb{R}^+, \omega)$ both contain weak-$*$ closed nonstandard ideals I (with $\alpha(I) = 1 + \alpha(f) > 0$).

8 Nonstandard Ideals $I \subset L^1(\mathbb{R}^+, \omega)$ with $\alpha(I) = 0$.

We now restrict to the continuous case $\mathcal{A} = L^1(\mathbb{R}^+, \omega)$, and seek ideals I with $\alpha(I)=0$. We can use the same weight function ω, but we need a different choice of nonstandard dual pair (f, ϕ). Note that in the proof of Lemma 7.2 it was important that $L|p_n(L)$ for all n, but this fact leads to an ideal with $\alpha(I) \ge 1$. To get ideals with $\alpha(I) < 1$ we need to use generalised "polynomials" $p_n(L)$ involving fractional powers $L^{1/2}$, $L^{1/4}$ and so on and if we finally want $\alpha(I) = 0$, the minimum degree $\alpha(p_n)$ needs to tend to zero. This leads to the following alternative to Definition 5.1.

Definition 8.1 Let $(a_n)_{n=0}^\infty$ and $(A_n)_{n=0}^\infty$ be strictly increasing sequences of positive integers with $a_0 = 1$. Let us define compactly supported functions g_n $(n \ge 0)$ by

$$g_n(x) = \begin{cases} 1 & \text{if } x < 2^{-n}, \\ 0 & \text{otherwise.} \end{cases} \tag{8.1}$$

Note that $g_{n-1} = (1+R^{2^{-n}})g_n$ for all $n > 0$. Define $q_0'(R) = R^2$, $\rho_0'(R) = (A_0+A_0^{-1})R$ and for $n \in \mathbb{N}_0$ define

$$p_n'(R) = \sum_{k=0}^{n}(A_kR^{a_k} + A_k^{-1}R^{2^{-k}}) \cdot \prod_{l=k+1}^{n}(1 + R^{2^{-l}}). \tag{8.2}$$

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Note that $\alpha(p'_n) = 2^{-n}$, $\beta(p'_n) = a_n$, and

$$(8.3)\quad p'_n(R)g_n = p'_{n-1}(R)g_{n-1} + (A_n R^{a_n} + A_n^{-1}R^{2^{-n}})g_n = \sum_{k=0}^{n}(A_k R^{a_k} + A_k^{-1}R^{2^{-k}})g_k.$$

Then recursively define generalised polynomials $\rho'_n(R), q'_n(R), h'_n(R), \delta'_n(R)$ $(n \in \mathbb{N})$ as follows.

$$(8.4)\qquad h'_n(R) = -A_n^{-1}\Big((p'_{n-1}(L)(1 + L^{2^{-n}}) + A_n^{-1}L^{2^{-n}}\Big)R^{a_n},$$

(this is a generalised polynomial with $\alpha(h'_n) = a_n - \beta(p'_{n-1}) - 2^n = a_n - a_{n-1} - 2^{-n} \geq 1/2$)

$$(8.5)\qquad \delta'_n(R) = -A_n^{-1}\Big((1 + L^{2^{-n}})\rho'_{n-1}(R) + A_n^{-1}L^{2^{-n}}\sum_{k=0}^{n-1}q'_k(R)\Big)R^{a_n},$$

(a generalised polynomial with $\alpha(\delta'_n) \geq a_n - 2^{-n}$)

$$(8.6)\qquad q'_n(R) = \sum_{k=0}^{a_n^2-1} h'_n(R)^k \delta'_n(R),$$

(again, $\alpha(q'_n) \geq a_n - 2^{-n}$) and

$$(8.7)\qquad \rho'_n(R) = h'_n(R)^{a_n^2}\Big((1 + L^{2^{-n}})\rho'_{n-1}(R) + A_n^{-1}L^{2^{-n}}\sum_{k=0}^{n-1}q'_k(R)\Big)$$

(a generalised polynomial with $\alpha(\rho'_n) \geq a_n^2/2 - 2^{-n} + \min(\alpha(\rho'_{n-1}), a_n)$).

The reader will observe that these definitions are quite close to Definition 5.1 but not quite the same; the nature of the nonstandard dual pair is slightly different, as this alternative to Lemma 5.2 shows:

Lemma 8.2 *With the notation of Definition* 8.1*, let us assume the a_n increase sufficiently rapidly that $a_{n+1} > 2a_n$ for all $n \geq 0$. Noting that the degree $\beta(q'_n)$ is bounded by a function of $a_0, a_1, \dots a_n$, let us assume, as a further condition of rapid increase, that $\beta(q'_n) < a_{n+1} - a_1/2$ for all $n \in \mathbb{N}_0$.*

Suppose that the weight ω has been chosen in such as way that the sums

$$(8.8)\qquad g = \sum_{n=0}^{\infty}(A_n R^{a_n} + A_n^{-1}R^{2^{-n}})g_n = \lim_{n\to\infty} p'_n(R)g_n$$

and

$$(8.9)\qquad \phi' = \sum_{n=0}^{\infty} q'_n(R)g_0$$

are norm convergent in $\mathcal{A}$ and $\mathcal{A}^*$, respectively. Then (g, ϕ') is a nonstandard dual pair with $\alpha(g) = 0$.

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Proof In parts we give just a sketch proof, since the proof is very similar to that of Lemma 5.2.

Note that the sum (8.8) is a sum of positive functions $\gamma_n = (A_n R^{a_n} + A_n^{-1} R^{2^{-n}}) g_n$ with $\alpha(\gamma_n) = 2^{-n}$. Therefore, g is nonzero with $\alpha(g) = 0$. And as before ϕ' is nonzero with $\alpha(\phi') = 2 + \alpha(g_0) = 2$. We claim that for all $n \geq 0$,

$$\left[p_n'(R)g_n : \sum_{k=0}^{n} q_k'(R)g_0\right] = \left[g_n : \rho_n'(R)g_0\right]. \tag{8.10}$$

This proves the lemma because $\alpha(\rho_n') \geq a_n^3/2$ for similar reasons as before, and then we have $\alpha([p_n'(R)g_n : \sum_{k=0}^{n} q_k'(R)g_0]) \to \infty$, so $[g : \phi'] = 0$ provided the sums (8.8) and (8.9) converge.

As before, one proves (8.10) by induction. When handling the geometric progression q_n', one notes that for any compactly supported $\psi \in \mathcal{A}^*$ and any $k \geq 0$ one has $[p_{n-1}'(R)g_{n-1} + A_n^{-1}R^{2^{-n}}g_n : h_n'(R)^k\psi] = -[A_n R^{a_n} g_n : h_n'(R)^{k+1}\psi]$. Hence, most of the terms in the geometric progression $[p_n'(R)g_n : q_n'(R)g_0]$ cancel, leaving $[p_n'(R)g_n : q_n'(R)g_0]$ equal to

$$[A_n R^{a_n} g_n : \delta_n'(R)g_0] + \left[p_{n-1}'(R)g_{n-1} + A_n^{-1}R^{2^{-n}}g_n : h_n'(R)^{a_n^2-1}\delta_n'(R)g_0\right]. \tag{8.11}$$

The first term in (8.11) above is precisely $-[p_n'(R)g_n : \sum_{k=0}^{n-1} q_k'(R)\, g_0]$ because δ_n' is chosen in such a way as to give

$$\begin{aligned}
[A_n R^{a_n} g_n : \delta_n'(R)g_0] &= -\Big[g_n : \Big((1 + L^{2^{-n}})\rho_{n-1}'(R) + A_n^{-1}L^{2^{-n}}\sum_{k=0}^{n-1} q_k'(R)\Big)\, g_0\Big] \\
&= -[g_{n-1} : \rho_{n-1}'(R)g_0] - \Big[A_n^{-1}R^{2^{-n}}g_n : \sum_{k=0}^{n-1} q_k'(R)g_0\Big] \\
&= -\Big[p_{n-1}'(R)g_{n-1} + A_n^{-1}R^{2^{-n}}g_n : \sum_{k=0}^{n-1} q_k'(R)g_0\Big]
\end{aligned}$$

by induction hypothesis. The difference between this and $-[p_n'(R)g_n : \sum_{k=0}^{n-1} q_k'(R)g_0]$ consists of the term $[A_n R^{a_n} g_n : \sum_{k=0}^{n-1} q_k'(R)g_0]$, which is zero because $\sum_{k=0}^{n-1} q_k'(R)g_0 \ll a_n$ as before. Hence the interaction $[p_n'(R)g_n : \sum_{k=0}^{n} q_k'(R)g_0]$ consists of the single term

$$\begin{aligned}
&[p_{n-1}'(R)g_{n-1} + A_n^{-1}R^{2^{-n}}g_n : h_n'(R)^{a_n^2-1}\delta_n'(R)g_0] \\
&\qquad = [g_n : (p_{n-1}'(L)(1 + L^{2^{-n}}) + A_n^{-1}L^{2^{-n}})h_n'(R)^{a_n^2-1}\delta_n'(R)g_0] \\
&\qquad = \Big[g_n : h_n'(R)^{a_n^2}\Big((1 + L^{2^{-n}})\rho_{n-1}'(R) + A_n^{-1}L^{2^{-n}}\sum_{k=0}^{n-1} q_k'(R)\Big)\, g_0\Big] \\
&\qquad = [g_n : \rho_n'(R)g_0],
\end{aligned}$$

and this completes the proof by induction. ∎

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9 Continuing the Proof of Theorem 6.3

Once again, the weight function ω as in Definition 6.1 can be used to satisfy the conditions of Lemma 8.2, just as it satisfied the conditions of Lemma 5.2 before. In this section, then, ω is the specific weight of Definition 6.1, obtained from some underlying sequence (a_n). We may assume the rapid increase conditions on (a_n) which are required by Lemma 8.2, and also those required by Lemma 6.2.

Lemma 9.1 *Provided the sequence (a_n) increases sufficiently rapidly, the following is true. For all $n \in \mathbb{N}$ and $0 \le k < a_n^2$ one has*

$$\|h_n'(R)^k \delta_n'(R) g_0\|_{\mathcal{A}^*} \le \big\| \big(p_{n-1}'(L)(1+L^{2^{-n}}) + A_n^{-1} L^{2^{-n}} \big)^k \big\|_{\mathcal{A}^*} \cdot 2^{n(k+1)} \|\gamma\|_{\mathcal{A}^*}, \tag{9.1}$$

where

$$\gamma = \Big((1+L^{2^{-n}}) \rho_{n-1}'(R) + A_n^{-1} L^{2^{-n}} \sum_{k=0}^{n-1} q_k'(R) \Big) g_0. \tag{9.2}$$

Also

$$\|\rho_n'(R) g_0\|_{\mathcal{A}^*} \le \big\| \big(p_{n-1}'(L)(1+L^{2^{-n}}) + A_n^{-1} L^{2^{-n}} \big)^{a_n^2} \big\|_{\mathcal{A}^*} \cdot 2^{n a_n^2} \|\gamma\|_{\mathcal{A}^*}. \tag{9.3}$$

Proof One may calculate (as one did when proving Lemma 7.1)

$$h_n'(R)^k \delta_n'(R) g_0 = \big(p_{n-1}'(L)(1+L^{2^{-n}}) + A_n^{-1} L^{2^{-n}} \big)^k (-A_n^{-1} R^{a_n})^{k+1} \gamma \tag{9.4}$$

and as with (7.1), the result (9.1) follows because the norm of $(-A_n^{-1} R^{a_n})^{k+1}\gamma$ is precisely $2^{n(k+1)} \|\gamma\|_{\mathcal{A}^*}$ by Lemma 6.2. Similarly one may obtain (9.3). ■

Lemma 9.2 *Provided the sequence (a_n) increases sufficiently rapidly, the following is true. Let $p_{n-1}''(L) = p_{n-1}'(L)(1+L^{2^{-n}}) + A_n^{-1} L^{2^{-n}}$. Then for all $n \in \mathbb{N}$ we have*

$$\|p_{n-1}''(L)\|_{\mathcal{A}^*} \le 2^{n+2}; \tag{9.5}$$

furthermore, for $n > 1$ we have

$$\|p_{n-1}''(L)^{2^n a_n}\|_{\mathcal{A}^*} \le 1/\sqrt{A_n}; \tag{9.6}$$

and for every $k \ge 2^n a_n$ we have

$$\|p_{n-1}''(L)^k\| \le A_n^{-k/2^{n+3} a_n}. \tag{9.7}$$

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Proof Using the fact that $\|L^t\|_{\mathcal{A}^*} \le 1$ for all t, and the fact that $\|L^{a_n}\|_{\mathcal{A}^*} \le \omega(a_n) = 2^{-n}A_n$, we obtain

$$(9.8) \quad \|p''_{n-1}(L)\|_{\mathcal{A}^*} = \|A_n^{-1}L^{2^{-n}} + \sum_{k=0}^{n-1}(A_k L^{a_k} + A_k^{-1}L^{2^{-k}}) \cdot \prod_{l=k+1}^{n}(1 + L^{2^{-l}})\|_{\mathcal{A}^*}$$
$$\le A_n^{-1} + \sum_{k=0}^{n-1}(2^{-k} + A_k^{-1}) \cdot 2^{n-k} \le 2^{-n} + \sum_{k=0}^{n-1} 2^{n-2k+1} < 2^{n+2},$$

because $A_k^{-1} < 2^{-k}$. For (9.6) we note that $\alpha(p''_{n-1}) = 2^{-n}$, so L^{a_n} divides $p''_{n-1}(L)^{2^n a_n}$, whence $\|p''_{n-1}(L)^{2^n a_n}\|_{\mathcal{A}^*} \le 2^{-n}A_n^{-1}|p''_{n-1}|^{2^n a_n}$. Given another rapid increase condition, the A_n^{-1} factor can be assumed to dominate the others, hence we can obtain (9.6). If now $k = r \cdot 2^n a_n + l$ with $r \in \mathbb{N}$, $0 \le l < 2^n a_n$, our two estimates give $\|p''_{n-1}(L)^k\|_{\mathcal{A}^*} \le 2^{(n+2)l} A_n^{-r/2}$. Now certainly $r > k/2^{n+1}a_n$, so $\|p''_{n-1}(L)^k\| \le 2^{(n+2)\cdot 2^n a_n} A_n^{-k/2^{n+2}a_n}$, and given rapid increase this is at most $A_n^{-k/2^{n+3}a_n}$ for every $k \ge 2^n a_n$. ■

Lemma 9.3 *Provided the sequence (a_n) increases sufficiently rapidly, the following is true. For all $n \in \mathbb{N}$ we have*

$$(9.9) \qquad \|\rho'_n(R)g_0\|_{\mathcal{A}^*} \le A_n^{-a_n/2^{n+4}}$$

and

$$(9.10) \qquad \|q'_n(R)g_0\|_{\mathcal{A}^*} \le \begin{cases} 4A_0^2 \cdot 2^{5a_1^2} & \text{if } n = 1, \\ 2^{-n} & \text{if } n > 1. \end{cases}$$

Proof One may use the fact that $\omega(t) = A_0^{-1}$ for $t \in [1, 2)$ to compute

$$\|\rho'_0(R)g_0\|_{\mathcal{A}^*} = A_0(A_0 + A_0^{-1}).$$

Also $\omega(t) = A_0^{-2}$ for $t \in [2, 3)$, hence $\|q'_0(R)g_0\|_{\mathcal{A}^*} = A_0^2$. Let us prove (9.9) and (9.10) together, proceeding by induction on n.

Let γ be the vector defined in (9.2). Now γ is a sum of two terms, the first of which is $(1 + L^{2^{-n}})\rho'_{n-1}(R)g_0$. This first term has $\mathcal{A}^*$ norm at most $2\|\rho'_{n-1}(R)g_0\|_{\mathcal{A}^*}$ which for $n = 1$ is $2A_0(A_0 + A_0^{-1}) < 3A_0^2$, and for $n > 1$ may be assumed (by induction hypothesis) to be at most $2A_{n-1}^{-a_{n-1}/2^{n+3}}$. The second term in γ is $A_n^{-1}L^{2^{-n}}\sum_{k=0}^{n-1} q'_k(R)g_0$ which (since $\|L^{2^{-n}}\|_{\mathcal{A}^*} \le 1$) has norm at most $A_n^{-1}\sum_{k=0}^{n-1}\|q'_k(R)g_0\|$. When $n = 1$, that is $A_0^2/A_1 < A_0^2$. So certainly

$$(9.11) \qquad \|\gamma\|_{\mathcal{A}^*} \le \begin{cases} 4A_0^2 & \text{if } n = 1, \\ 2A_{n-1}^{-a_{n-1}/2^{n+3}} + A_n^{-1}\sum_{k=0}^{n-1}\|q'_k(R)g_0\|_{\mathcal{A}^*} & \text{if } n > 1. \end{cases}$$

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Now (9.3) gives us $\|\rho_n'(R)g_0\|_{\mathcal{A}^*} \leq \|p_{n-1}''(L)^{a_n^2}\| \cdot 2^{na_n^2} \cdot \|\gamma\|_{\mathcal{A}^*}$. We know $a_n^2 \geq 2^n a_n$, so by (9.7) we have $\|p_{n-1}''(L)^{a_n^2}\| \leq A_n^{-a_n/2^{n+3}}$. So $\|\rho_n'(R)g_0\|_{\mathcal{A}^*} \leq A_n^{-a_n/2^{n+3}} \cdot 2^{na_n^2} \cdot \|\gamma\|_{\mathcal{A}^*}$. Substituting our estimate (9.11) we have

(9.12)

$$\|\rho_n'(R)g_0\|_{\mathcal{A}^*} \leq \begin{cases} 4A_0^2 A_1^{-a_1/16} 2^{a_1^2} & \text{if } n = 1, \\ (2A_{n-1}^{-a_{n-1}/2^{n+3}} + A_n^{-1}\sum_{k=0}^{n-1}\|q_k'(R)g_0\|_{\mathcal{A}^*})A_n^{-a_n/2^{n+3}} 2^{na_n^2} & \text{if } n > 1. \end{cases}$$

For $n = 1$, a rapid increase condition on $A_1 = a_3$ ensures that the right-hand side is at most $A_1^{-a_1/32}$ as required by (9.9). For $n > 1$, our induction hypothesis tells us that the right-hand side is at most

$$(2A_{n-1}^{-a_{n-1}/2^{n+3}} + A_n^{-1}(4A_0^2 \cdot 2^{5a_1^2} + \sum_{k=1}^{n-1} 2^{-k})) \cdot A_n^{-a_n/2^{n+3}} \cdot 2^{na_n^2}.$$

Once again the dominant factor in this is $A_n^{-a_n/2^{n+3}}$, and we can assume

$$\|\rho_n'(R)g_0\|_{\mathcal{A}^*} \leq A_n^{-a_n/2^{n+4}},$$

given a suitable condition of rapid increase. Thus we obtain (9.9). For (9.10), we sum our estimates (9.1) for $k = 0$ to $a_n^2 - 1$, which, given the definition (8.6) of q_n', tells us

$$\text{(9.13)} \qquad \|q_n'(R)g_0\|_{\mathcal{A}^*} \leq \sum_{k=0}^{a_n^2-1} \|p_{n-1}''(L)^k\|_{\mathcal{A}^*} \cdot 2^{n(k+1)} \|\gamma\|_{\mathcal{A}^*}$$
$$\leq \sum_{k=0}^{a_n^2-1} 2^{k(n+2)+n(k+1)} \|\gamma\|_{\mathcal{A}^*} \leq 2^{5na_n^2} \|\gamma\|_{\mathcal{A}^*}.$$

When $n = 1$ this gives an estimate of $4A_0^2 \cdot 2^{5a_1^2}$ as required by (9.10). When $n > 1$ we use our estimate (9.11) and our induction hypothesis about previous $\|q_k(R)g_0\|_{\mathcal{A}^*}$ to obtain the estimate

$$\text{(9.14)} \qquad \|q_n'(R)g_0\|_{\mathcal{A}^*} \leq 2^{5na_n^2} \cdot \left(2A_{n-1}^{-a_{n-1}/2^{n+3}} + A_n^{-1}\left(4A_0^2 \cdot 2^{5a_1^2} + \sum_{k=1}^{n-1} 2^{-k}\right)\right),$$

and a final rapid increase condition tells us the right-hand side is at most 2^{-n} for all n. (Although note that we again use the fact that $A_{n-1} = a_{n+1}$, so the factor $A_{n-1}^{-a_{n-1}/2^{n+3}}$ "kills" factors like $2^{5na_n^2}$ provided we assume rapid increase of the sequence (a_n).) Thus Lemma 9.3 is proved. ■

10 Completing the Proof of Theorem 6.3.

We have proved everything in Theorem 6.3 except the part about nonstandard ideals with $\alpha(I) = 0$. But (9.10) implies that the sum (8.9) is norm convergent.

The sum (8.8) is obviously convergent because $\|A_n R^{a_n}\|_{\mathcal{A}} \le 2^{-n}$, $\|g_n\|_{\mathcal{A}} \le 2^{-n}$ and $\|R^{2^{-n}}\|_{\mathcal{A}} \le 1$. The rapid increase conditions of Lemma 8.2 may also be assumed, so for suitably rapid increasing sequences (a_n), the Lemma applies and tells us there is a nonstandard dual pair (g, ϕ') on $L^1(\mathbb{R}^+, \omega)$, with g and ϕ' given by (8.8) and (8.9), respectively. Since $\alpha(g) = 0$, we have nonstandard ideals $I \subset L^1(\mathbb{R}^+, \omega)$ with $\alpha(I) = 0$. To see that we can get weak-$*$ closed such ideals, note that while ϕ' itself is not a C_0 function because of the norm convergence of (8.9), it is obvious that $h * \phi$ will be C_0 if h is a continuous function supported on a compact subset of $[-\infty, 0]$. Titchmarsh tells us $\phi' \ne 0$ provided h is supported on $[-\alpha(\phi'), 0]$, and finally $[g : h * \phi'] = 0$ because $\phi' * h = \int_{s=0}^{\infty} (L^s \phi') h(-s)\, ds$ ($L^s \phi'$ is a bounded continuous function of s taking values in the Banach algebra $\mathcal{A}^*$, so the integral makes sense) and $[g : \phi' * h](t) = \int_{s=0}^{\infty} [g : L^s \phi'](t) h(-s)\, ds = \int_{s=0}^{\infty} [g : \phi'](t+s) h(s)\, ds = 0$. So $L^1(\mathbb{R}^+, \omega)$ has even got weak-$*$ closed nonstandard ideals with $\alpha(I) = 0$. Thus the theorem is proved.

11 Conclusion

The last argument above is quite general. If (g, ϕ) is a nonstandard dual pair on $L^1(\mathbb{R}^+, \omega)$ with $\alpha(g) = 0$, then so is $(g, h * \phi)$ for any nonzero, compactly supported $h \in L^1(\mathbb{R}^-)$ whose support includes zero. I conjecture that a dual pair $(g, h * \phi)$ with $h * \phi \in C_0(\mathbb{R}^+, 1/\omega)$ may be found; hence (I conjecture) every algebra $L^1(\mathbb{R}^+, \omega)$ that has nonstandard ideals has weak-$*$ closed nonstandard ideals.

Concerning the issue of compact multiplication, note that the simple Definition 6.1, the "greatest weight function satisfying conditions", does not give a weight having compact multiplication. But it can easily be adjusted in such a way that one does have compact multiplication (*i.e.*, $\omega(s+t)/\omega(s) \to 0$ as $s \to \infty$ for each $t > 0$). One simply defines $n(t) = \max\{1\} \cup \{n : t \ge a_n\}$ and uses $\omega'(t) = n(t)^{-t} \omega(t)$ in place of the weight ω. One may check that, *mutatis mutandis*, the conclusions of Theorem 6.3 still apply with ω replaced by ω'. (Specifically, one no longer has $\omega(t + ra_n) = \omega(a_n)^r \omega(t)$ for $t + ra_n < a_{n+1}$, but rather the weaker $\omega'(t + ra_n) \ge n^{-a_n} \omega'(a_n)^r \omega'(t)$. That means one needs extra factors n^{a_n} on the right-hand sides of (7.1), (7.2), (9.1) and (9.3) to make them still true for $\mathcal{A} = L^1(\mathbb{R}^+, \omega')$. These factors can however be absorbed into the rapid increase conditions of Lemmas 7.3 and 9.3, leaving those lemmas still valid in the new situation. The rest of the proof then follows as before, with the same dual pairs.)

We expect that there are a fair number of weight functions ω such that $l^1(\omega)$ and $L^1(\mathbb{R}^+, \omega)$ have nonstandard ideals. The main requirement seems to be a "staircase" property that there is a sequence a_n such that $\omega(a_n)$ is very small — much smaller than $\omega(t)$ for t significantly smaller than a_n — and then $\omega(ra_n + t)$ is "roughly" $\omega(a_n)^r \omega(t)$ for small r and t. Then $\omega(a_n)$ must be much smaller than a_n itself, more so than is required for ω to be a radical weight (remember in the explicit construction we had $\omega(a_n) = 2^{-n}/a_{n+2}$). Possibly there is a "neat" condition on a weight which guarantees nonstandard ideals, hopefully not too far from the negation of Domar's "neat" star-shaped condition that guarantees all ideals are standard. Thus the structure of $L^1(\mathbb{R}^+, \omega)$ and of $l^1(\omega)$ is taking shape.

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