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DYNAMICS OF THE ANTONOV-NURITDINOV PLANAR GALAXY MODEL

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ABSTRACT

A time dependent solution of the collisionless Boltzmann equation (CBE) in two space dimensions first given by Antonov and Nuritdinov is discussed further. The solution is self consistent with a quadratic potential and represents a generalised Freeman bar characterised by 10 parameters. Because of two conserved quantities, they form an 8-dimensional phase space. Geometric and group theoretical aspects of this dynamical system are discussed. Systems of this type can show chaotic oscillations in general. This study raises interesting general questions about time dependent solutions of the CBE.

INTRODUCTION

The dynamical structure of the stellar and dark matter components of galaxies is described by the collisionless Boltzmann equation (CBE) for the phase space distribution function $f(\underset{\sim}{r},\underset{\sim}{v},t)$ with a self consistent potential $\phi(\underset{\sim}{r},t)$ calculated from the real space density associated with f,

$$\frac{\partial f}{\partial t} + \underset{\sim}{v}\cdot\frac{\partial f}{\partial r} + \underset{\sim}{a}\cdot\frac{\partial f}{\partial v} = 0,\quad \underset{\sim}{a} = -\nabla\phi,\quad \nabla^2\phi = 4\pi G\int f(\underset{\sim}{r},\underset{\sim}{v})d\underset{\sim}{v} \qquad (1)$$

The normalisation of f is chosen here to give the mass per unit phase volume. Time independent solutions have been extensively studied (see the texts by Binney and Tremaine, Fridman

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and Polyachenko) but there are fewer analytical studies of time dependent solutions. We list some below.

Kalnajs (1973) gave a one dimensional self consistent oscillating solution while Antonov and Nuritdinov (AN, 1977) constructed a two dimensional oscillating elliptic disc and Malkov (1984) discussed the stability of an oscillating sphere. Unaware of this early work, Sridhar (1989) and Sridhar and Nityananda (1989, 1990 (SN)) discussed oscillations of slabs, spheres, spheroids, and discs, all of which have self consistent time dependent harmonic potentials. This paper is concerned with the planar model discovered by AN which was elaborated and used by SN in a study of a tidal encounter between galaxies: we will try to bring out its mathematical structure more explicitly than in the earlier papers by AN and SN.

CONSTRUCTION OF THE SELF CONSISTENT TIME DEPENDENT ELLIPTIC DISC

The basic idea behind the model is that in three dimensions, a uniform density ellipsoid produces a quadratic potential within its boundary. A limiting case, when the c-axis shrinks to zero, is an elliptic disc with a surface density $\Sigma(x,y)$ given by

$$\Sigma(x,y) = (1 - \frac{x^2}{a^2} - \frac{y^2}{b^2})^{1/2} \qquad (2)$$

which is obtained by projection. A surface density of this form can be obtained by integrating the following phase space density f over velocities

$$f = (1-Q(x,y,v_x,v_y))^{-1/2},\ Q < 1,\ f = 0. \qquad (3)$$

The quadratic form Q in the four phase space variables can be written compactly as $Q = z^T Q z$ where the row vector z^T of phase space coordinates $z^T = (x,y,v_x,v_y)$ and where Q is a 4×4 real symmetric positive definite matrix and T stands for transpose. (Incidentally, this idea does not work for three dimensions. Power counting shows that to get constant real space density, one needs $f \propto (1-Q)^{-3/2}$ which is not normalisable). The final remark is that with a quadratic potential, even a time dependent one, the equations of motion are linear. Under a linear transformation of the phase space variables, the distribution function (3) maintains its form and the potential remains quadratic so self consistency has been achieved. The detailed equations of motion for the phase space variables z and for the matrix Q are given in SN. We now want to emphasize the general structure of these equations.

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A linear transformation of the phase space variables z is described by a so called symplectic matrix M where M is a function of time.

$$z = Mz' \tag{4}$$

The Poisson brackets $[z_i, z_j] = \beta_{ij}$ can be arranged in a matrix

$$\beta = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} , \quad \beta^{-1} = \beta^T = -\beta$$

Since z's have the same Poisson brackets, we have

$$\beta = M\beta M^T \quad \text{or} \quad M^{-1}\beta = \beta M^T \tag{5}$$

This condition defines the symplectic property of M. In the original work of AN, this property was built in by using a quadratic generating function for the canonical transformation relating particle coordinates at different times.

The CBE implies that f is a constant along the phase trajectories. From (3), this means that

$$Q = M^T Q' M \tag{6}$$

Thus, we need a time dependent Q to ensure that f is a constant of the motion. The physical meaning of the matrix Q becomes clear if we calculate the expectation value of a product of two phase space variables with the distribution function f. The result is (SN)

$$\langle z_i z_j \rangle \equiv \int f z_i z_j \, d\Gamma \Big/ \int f \, d\Gamma = \frac{1}{5} (Q^{-1})_{ij} \equiv P_{ij} \tag{7}$$

where P_{ij} is an element of the matrix inverse to Q. For example, P_{11}, P_{22} and P_{12} determine the shape of the system, $P_{14} - P_{23}$ the angular momentum $P_{13} + P_{24}$ expansion and P_{33}, P_{44}, P_{34} the peculiar velocity ellipse.

One can also visualise this distribution function and its dynamics geometrically. The boundary in phase space of the system is the four dimensional ellipsoid $z^T Q z = 1$. The phase space density is constant on the concentric ellipsoids $z^T Q z = c$, $0 \leq c \leq 1$ and in fact increases outwards. The real space density is clearly nonzero inside an elliptic area

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which is the projection of this ellipsoid. Under linear canonical transformations of the phase space produced by the quadratic (i.e. harmonic oscillator) Hamiltonian, the ellipsoidal surfaces of constant phase density clearly retain their form and the model remains self consistent.

CONSERVED QUANTITIES AND PHASE SPACE STRUCTURE

Without looking at the detailed structure of the matrix M in equation (6), it is still possible to bring out some general properties. For this purpose it is more convenient to deal with the matrices βQ and $P\beta$ which evolve in time according to

$$\beta Q = \beta M^T Q' M = M^{-1} \beta Q M \quad (8a)$$

Taking inverses

$$-P\beta = M^{-1}(-P'\beta)M \quad (8b)$$

The significance of a matrix like $P\beta$ is that it can be regarded as an infinitesimal symplectic matrix. In other words, the matrix $I + \varepsilon P\beta$ (I = identity matrix) satisfies the condition (5) to first order in ε. More technically, we can say that $P\beta$ is an element of sp(4,R), the Lie algebra of the symplectic group in four real variables. In such a group, the operation of conjugation by an element M is defined as multiplication by M^{-1} on the left and M on the right. Now our galaxy models are described by 10 parameters, as is sp(4,R).Under conjugation by all possible group elements, a given infinitesimal element $P\beta$ moves over an "orbit" in the Lie algebra. The physical meaning of this orbit in the context of our model is that it represents the total set of all distribution functions (i.e. dynamical states) which are accessible starting from a given initial state, the motion of the stars being described by a general time dependent quadratic Hamiltonian. One can see that there only are two quantities C_1 and C_2 which are conserved under conjugation.

$$C_1 = \mathrm{Tr}(P\beta)^2 , \quad C_2 = \mathrm{Tr}(P\beta)^4 \quad (9)$$

Replacing $P\beta$ by $M^{-1}P\beta M$ inside the trace and using cyclic invariance, it is easily checked that these are invariant.The traces of odd powers vanish, while even powers higher than the fourth can be reduced to fourth or lower by the Cayley-Hamilton theorem. Thus, with these two conserved quantities, the orbit is 8 dimensional.

It may be worth pointing out that the same mathematical structure is encountered in paraxial optics of Gaussian beams (Simon and Mukunda 1991). The reasons are really similar -

the distribution function for the light rays (or the corresponding object in the wave theory) depends on a quadratic function of the two coordinates and two momenta transverse to the axis of the light beam.

The two conserved quantities we have introduced algebraically can also be interpreted in terms of the geometry of the ellipsoid $z^T Qz = 1$ which defines the boundary of the system in phase space. One can draw a (two dimensional) plane through the origin which intersects the three dimensional boundary in a one dimensional curve (an ellipse). The integral $I_1 = \int p_1\, dq_1 + p_2\, dq_2$ around this closed curve is called the first Poincare' invariant and has the property that it is preserved under canonical transformation in general and time evolution in particular. One can therefore ask for the two extremal sections for which this invariant has the largest and smallest values. One could choose these values as the two constants of motion. More precisely, we can denote the generalised "areas" of these two sections by πA_1 and πA_2. Then it can be shown that A_1 and A_2 are the roots of the polynomial $x^2 + (C_1/2)x + (C_2^2 - C_4)/4$. The product $(\pi^4/4)A_1^2 A_2^2$ is the square of the four dimensional volume enclosed by the boundary of the ellipsoid, which is of course a conserved quantity by Liouville's theorem. The next question is whether the evolution of the P's given by equations (6) or (8) is Hamiltonian. In other words, can we define Poisson brackets (P,B's for short) between the P's so that we can write

$$\frac{d}{dt} P_{ij} = [P_{ij}, H]_{P.B.} \tag{10}$$

where H is a function of the P's? In this case, we could regard these parameters as (noncanonical) coordinates in a phase space. The advantage of such a Hamiltonian formulation would be that well known properties of Hamiltonian systems and techniques for studying them could be used for the study of the dynamics of the collective variables P_{ij} which describe our model.

A straightforward and physically motivated solution to this problem can be given using the idea that our system is just the limit of an N body system. A quantity like $5 \langle z_i z_j \rangle = P_{ij}$ can also be written as

$$P_{ij} = 5 \langle z_i z_j \rangle = \frac{5}{N} \sum_{a=1}^{N} z_i^a z_j^b$$

where z_i^a's are the four phase space variables for particle number a. We can now use the well known notion of Poisson brackets for the N particles,

$$[z_i^a, z_j^b] = \beta_{ij}$$

for all a = b and zero otherwise. This straightaway give

$$[P_{ij}, P_{k\ell}]_{P.B} = 5(\beta_{ik} P_{j\ell} + \beta_{i\ell}P_{jk} + \beta_{jk}P_{i\ell} + \beta_{j\ell}P_{ik})$$

i.e. enough terms to ensure symmetry with respect to interchange of ij, and kℓ. The factor of 5 can be eliminated by taking $P'_{ij} \equiv 5P_{ij}$. The Poisson bracket then reads

$$[P'_{ij}, P'_{k\ell}]_{P.B} = \beta_{ik}P'_{j\ell} + \beta_{i\ell}P'_{jk} + \beta_{jk}P'_{i\ell} + \beta_{j\ell}P'_{ik}$$

The advantage of proceeding from the P.B. for an N-particle system is that the basic properties of the PB are formally guaranteed. One also sees that the Hamiltonian for the N-particle system can be reexpressed in terms of P_{ij}'s (since both the kinetic and potential energies can). Then the time derivatives of the P_{ij}'s can be written in the desired form (10).

In the ten dimensional space of the P_{ij}'s, one can ask if the PB is degenerate i.e. are there any combinations of the P's whose PB's with all the P's vanish. These would then be conserved quantities for any Hamiltonian which is a function of the P's. The two conserved quantities C_1 and C_2 have been written down. It is therefore clear that we are dealing with a ten dimensional space of P's which, by fixing the values of C_1 and C_2, is divided into eight dimensional subspaces. The dimension of the appropriate phase space is therefore eight. The notion of making the orbit (under conjugation) of a Lie algebra into a phase space is well known to mathematicians (Arnold 1978).

The Poisson bracket used above is just a special case of that introduced by Morrison (1982) for the CBE in general. Incidentally his more general result can also be viewed as a limiting case of the N-particle PB by the method outlined above.

CONCLUSION AND GENERAL COMMENTS

The AN planar model discussed in this paper, and related exact time dependent solutions can all be regarded as successful descriptions in terms of collective variables such as the moments P_{ij}, or the semiaxes of a spheroid, etc. For harmonic potentials, these variables obey a closed finite dimensional Hamiltonian dynamics of their own. This dynamics is not trivial as can be checked by referring to AN or SN. For example, in the case of spheroids, the dynamics of the two axes could be chaotic in general. The AN model is quite rich having an eight dimensional phase space, equivalent to four coordinates and their time derivatives. The full phase space of the model has not been explored but it would be surprising indeed if it did not contain regions of chaotic motion. However, in a finite dimensional Hamiltonian system, there is no question of relaxation to a steady state. A challenging open question is whether there are useful approximate descriptions of more general time dependent stellar systems based on phase space moments or other collective variables like the quantities P_{ij} of the AN model

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