BULL. AUSTRAL. MATH. SOC. 30C15
VOL. 36 (1987) 449-460

ON COMPOSITE POLYNOMIALS WHOSE ZEROS ARE IN A HALF-PLANE

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Let $P(z)$ and $Q(z)$ be two polynomials of the same degree n. If $P(z)$ and $Q(z)$ are apolar and if one of them has all its zeros in a circular region C, then according to a famous result known as Grace's Apolarity Theorem, the other will have at least one zero in C. In this paper we relax the condition that $P(z)$ and $Q(z)$ are of the same degree and present some generalizations of Grace's Apolarity theorem for the case when the circular region C is a closed half-plane. As an application of these results, we also generalize some results of Walsh and Szegö.

1. Introduction

Two polynomials

$$P(z) = \sum_{j=0}^{n} C(n,j)A_j z^j \quad \text{and} \quad Q(z) = \sum_{j=0}^{n} C(n,j)B_j z^j, \quad A_n B_n \neq 0$$

of the same degree n are said to be apolar if their coefficients satisfy the relation

$$C(n,0)A_0B_n - C(n,1)A_1B_{n-1} + \ldots + (-1)^n C(n,n)A_nB_0 = 0 .$$

Received 6 January 1987.

As to the relative location of the zeros of the polynomials $P(z)$ and $Q(z)$, we have the following fundamental result known as Grace's Apolarity theorem [4, p. 61].

THEOREM A. *If* $P(z)$ *and* $Q(z)$ *are apolar polynomials, then any circular region* C *containing all the zeros of* $P(z)$ *or* $Q(z)$ *contains at least one zero of the other polynomial.*

By a circular region we mean the closure of not merely the interior of a circle but also the exterior of a circle or a half-plane.

Recently in [2] and [3], (see also [7]), the author has presented certain generalizations of Theorem A and their applications in the case when the circular region C is the closed interior or exterior of a circle, by studying the relative location of the zeros of the two polynomials

$$(1) \qquad P(z) = \sum_{j=0}^{n} C(n,j)A_j z^j \quad \text{and} \quad Q(z) = \sum_{j=0}^{m} C(m,j)B_j z^j , \quad A_n B_m \neq 0 ,$$

of degree n and m respectively, $m \leq n$, when the coefficients of these polynomials satisfy an apolar type relation. In this paper we study the relative location of the zeros of the two polynomials $P(z)$ and $Q(z)$ defined by (1) with their coefficients satisfying an apolar type relation and obtain some generalizations of Theorem A for the case when the circular region C is a closed half-plane. As an application of these results, we present certain generalizations of results of Walsh and Szegö.

2. Some generalizations of Theorem A for half-planes

THEOREM 1. *If*

$$P(z) = \sum_{j=0}^{n} C(n,j)A_j z^j , \quad A_0 \neq 0 , \quad \text{and} \quad Q(z) = \sum_{j=0}^{m} C(m,j)B_j z^j ,$$

are two polynomials of degree n *and* m *respectively,* $m \leq n$, *such that*

$$(2) \qquad C(m,0)A_0B_m - C(m,1)A_1B_{m-1} + \ldots + (-1)^m C(m,m)A_m B_0 = 0 ,$$

then the following holds.

(i) If all the zeros of $P(z)$ *or* $Q(z)$ *lie in the half-plane* $Re(z) \leq 0$, *then at least one zero of the other polynomial lies in* $Re(z) \leq 0$.

https://doi.org/10.1017/S0004972700003749 Published online by Cambridge University Press

(ii) If all the zeros of $P(z)$ or $Q(z)$ lie in the half-plane $\mathrm{Re}(z) \geq 0$, then at least one zero of the other polynomial lies in $\mathrm{Re}(z) \geq 0$.

(i) and (ii) hold equally well if $\mathrm{Re}(z)$ is replaced by $\mathrm{Im}(z)$.

For the proof of Theorem 1, we need the following lemmas.

LEMMA 1. *If all the zeros of a polynomial $P(z)$ of degree n lie in $\mathrm{Re}(z) \leq a$ $(\mathrm{Re}(z) \geq a)$ and $\mathrm{Re}(\alpha) > a$ $(\mathrm{Re}(\alpha) < a)$, then all the zeros of the first polar derivative*

$$P_1(z) = nP(z) + (\alpha - z)P'(z) ,$$

of $P(z)$ lie in $\mathrm{Re}(z) \leq a$ $(\mathrm{Re}(z) \geq a)$. Furthermore, under the given hypothesis with $a = 0$, if $P(0) \neq 0$, then $P_1(0) \neq 0$.

The first part of Lemma 1 is a special case of a result due to Laguerre [4, p. 49] or [6]. A new, simple and purely analytic proof of Laguerre's theorem is given in [1]. Here we prove the second part of Lemma 1.

Proof of the 2nd part of Lemma 1. Suppose that all the zeros z_1, z_2, ..., z_n of $P(z)$ lie in $\mathrm{Re}(z) \leq 0$, $P(0) \neq 0$ and $\mathrm{Re}(\alpha) > 0$. Then $\mathrm{Re}(z_j) \leq 0$ and $z_j \neq 0$ for all $j = 1, 2, \ldots, n$. We have to show that $P_1(0) \neq 0$. Assume that $P_1(0) = 0$, then $nP(0) + \alpha P'(0) = 0$. Since $P(0) \neq 0$, this implies

$$\sum_{j=1}^{n} \frac{1}{z_j} = -\frac{P'(0)}{P(0)} = \frac{n}{\alpha} ,$$

which gives

$$n\,\mathrm{Re}\left(\frac{1}{\alpha}\right) = \sum_{j=1}^{n} \mathrm{Re}\left(\frac{1}{z_j}\right) = \sum_{j=1}^{n} \mathrm{Re}\,\frac{\overline{z_j}}{|z_j|^2}$$

$$= \sum_{j=1}^{n} \frac{\mathrm{Re}(z_j)}{|z_j|^2} \leq 0 .$$

Hence

$$\frac{\mathrm{Re}(\alpha)}{|\alpha|^2} = \frac{\mathrm{Re}(\bar{\alpha})}{|\alpha|^2} = \mathrm{Re}\left(\frac{1}{\alpha}\right) \leq 0 .$$

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This implies that $Re(\alpha) \le 0$, which is a contradiction to the hypothesis that $Re(\alpha) > 0$. Thus $P_1(0) \ne 0$. Now if all the zeros of $P(z)$ lie in $Re(z) \ge 0$, $P(0) \ne 0$ and $Re(\alpha) < 0$, a similar proof shows that $P_1(0) \ne 0$. This completes the proof.

The following lemma can be proved in the same way as Lemma 1.

LEMMA 2. *If all the zeros of a polynomial* $P(z)$ *of degree* n *lie in* $Re(z) < a (Re(z) > a)$ *and* $Re(\alpha) \ge a (Re(\alpha) \le a)$, *then all the zeros of the polynomial* $P_1(z) = nP(z) + (\alpha - z)P'(z)$ *lie in* $Re(z) < a (Re(z) > a)$.

Remark 1. It can be easily seen that both Lemma 1 and Lemma 2 remain true if $Re(z)$ and $Re(\alpha)$ are throughout replaced by $Im(z)$ and $Im(\alpha)$ respectively.

We also need

LEMMA 3 [4, p. 52]. *If* $P(z) = \sum_{j=0}^{n} C(n,j)A_j z^j$ *is a polynomial of degree* n *and* $\alpha_1, \alpha_2, \ldots, \alpha_m$ *are* m, $m \le n$, *arbitrary real or complex numbers, then the* kth *polar derivative*

$$P_k(z) = (n-k+1)P_{k-1}(z) + (\alpha_k - z)P'_{k-1}(z), \quad k = 1,2, \ldots, m,$$

of $P(z)$, *with* $P_0(z) = P(z)$, *can be written in the form*

$$P_k(z) = \sum_{j=0}^{n-k} C(n-k,j)A_j^{(k)} z^j,$$

where

$$A_j^{(k)} = n(n-1)\ldots(n-k+1) \sum_{i=0}^{k} S(k,i)A_{i+j}$$

and $S(k,i)$ *is the symmetric function consisting of the sum of all possible products of* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *taken* i *at a time.*

Proof of Theorem 1. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the zeros of the polynomial $Q(z)$, then we have

$$\sum_{j=0}^{m} C(m,j)B_j z^j = B_m(z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_m). \tag{3}$$

Comparing the coefficients of the like powers of z on the two sides of (3), we obtain

(4) $$C(m,j)B_{m-j} = C(m,m-j)B_{m-j} = (-1)^j S(m,j)B_m .$$

Now suppose first that all the zeros of the polynomial $P(z)$ lie in $\text{Re}(z) \le 0$. We have to show that at least one zero of $Q(z)$ lies in $\text{Re}(z) \le 0$. Assume that all the zeros of $Q(z)$ lie in $\text{Re}(z) > 0$, then $\text{Re}(\alpha_i) > 0$ for all $i = 1, 2, \ldots, m$. Since $P(0) \ne A_0 = 0$ and all the zeros of $P(z)$ lie in $\text{Re}(z) \le 0$, it follows by repeated application of Lemma 1 that all the zeros of each polar derivative

(5) $$P_k(z) = (n-k+1)P_{k-1}(z) + (\alpha_k - z)P'_{k-1}(z) , \quad k = 1,2,\ldots,m ,$$

also lie in $\text{Re}(z) \le 0$ and $P_k(0) \ne 0$. Hence in particular all the zeros of $P_m(z)$ lie in $\text{Re}(z) \le 0$ and $P_m(0) \ne 0$. But by Lemma 3, $P_m(z)$ can be written as

(6) $$P_m(z) = \sum_{j=0}^{n-m} C(n-m,j)A_j^{(m)} z^j ,$$

where

$$A_j^{(m)} = n(n-1)\ldots(n-m+1) \sum_{i=0}^{m} S(m,i)A_{i+j} ,$$

$$= \frac{n(n-1)\ldots(n-m+1)}{B_m} \sum_{i=0}^{m} (-1)^i C(m,i)B_{m-i}A_{i+j} .$$

Since by hypothesis

$$\sum_{i=0}^{m} (-1)^i C(m,i)B_{m-i}A_i = 0 ,$$

therefore, if $n > m$, then from (6) we get

$$P_m(0) = A_0^{(m)} = \frac{n(n-1)\ldots(n-m+1)}{B_m} \sum_{i=0}^{m} (-1)^i C(m,i)B_{m-i}A_i$$

$$= 0 ,$$

which clearly contradicts (5). In the case $n = m$, from (6) we have

$$P_m(z) \equiv A_0^{(m)} = 0 .$$

Since

$$P_m(z) = P_{m-1}(z) + (\alpha_m - z)P'_{m-1}(z) ,$$

it follows that $P_{m-1}(\alpha_m) = 0$. But $\text{Re}(\alpha_m) > 0$, which contradicts (5)

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again. Hence we conclude that $Q(z)$ must have a zero in $\mathrm{Re}(z) \le 0$.

We next suppose that all the zeros of the polynomial $Q(z)$ lie in $\mathrm{Re}(z) \le 0$. We have to show that $P(z)$ has at least one zero in $\mathrm{Re}(z) \le 0$. Assume the contrary, that is, assume that all the zeros of $P(z)$ lie in $\mathrm{Re}(z) > 0$. Since in the present case $\mathrm{Re}(\alpha_j) \le 0$ for all $j = 1, 2, \ldots, m$, it follows by repeated application of Lemma 2 that all the zeros of each polar derivative

(7) $$P_k(z) = (n-k+1)P_{k-1}(z) + (\alpha_k - z)P'_{k-1}(z), \quad k = 1, 2, \ldots, m,$$

lie in $\mathrm{Re}(z) > 0$. Hence in particular all the zeros of $P_m(z)$ lie in $\mathrm{Re}(z) > 0$. Now if $n > m$, then with the help of (2) it follows from (6) that $P_m(0) = A_0^{(m)} = 0$. This shows that $z = 0$ is a zero of $P_m(z)$, which contradicts (7). If $n = m$, then from

$$P_{m-1}(z) + (\alpha_m - z)P'_{m-1}(z) = P_m(z) \equiv A_0^{(m)} = 0,$$

we get as before $P_{m-1}(\alpha_m) = 0$. Since $\mathrm{Re}(\alpha_m) \le 0$, this contradicts (7) once again. Thus we conclude that $P(z)$ must have at least one zero in $\mathrm{Re}(z) \le 0$. This completes the proof of the first part of Theorem 1. With the help of repeated applications of Lemma 1 and Lemma 2, part (ii) of Theorem 1 can be proved in a similar way to part (i) above. Part (ii) of this theorem also follows by applying part (i) to the polynomials $P(-z)$ and $Q(-z)$. Finally applying part (i) and part (ii) to the polynomials $P(iz)$ and $Q(iz)$, it can be easily seen that these results hold equally well if $\mathrm{Re}(z)$ is replaced by $\mathrm{Im}(z)$. This completes the proof of Theorem 1.

Remark 2. If in Theorem 1, the polynomial $P(z)$ has all its zeros in $\mathrm{Re}(z) \ge a$ where $a \ne 0$ is a real number and $n > m$, then the polynomial $Q(z)$ need not have any zero in $\mathrm{Re}(z) \ge a$. For example, consider the polynomials

$$P(z) = 1 + z + z^2 + \ldots + z^n = \sum_{j=0}^{n} C(n,j)A_j z^j, \qquad n > 1$$

and

$$Q(z) = n + z,$$

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then $n > 1 = m$ and the relation (2) is satisfied. But $P(z)$ has all its zeros in $\mathrm{Re}(z) \geq -1$, whereas the only zero of $Q(z)$ lies on $\mathrm{Re}(z) = -n < -1$. However, in this case we establish the following result.

THEOREM 2. *If*

$$P(z) = \sum_{j=0}^{n} C(n,j)A_j z^j \quad \text{and} \quad Q(z) = \sum_{j=0}^{m} C(m,j)B_j z^j ,$$

are two polynomials of degree n *and* m *respectively,* $m \leq n$ *, such that*

$$C(m,0)B_0A_n - C(m,1)B_1A_{n-1} + \ldots + (-1)^m C(m,m)B_mA_{n-m} = 0 ,$$

then the following holds.

(i) If all the zeros of $P(z)$ *or* $Q(z)$ *lie in the half-plane* $\mathrm{Re}(z) \leq a$ *, then at least one zero of the other polynomial lies in* $\mathrm{Re}(z) \leq a$ *.*

(ii) If all the zeros of $P(z)$ *or* $Q(z)$ *lie in the half-plane* $\mathrm{Re}(z) \geq b$ *, then at least one zero of the other polynomial lies in* $\mathrm{Re}(z) \geq b$ *.*

The results (i) and (ii) hold equally well if $\mathrm{Re}(z)$ *is replaced by* $\mathrm{Im}(z)$ *.*

Proof of Theorem 2. Since $P(z)$ is a polynomial of degree n and therefore, $P^{(k)}(z)$ is a polynomial of degree $n-k$ and hence in particular $R(z) = (m!/n!)P^{(n-m)}(z)$ is a polynomial of degree m. It is an easy matter to see that the polynomial $R(z)$ can be written as

$$R(z) = \sum_{j=0}^{m} C(m,j)A_{n-m+j}z^j .$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the zeros of $R(z)$, then we have

$$\sum_{j=0}^{m} C(m,j)A_{n-m+j}z^j = A_n(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_m) . \tag{9}$$

Equating the coefficients of the like powers of z on the two sides of (9), we get

$$C(m,j)A_{n-j} = C(m,m-j)A_{n-j} = (-1)^j S(m,j)A_n . \tag{10}$$

Now suppose first that all the zeros of $P(z)$ lie in $\mathrm{Re}(z) \leq a$, then it follows by the Gauss-Lucas Theorem that all the zeros of $R(z)$

also lie in $\mathrm{Re}(z) \le a$. We have to show that the polynomial $Q(z)$ has at least one zero in $\mathrm{Re}(z) \le a$. Assume that all the zeros of $Q(z)$ lie in $\mathrm{Re}(z) > a$. Since

(11) $$\mathrm{Re}(\alpha_j) \le a \quad \text{for all } j = 1, 2, \ldots, m ,$$

it follows by repeated application of Lemma 2 that all the zeros of

(12) $$Q_k(z) = (m-k+1)Q_{k-1}(z) + (\alpha_k - z)Q'_{k-1}(z) , \; k = 1, 2, \ldots, m-1 ,$$

also lie in $\mathrm{Re}(z) > a$. Hence in particular all the zeros of $Q_{m-1}(z)$ lie in $\mathrm{Re}(z) > a$. But by Lemma 3, (10) and (8) we have

(13) $$Q_m(z) \equiv B_0^{(m)} = m(m-1) \ldots 2.1 \sum_{i=0}^{m} S(m,i)B_i$$

$$= \frac{m!}{A_n} \sum_{i=0}^{m} (-1)^i \, C(m,i)A_{n-i}B_i = 0 .$$

Since $Q_m(z) = Q_{m-1}(z) + (\alpha_m - z)Q'_{m-1}(z)$, it follows that $Q_{m-1}(\alpha_m) = 0$. But by (11) $\mathrm{Re}(\alpha_m) \le a$, which contradicts (12). Hence $Q(z)$ must have at least one zero in $\mathrm{Re}(z) \le a$.

Next suppose that all the zeros of $Q(z)$ lie in $\mathrm{Re}(z) \le a$. We have to show that $P(z)$ has at least one zero in $\mathrm{Re}(z) \le a$. Assume that all the zeros of $R(z)$ lie in $\mathrm{Re}(z) > a$, then by the Gauss-Lucas Theorem all the zeros of $P(z)$ lie in $\mathrm{Re}(z) > a$, so that we have

(14) $$\mathrm{Re}(\alpha_j) > a \quad \text{for all } j = 1, 2, \ldots, m .$$

Since $Q(z)$ has all its zeros in $\mathrm{Re}(z) \le a$, it follows by repeated application of Lemma 1 that all the zeros of $Q_k(z)$ defined by (12) lie in $\mathrm{Re}(z) \le a$. Hence in particular $Q_{m-1}(z)$ has all its zeros in $\mathrm{Re}(z) \le a$. But by (13),

$$Q_{m-1}(z) + (\alpha_m - z)Q'_{m-1}(z) = Q_m(z) \equiv B_0^{(m)} = 0 .$$

and therefore $Q_{m-1}(\alpha_m) = 0$, which implies $\mathrm{Re}(\alpha_m) \le a$. This clearly contradicts (14). Thus we conclude that $P(z)$ must have at least one zero in $\mathrm{Re}(z) \le a$. This completes the proof of the part (i) of the theorem. Part (ii) of Theorem 2 can be proved in a similar way to part (i) above. Finally if we replace $\mathrm{Re}(z)$ by $\mathrm{Im}(z)$ throughout in the above proof, it can be easily seen that part (i) and part (ii) hold equally well when $\mathrm{Re}(z)$ is replaced by $\mathrm{Im}(z)$. This establishes Theorem 2 completely.

3. Some Applications

In the following, we denote by H any one of the half-planes $\mathrm{Re}(z) \le \alpha$, $\mathrm{Re}(z) \ge \beta$, $\mathrm{Im}(z) \le a$ or $\mathrm{Im}(z) \ge b$, where α, β, a, b are real numbers. As the first application of Theorem 2, we present the following result which is a generalization of the Coincidence Theorem of Walsh [5] for the case when the circular region C is a half-plane H. Since our method of proof of this result is similar to the proof of Theorem 2 of [3], we shall omit it.

THEOREM 3. *Let* $G(z_1, z_2, \ldots, z_n)$ *be a symmetric* n*-linear form of total degree* m, $m \le n$, *in* $z_1, z_2, \ldots, z_n$ *and let* H *be a half-plane containing the* n *points* $w_1, w_2, \ldots, w_n$. *Then in* H *there exists at least one point* w *such that*

$$G(w, w, \ldots, w) = G(w_1, w_2, \ldots, w_n) .$$

As our next application of Theorem 2, we deduce the following generalization of a result due to Szegö [4, p. 65] for half-planes.

THEOREM 4. *From the two given polynomials*

$$P(z) = \sum_{j=0}^{n} C(n,j)A_j z^j , \quad A_0 A_n \neq 0 \quad \text{and} \quad Q(z) = \sum_{j=0}^{m} C(m,j)B_j z^j ,$$

of degree n *and* m *respectively,* $m \le n$, *we form the third polynomial*

$$R(z) = \sum_{j=0}^{m} C(m,j)A_j B_j z^j ,$$

of degree m. *If all the zeros of* $Q(z)$ *lie in a half-plane* H, *then every zero* w *of* $R(z)$ *has the form* $w = -\alpha\beta$ *where* α *is a zero of* $P(z)$ *and* β *is a suitable chosen point in* H.

Proof of Theorem 4. If w is a zero of the polynomial $R(z)$, then the equation

$$R(w) = \sum_{j=0}^{m} C(m,j)A_j B_j w^j = 0 ,$$

shows that the polynomials

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$$z^n P(-w/z) = C(n,0)(-1)^n A_n w^n + \dots + C(n,m)(-1)^m A_m w^m z^{n-m} + \dots + C(n,n) A_0 z^n$$

and

$$Q(z) = C(m,0)B_0 + C(m,1)B_1 z + \dots + C(m,m)B_m z^m$$

satisfy the condition of Theorem 2. Since all the zeros of $Q(z)$ lie in H, it follows from Theorem 2 that $z^n P(-w/z)$ has at least one zero in H. If the zeros of $P(z)$ are denoted by $\alpha_1, \alpha_2, \dots, \alpha_n$, then the zeros of $z^n P(-w/z)$ will be $-w/\alpha_1, -w/\alpha_2, \dots, -w/\alpha_n$. One of these zeros must be β where $\beta \in H$. Therefore, we must have $w = -\alpha_j \beta$ for some $j = 1, 2, \dots, n$. This completes the proof.

By applying Theorem 2 to the polynomials $P(z)$ and $z^m Q(-w/z)$, we may deduce the following result in exactly the same way as Theorem 4.

THEOREM 5. *From the two given polynomials*

$$P(z) = \sum_{j=0}^{n} C(n,j) A_j z^j \quad \text{and} \quad Q(z) = \sum_{j=0}^{m} C(m,j) B_j z^j, \quad B_0 B_m \neq 0,$$

of degree n *and* m *respectively,* $m \leq n$, *we form the third polynomial*

$$R(z) = \sum_{j=0}^{m} C(m,j) A_{n-m+j} B_j z^j,$$

of degree m. *If all the zeros of* $P(z)$ *lie in a half-plane* H, *then every zero* w *of* $R(z)$ *has the form* $w = -\alpha\beta$ *where* β *is a zero of* $Q(z)$ *and* α *is a suitably chosen point in* H.

As an another application of Theorem 2, we obtain the following generalization of a result due to Walsh [5].

THEOREM 6. *From the two given polynomials*

$$P(z) = \sum_{j=0}^{n} a_j z^j = a_n \prod_{j=1}^{n} (z - \alpha_j)$$

and

$$Q(z) = \sum_{j=0}^{m} b_j z^j = b_m \prod_{j=1}^{m} (z - \beta_j),$$

of degree n *and* m *respectively,* $m \leq n$, *let us form the third polynomial*

$$R(z) = \sum_{j=0}^{m} (n-j)!\, a_{n-j} Q^{(j)}(z),$$

of degree m *, then the following holds:*

(i) If all the zeros of $P(z)$ *lie in a half-plane* H *, then every zero* w *of* $R(z)$ *has the form* $w = \alpha + \beta$ *where* α *is a suitably chosen point in* H *and* β *is a zero of* $Q(z)$ *.*

(ii) If all the zeros of $Q(z)$ *lie in a half-plane* H *, then every zero* w *of* $R(z)$ *has the form* $w = \alpha + \beta$ *where* β *is a suitably chosen point in* H *and* α *is a zero of* $P(z)$ *.*

Since the proof of Theorem 6 is analogous to the proof of Theorem 5 of [3], we omit it here. The following corollary is an immediate consequence of Theorem 6.

COROLLARY. *If all the zeros of a polynomial* $P(z) = \sum_{j=0}^{n} a_j z^j$ *of degree* n *lie in* $Re(z) \geq a$ *and all the zeros of a polynomial* $Q(z) = \sum_{j=0}^{m} b_j z^j$ *of degree* m, $m \leq n$ *, lie in* $Re(z) \geq b$ *, then all the zeros of the polynomial*

$$R(z) = \sum_{j=0}^{m} (n-j)!a_{n-j} Q^{(j)}(z) ,$$

of degree m *lie in* $Re(z) \geq a + b$ *.*

This follows from the fact that $Re(\alpha) \geq a$ *and* $Re(\beta) \geq b$ *imply* $Re(w) = Re(\alpha) + Re(\beta) \geq a + b$ *.*

Remark 3. In exactly the same way as Theorem 6, a result similar to Theorem 6 of [3] can be deduced from Theorem 1. Furthermore, in very much the same way as above, we can deduce from Theorems 1 - 6 many other interesting results.

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