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ON HANKEL TRANSFORMABLE SPACES AND A CAUCHY PROBLEM

R. S. PATHAK

1. Introduction. The classical Hankel transform of a conventional function ϕ on $(0, \infty)$ defined formally by

$$H_\mu[\phi] = H_{\mu,y}[\phi(x)]: = \int_0^\infty \phi(x)(xy)^{1/2} J_\mu(xy)dx, \quad \mu \geqq -1/2,$$

was extended by Zemanian [**21-23**] to certain generalized functions of one dimension. Koh [**9**, **10**] extended the work of [**21**] to n-dimensions, and that of [**22**] to arbitrary real values of μ. Motivated from the work of Gelfand and Shilov [**6**], Lee [**11**] introduced spaces of type H_μ and studied their Hankel transforms. The results of Lee [**11**] and Zemanian [**21**] are special cases of recent results obtained by the author and Pandey [**14**]. The aforesaid extensions are accomplished by using the so-called adjoint method of extending integral transforms to generalized functions. Dube and Pandey [**2**], Pathak and Pandey [**15**, **16**] applied a more direct method, the so-called kernel method, for extending the Hankel and other related transforms.

Recently, Eijndhoven and De Graaf [**4**] applied a functional analytic approach to discover certain spaces of test functions and generalized functions which are invariant under Hankel transforms. They could discover three such spaces of test functions: S_{X,A_μ}, $\tau(X, \log A_\mu)$, $\tau(X, A_\mu)$. It turns out that $\tau(X, \log A_\mu)$ is the same as Zemanian's space $\mathscr{H}_\mu$ [**21**]. The space S_{X,A_μ} is related to the Gelfand-Shilov space $S_{1/2}^{1/2}$ [**6**], and $\tau(X, A_\mu)$ also possesses a characterization similar to $S_{1/2}^{1/2}$. It is well-known that S_α^α and its generalization W_M^Ω-space [**7**] are invariant under Fourier transformation for certain values of M and Ω. This motivated us to investigate certain test function spaces of W-type which are suitable for Hankel transforms. These spaces are denoted by $U_{\mu,M,a}$, $U_\mu^{\Omega,b}$ and $U_{\mu,M,a}^{\Omega,b}$. Here M and Ω are arbitrary convex functions. That is why these spaces are able more exactly to discern singularities in the growth or decrease of functions at infinity. The space $U_{\mu,M,a}^{\Omega,b}$ yields a large class of Hankel invariant spaces.

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In this paper we define the aforesaid three U_μ-spaces and study their algebraic and topological properties. We discuss their relationships with other well-known test function spaces and mention their special cases. We study the Hankel transform of test functions in these spaces and define generalized Hankel transform of generalized functions belonging to their dual spaces. We extend the Hankel transform to negative values of μ. Finally, we apply the theory thus developed to study uniqueness and existence of the Cauchy problem for the operator

$$S_\mu: = d^2/dx^2 + (1 - 4\mu^2)/4x^2.$$

Theorem 5.3 of the present paper extends the inversion theorems contained in the papers [**21**] and [**22**] in one stroke, whereas Theorem 5.9 is one of the most general inversion theorems for the Hankel transform.

We shall use the notation and terminology of Schwartz as employed in [**20**]. I denotes the open interval $(0, \infty)$; x, y, u and t are real one-dimensional variables. $z = x + iy$ and $s = u + it$ denote complex variables. The functions of z and s will be restricted to their principal branches. We shall use the following linear operators:

$$D^k = D_z^k = d^k/dz^k, \ k = 0, 1, 2, \dots$$

$$M_\mu = M_{\mu,z} = z^{-\mu-1/2} D z^{\mu+1/2}$$

$$N_\mu = N_{\mu,z} = z^{\mu+1/2} D z^{-\mu-1/2}$$

$$N_\mu^{-1} = N_{\mu,z}^{-1} = z^{\mu+1/2} \int_\infty^z t^{-\mu-1/2} \dots dt$$

$$S_\mu = S_{\mu,z} = M_\mu N_\mu = D^2 + (1 - 4\mu^2)/4z^2.$$

2. The spaces $U_{\mu,M,a}$, $U_{\mu,M}$ and their duals. Let ξ be a continuous increasing function on $[0, \infty)$ such that $\xi(0) = 0$ and $\xi(\infty) = \infty$. For $x \geqq 0$ define an increasing, convex, continuous function M by

$$M(x) = \int_0^x \xi(t)dt, \quad M(-x) = M(x).$$

Then $M(0) = 0$, $M(\infty) = \infty$ and

(1) $\quad M(x_1) + M(x_2) \leqq M(x_1 + x_2), \quad x_1, x_2 \geqq 0.$

Now, the space $U_{\mu,M}$ is defined as the set of all complex-valued C^∞-functions ϕ on $I = (0, \infty)$ satisfying

(2) $\quad |(x^{-1}d/dx)^q x^{-\mu-1/2}\phi(x)| \leqq C_q \exp(-M(ax)), \quad \mu \in \mathbf{R},$

for each non-negative integer q, where the positive constants C_q and a depend on ϕ. Clearly $U_{\mu,M}$ is a linear space. The space $U_{\mu,M}$ can be regarded as the union of countably-normed spaces $U_{\mu,M,a}$ of all complex-valued C^∞-functions ϕ, which for any $\delta > 0$ satisfy the inequalities

(3) $|(x^{-1}d/dx)^q x^{-\mu-1/2}\phi(x)| \leqq C_{q\delta}\exp(-M([a-\delta]x))$,

for each $q = 0, 1, 2, \ldots$. The topology over $U_{\mu,M,a}$ is generated by the family of norms

(4) $$\|\phi\|_p = \|\phi\|_{p,\mu} = \sup_{\substack{x \\ q \leqq p}} M_p(x)|(x^{-1}d/dx)^q x^{-\mu-1/2}\phi(x)| < \infty,$$

where

$$M_p(x) = \exp(M[a(1 - 1/p)]x), \quad p = 2, 3, \ldots.$$

It is not hard to see that $U_{\mu,M,a}$ is sequentially complete [**6**]. The union of the spaces $U_{\mu,M,a}$ with $a = 1, 1/2, \ldots$ coincides with the space $U_{\mu,M}$. The dual spaces of $U_{\mu,M,a}$ and $U_{\mu,M}$ are denoted by $U'_{\mu,M,a}$ and $U'_{\mu,M}$ respectively. They are also complete with respect to their weak topologies.

For $M(x) = x^{1/\alpha}$, $0 < \alpha < 1$, the space $U_{\mu,M}$, $\mu \in \mathbf{R}$, coincides with the space $H_{\mu,\alpha}$ considered by Lee [**11**].

We now list some properties of the spaces $U_{\mu,M,a}$, $U_{\mu,M}$ and their duals.

(i) $\mathscr{D}(I)$ the space of C^∞-functions of compact supports on I is a subspace of $U_{\mu,M,a}(I)$, and the topology of $\mathscr{D}(I)$ is stronger than that induced on it by $U_{\mu,M,a}(I)$. Hence, the restriction of any $f \in U'_{\mu,M,a}(I)$ to $\mathscr{D}(I)$ is in $\mathscr{D}'(I)$, and convergence in $U'_{\mu,M,a}(I)$ implies convergence in $\mathscr{D}'(I)$.

(ii) For every choice of μ and $a > 0$,

$$U_{\mu,M,a}(I) \subset \mathscr{E}(I).$$

Moreover, it is dense in $\mathscr{E}(I)$ because $\mathscr{D}(I) \subset U_{\mu,M,a}(I)$ and $\mathscr{D}(I)$ is dense in $\mathscr{E}(I)$. The topology of $U_{\mu,M,a}(I)$ is stronger than that induced on it by $\mathscr{E}(I)$. Hence, $\mathscr{E}'(I)$ can be identified with a subspace of $U'_{\mu,M,a}(I)$.

(iii) The Zemanian's space $\mathscr{B}_{\mu,b}$, $b > 0$, consists of all smooth complex valued functions ϕ on $(0, \infty)$ such that $\phi(x) = 0$ on $b < x < \infty$ and

$$\nu_k^\mu(\phi) = \sup_{0<x<\infty} |(x^{-1}D)^k x^{-\mu-1/2}\phi(x)| < \infty, \quad k = 0, 1, 2, \ldots.$$

The strict inductive limit of $\mathscr{B}_{\mu,b}$ is denoted by $\mathscr{B}_\mu$. Then $\mathscr{B}_\mu$ is a subspace of $U_{\mu,M}$. The topology of $\mathscr{B}_\mu$ is stronger than the topology induced on it by $U_{\mu,M}$. With the induced topology $\mathscr{B}_\mu$ is everywhere dense in $U_{\mu,M}$.

(iv) If $a > b > 0$, then $U_{\mu,M,a} \subset U_{\mu,M,b}$ and the topology of $U_{\mu,M,a}$ is stronger than that induced on it by $U_{\mu,M,b}$. Hence, the restriction of $f \in U'_{\mu,M,b}$ to $U_{\mu,M,a}$ is in $U'_{\mu,M,b}$ and convergence in $U'_{\mu,M,b}$ implies convergence in $U'_{\mu,M,a}$.

(v) If q is an even positive integer, then $U_{\mu+q,M,a} \subset U_{\mu,M,a}$, and the topology of $U_{\mu+q,M,a}$ is stronger than that induced on it by $U_{\mu,M,a}$. For, let $\phi \in U_{\mu+2,M,a}$, then after a little computation, we have

$$(x^{-1}D)^q x^{-\mu-1/2}\phi(x) = 2k(x^{-1}D)^{q-1}x^{-\mu-5/2}\phi(x) + x^2(x^{-1}D)^q x^{-\mu-5/2}\phi(x).$$

Since $x^2 \leqq C \exp(ax/2p)$, using convexity of M one can conclude that

$$U_{\mu+2,M,a} \subset U_{\mu,M,a}.$$

(vi) The operation $\phi \to x\phi$ is an isomorphism from $U_{\mu,M,a}$ onto $U_{\mu+1,M,a}$.

(vii) The operation $\phi \to N_\mu\phi$ is an isomorphism from $U_{\mu,M,a}$ onto $U_{\mu+1,M,a}$, the inverse mapping being $\phi \to N_\mu^{-1}\phi$.

(viii) The operation $\phi \to M_\mu\phi$ is a continuous linear mapping of $U_{\mu+1,M,a}$ into $U_{\mu,M,a}$.

The proof can be given by using the equality

$$(x^{-1}D)^k x^{-2\mu-1}Dx^{2\mu+2}x^{-\mu-3/2}\phi = 2(\mu + k + 1)(x^{-1}D)^k x^{-\mu-3/2}\phi + x^2(x^{-1}D)^{k+1}x^{-\mu-3/2}\phi.$$

The adjoint operator theory [**20**, pp. 25-29], when applied to (v)-(viii), yields the following:

(ix) If q is an even positive integer, then the restriction of $f \in U'_{\mu,M}$ to $U_{\mu+q,M}$ is in $U'_{\mu+q,M}$; moreover convergence in $U'_{\mu+q,M}$ implies convergence in $U'_{\mu+q,M}$.

(x) The operation $f \to xf$ is an isomorphism from $U'_{\mu+1,M}$ onto $U'_{\mu,M}$.

(xi) The operation $f \to N_\mu f$ is an isomorphism from $U'_{\mu+1,M}$ onto $U'_{\mu,M}$, the inverse mapping being $f \to N_\mu^{-1}f$.

(xii) The operation $f \to M_\mu f$ is a continuous linear mapping of $U'_{\mu,M}$ into $U'_{\mu+1,M}$.

LEMMA 2.1. *$U_{\mu,M}$ is a subspace of Zemanian's space $\mathscr{H}_\mu$. The topology of $U_{\mu,M}$ is stronger than the topology induced on it by $\mathscr{H}_\mu$. With the induced topology $U_{\mu,M}$ is everywhere dense in $\mathscr{H}_\mu$.*

Proof. Clearly $U_{\mu,M}$ is a subspace of $\mathscr{H}_\mu$. Let $\phi \in U_{\mu,M}$; then there exists $a > 0$ such that

$$\max_{0 \leqq k \leqq p} \nu^\mu_{m,k}(\phi) \leqq \|\phi\|_{p,\mu}$$

for each $m, p = 0, 1, 2, \ldots$, where $\nu^\mu_{m,k}$ is a seminorm on $\mathscr{H}_\mu$. This implies our second assertion.

Finally, to show that $U_{\mu,M}$ is everywhere dense in $\mathscr{H}_\mu$, let $\lambda(x)$ be a smooth function that is equal to 1 on $-\infty < x < 1$ and equal to 0 on $2 < x < \infty$. Then,

$$\lambda_\nu \phi := \lambda(x - \nu)\, \phi \in U_{\mu,M}, \quad \nu = 1, 2, 3, \ldots,$$

whenever $\phi \in \mathscr{H}_\mu$. This implies that

$$\nu^{\mu}_{\substack{m,k\\0 \leq k \leq p}} (\phi - \lambda_\nu \phi) \to 0 \quad \text{as } \nu \to \infty,$$

for each $m, p = 0, 1, 2, \ldots$.

As a consequence of the above we have

LEMMA 2.2. *$\mathscr{H}'_\mu$ is a subspace of $U'_{\mu,M}$. The topology of $\mathscr{H}'_\mu$ is stronger than the topology induced on $\mathscr{H}'_\mu$ by $U'_{\mu,M}$. With the induced topology, $\mathscr{H}'_\mu$ is everywhere dense in $U'_{\mu,M}$.*

THEOREM 2.3. *Let $\theta(x)$ be a smooth function on $0 < x < \infty$ such that*

$$|(x^{-1}D)^q \theta(x)| \leqq D_q \exp(M(a_1 x)), \quad 0 < a_1 < a.$$

Then $\phi \to \theta\phi$ is a continuous linear mapping from $U_{\mu,M,a}$ into $U_{\mu,M,a-a_1}$.

Proof. We have

$$|(x^{-1}D)^q (x^{-\mu-1/2}\theta\phi)|$$

$$= \left| \sum_{n=0}^{k} \binom{k}{n} (x^{-1}D)^n \theta \cdot (x^{-1}D)^{k-n} x^{-\mu-1/2} \phi \right|$$

$$\leqq \left| \sum_{n=0}^{k} \binom{k}{n} D_n C_{q\delta} \exp(M(a_1 x) - M((a - \delta)x)) \right.$$

$$\leqq \sum_{n=0}^{k} \binom{k}{n} D_n C_{q\delta} \exp(-M((a - a_1 - \delta)x)),$$

from which the result follows.

THEOREM 2.4. *Let θ be the same as in Theorem 2.3. Then $f \to \theta f$ is a continuous linear mapping from $U'_{\mu,M,a-a_1}$ into $U'_{\mu,M,a}$ defined by*

$$\langle \theta f, \phi \rangle := \langle f, \theta\phi \rangle, \quad \phi \in U_{\mu,M,a}.$$

3. The spaces $U_\mu^{\Omega,b}$, U_μ^{Ω} and their duals. Let ω be a continuous increasing function on $[0, \infty)$ with $\omega(0) = 0$ and $\omega(\infty) = \infty$. For $y \geqq 0$ define an increasing, convex, continuous function Ω by

$$\Omega(y) = \int_0^y \omega(t)dt, \quad \Omega(-y) = \Omega(y).$$

Then $\Omega(0) = 0$, $\Omega(\infty) = \infty$ and

(5) $\quad \Omega(y_1) + \Omega(y_2) \leqq \Omega(y_1 + y_2), \quad y_1, y_2 \geqq 0.$

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We define the linear space U_μ^Ω as follows: ψ belongs to U_μ^Ω if and only if $s^{-\mu-1/2}\,\psi(s)$ is an even entire function of $s(= u + it)$ and for each nonnegative integer k,

(6) $$|s^{2k-\mu-1/2}\psi(s)| \leqq C_k \exp(\Omega[by]),$$

where the positive constants C_k and b depend on ψ. Or, in other words $s^{-\mu-1/2}\,\psi(s)$ is even, entire and belongs to W^Ω-space [**7**].

A sequence $\{\psi_\nu(s)\} \in U_\mu^\Omega$ is said to converge to zero if the functions $\psi_\nu(s)$ converge uniformly to zero in any bounded domain of the s-plane and satisfy the inequalities

$$|s^{2k-\mu-1/2}\psi_\nu(s)| \leqq C_k \exp(\Omega[(by]),$$

where the constants C_k and b do not depend on the index ν.

The space U_μ^Ω can be regarded as a union of countably normed spaces $U_\mu^{\Omega,b}$. The set of all those functions in U_μ^Ω which satisfy the inequalities

(7) $$|s^{2k-\mu-1/2}\psi(s)| \leqq C_{k\rho} \exp(\Omega[b + \rho]y), \quad \rho > 0,$$

is denoted by $U_\mu^{\Omega,b}$. This is a linear space. The topology over $U_\mu^{\Omega,b}$ is generated by the norms

(8) $$\|\phi\|_{k\rho} = \|\phi\|_{k\rho,b} = \sup_{s\in\mathbf{C}} |s^{2k-\mu-1/2}\psi(s)| \exp(-\Omega[(b + \rho)y]).$$

Using the method used in [**6**] it can be proved that $U_\mu^{\Omega,b}$ is a complete, perfect, countably normed space.

Evidently, the union of all countably normed spaces $U_\mu^{\Omega,b}$ with $b = 1, 2, \ldots$ coincides with the space U_μ^Ω. Therefore, a sequence $\{\phi_\nu\} \in U_\mu^\Omega$ is convergent to zero in U_μ^Ω if it converges to zero in one of the spaces $U_\mu^{\Omega,b}$. U_μ^Ω is also a sequentially complete space. The dual of the spaces $U_\mu^{\Omega,b}$, U_μ^Ω are denoted by $(U_\mu^{\Omega,b})'$ and $(U_\mu^\Omega)'$ respectively. These are also complete with respect to their weak topologies.

We now list some properties of these spaces:

(i) If $0 < b < c$, then $U_\mu^{\Omega,b} \subset U_\mu^{\Omega,c}$ and the topology generated on $U_\mu^{\Omega,b}$ by $U_\mu^{\Omega,c}$ is identical to the topology of $U_\mu^{\Omega,b}$.

The proof can be given by using Phragmen-Lindelöf theorem [**18**, p. 177].

(ii) If q is an even positive integer, then $U_{\mu+q}^{\Omega,b} \subset U_\mu^{\Omega,b}$ and the topology of $U_{\mu+q}^{\Omega,b}$ is stronger than the topology induced on it by $U_\mu^{\Omega,b}$.

(iii) The operation $\phi \to z\phi$ is an isomorphism from $U_\mu^{\Omega,b}$ onto $U_{\mu+1}^{\Omega,b}$.

Again using the theory of adjoint operators, from (ii) and (iii), we derive

(iv) If q is an even positive integer, then the restriction of $f \in (U_\mu^\Omega)'$ to

$U^{\Omega}_{\mu+q}$ is in $(U^{\Omega}_{\mu+q})'$; moreover, convergence in $(U^{\Omega}_{\mu})'$ implies convergence in $(U^{\Omega}_{\mu+q})'$.

(v) The operation $f \to zf$ is an isomorphism from $(U^{\Omega}_{\mu+1})'$ onto $(U^{\Omega}_{\mu})'$.

The Zemanian's $\mathscr{Y}_{\mu,b}$-space [**22**] consists of all those ψ in **C** such that $z^{-\mu-1/2}\psi(z)$ is entire and

$$\alpha^{\mu}_{b,k}(\psi) = \sup_{z} |e^{-b|y|} z^{2k-\mu-1/2}\psi(z)| < \infty.$$

Then obviously $\mathscr{Y}_{\mu,b} \subset U^{\Omega,b}_{\mu}$.

THEOREM 3.1. *For any two positive real numbers b and d, the space* $U^{\Omega,b}_{\mu}$ *is dense in* $U^{\Omega,b+d}_{\mu}$, *and* $\mathscr{Y}_{\mu,b}$ *is also a dense subspace of* $U^{\Omega,b}_{\mu}$.

Proof. Let $S_e(\mathbf{R}^2)$ denote the space of even, smooth functions of rapid descent and let $S_{e,\mu}(z)$ be the subspace of the functions $z^{\mu+1/2}f(z)$, where $f \in S_e(\mathbf{C})$. Then, in view of the property (i), we have

$$S_{e,\mu}(z) \subset \mathscr{Y}_{\mu,b} \subset U^{\Omega,b}_{\mu} \subset U^{\Omega,b+d}_{\mu} \subset z^{\mu+1/2} \cdot S_e(\mathbf{R}^2).$$

The proof is now an immediate consequence of [**19**, Theorem 15.5, p. 160].

THEOREM 3.2. *For* $d > 0$ *let* Φ *be an even entire function such that*

$$|\Phi(z)| \leqq C \exp(\Omega[dy])(1 + |z|^{2m}),$$

where C is a constant and m is a non-negative integer. Then $\psi \to \Phi\psi$ *is a continuous linear mapping from* $U^{\Omega,b}_{\mu}$ *into* $U^{\Omega,d+b}_{\mu}$.

Proof. From the convexity of Ω it follows that

$$\|\Phi\psi\|_{k\rho,d+b} \leqq C(\|\psi\|_{k\rho,b} + \|\psi\|_{(k+m)\rho,b}).$$

This gives the result.

THEOREM 3.3. *Let* Φ *be the same as in Theorem* 3.2. *Then,* $f \to \Phi f$ *is a continuous linear mapping of* $(U^{\Omega,d+b}_{\mu})'$ *into* $(U^{\Omega,b}_{\mu})'$.

Some additional properties of U^{Ω}_{μ}-spaces are derived after Theorem 5.9.

4. The spaces $U^{\Omega,b}_{\mu,M,a}$, $U^{\Omega}_{\mu,M}$ and their duals. Let M and Ω be the same functions as defined in Sections 2 and 3, and let $a, b > 0$. We define the test function space $U^{\Omega,b}_{\mu,M,a}$ as follows: ψ is a member of $U^{\Omega,b}_{\mu,M,a}$ if and only if $s^{-\mu-1/2}\psi(s)$, $s = u + it$, is an even entire function and

$$(9) \quad \|\psi\|_{\delta\rho} = \sup_{s\in\mathbf{C}} |s^{-\mu-1/2}\psi(s)| \times \exp(M[(a-\delta)x] - \Omega[(b+\rho)y]) < \infty.$$

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$U^{\Omega,b}_{\mu,M,a}$ is a linear space and the topology over this space is generated by the norms $\| \|_{\delta\rho}$. It can be seen that this is also a complete, perfect, countably normed space.

The union of all countably normed spaces $U^{\Omega,b}_{\mu,M,a}$ with $a = 1, \frac{1}{2}, \ldots$ and $b = 1, 2, \ldots$ is the space $U^{\Omega}_{\mu,M}$ whose elements ψ satisfy the inequality

(10) $|s^{-\mu-1/2}\psi(s)| \leqq C \exp(-M(ax) + \Omega(by))$.

Or, in other words $s^{-\mu-1/2}\psi(s)$ is even, entire and belongs to W^{Ω}_{M}-space [7].

The convergence and completeness of $U^{\Omega}_{\mu,M}$ are defined as usual.

Examples (1). The following functions are dual in the sense of Young:

$$M(x) = x^p/p, \Omega(y) = y^q/q, \quad (x, y > 0)$$

with $1/p + 1/q = 1$. The corresponding spaces are denoted by

$$U_{\mu,M,a} = U_{\mu,p,a}, \quad U^{\Omega,b}_{\mu} = U^{q,b}_{\mu}, \quad U^{\Omega,b}_{\mu,M,a} = U^{q,b}_{\mu,p,a}.$$

(2). Taking $M(x) = x^{1/\alpha}$, $\Omega(y) = y^{1/(1-\beta)}$, $(\alpha < 1, \beta < 1)$, we obtain the space $U^{\Omega}_{\mu,M}$ which consists of even entire functions $s^{-\mu-1/2}\ \psi(s)$ satisfying

$$|s^{-\mu-1/2}\psi(s)| \leqq C \exp(-a|x|^{1/\alpha} + b|y|^{1/(1-\beta)}).$$

This growth implies that

$$s^{-\mu-1/2}\psi(s) \in S^{\beta}_{\alpha}.$$

(3). Setting $a = A/2$, $b = B/2$, $\alpha = \beta = 1/2$, we see that

$$U^{\Omega}_{\mu,M} = S_{X,A_\mu},$$

where S_{X,A_μ} is the Hankel invariant test function space investigated by Eijndhoven and De Graaf [**4**].

(4). Furthermore, if we take $M(x) = L/2x^2$, $0 < L < 1$, then

$$\Omega(y) = 1/(2L)y^2.$$

Hence, from (9), we derive

$$|s^{-\mu-1/2}\psi(s)| \leqq C_{\delta\rho} \exp(-\tfrac{1}{2}L(a - \delta)^2x^2 + \tfrac{1}{2}L^{-1}(b + \rho)^2y^2).$$

Now, set $b = 1/a$ with $0 < a < 1$,

$$A = L(a - \delta)^2 \quad \text{and} \quad B = L^{-1}(1/a + \rho)^2.$$

So that $0 < A < 1$ and $B > 1$. Then the space $U^{\Omega,1/a}_{\mu,M,a}$ coincides with the Hankel invariant space $\tau(X, A_\mu)$ considered in [**4**].

The following theorems are of great importance to us whose proofs are analogous to [**7**, pp. 15-17].

THEOREM 4.1. *Differentiation and multiplication by polynomials are bounded operators on the* $U^{\Omega,b}_{\mu,M,a}$*-spaces.*

THEOREM 4.2. *Let* $\Omega(y)$ *be the dual of* $M(x)$. *Assume that* Φ *is an entire function satisfying the inequality*

$$|\Phi(z)| \leqq C \exp(M[a_0x] + \Omega[b_0y]).$$

Then $\psi \rightarrow \Phi\psi$ *is a continuous linear mapping from* $U^{\Omega,b}_{\mu,M,a}$ *into* $U^{\Omega,b+b_0}_{\mu,M,a-a_0}$ *for* $a > a_0$.

5. Hankel transforms of U_μ-spaces. We shall now find the Hankel transforms of U_μ-spaces. In what follows we shall assume that the functions M and Ω are dual in the sense of Young, i.e., they satisfy the inequality

(11) $\quad xy \leqq M(x) + \Omega(y), \quad x, y \geqq 0.$

THEOREM 5.1. *Let* $\Omega(y)$ *be the dual of* $M(x)$. *Then for* $\mu \geqq -1/2$,

$$H_\mu[U_{\mu,M,a}] \subset U^{\Omega,1/a}_{\mu}.$$

Proof. Let

$$\psi(u) := \int_0^\infty \phi(x)(xu)^{1/2}J_\mu(xu)dx, \quad \mu \geqq -1/2,$$

be the Hankel transform of $\phi \in U_{\mu,M,a}$. Since

(12) $\quad |[x(u + it)]^{-\mu}J_\mu[x(u + it)]| \leqq A_\mu(e^{-xt} + e^{xt}),$

we have

$$\left|\int_0^\infty \phi(x)[x(u + it)]^{1/2}J_\mu[x(u + it)]dx\right|$$

$$\leqq \int_0^\infty |\phi(x)[x(u + it)]^{1/2+\mu}|\,|[x(u + it)]^{-\mu}J_\mu[x(u + it)]|dx$$

$$\leqq C_0A_\mu \int_0^\infty \exp(-M[(a - \delta)x])(e^{-xt} + e^{xt})$$

$$\times |[x(u + it)]^{\mu+1/2}|dx.$$

The last integral is absolutely convergent because $M(x)$ increases faster than any power of x and $\mu + 1/2 \geqq 0$. Therefore, we can define

(13) $\quad \psi(u + it) := \int_0^\infty \phi(x)[x(u + it)]^{1/2}J_\mu[x(u + it)]dx.$

Now, differentiating formally with respect to $s = u + it$ and using the inequality (11), we have

(14) $$|d/ds[s^{-\mu-1/2}\psi(s)]| = \left|\int_0^\infty \phi(x)x^{\mu+3/2}[-(xs)^{-\mu}J_{\mu+1}(xs)]dx\right| \leqq A'_\mu C_0\,|s| \int_0^\infty \exp(-M[(a-\delta)x])x^{\mu+5/2} \times [e^{-xt} + e^{xt}]dx < \infty.$$

Thus from (13) and (14) it follows that $s^{-(\mu+1/2)}\psi(s)$ is an even entire function of s. Furthermore,

(15) $$s^{2q-\mu-1/2}\psi(s) = \int_0^\infty x^{2q+2\mu+1}[(x^{-1}D_x)^{2q}x^{-\mu-1/2}\phi(x)] \times (xs)^{-\mu}J_{\mu+2q}(xs)dx,$$

for each $q = 0, 1, 2, \ldots$. Since $\phi(x) \in U_{\mu,M,a}$, and $z^{-\mu}J_{\mu+2q}(z)$ is an entire function and hence bounded on every bounded domain of the z-plane; therefore for all x and s,

$$|e^{-x|t|}(xs)^{-\mu}J_{\mu+2q}(xs)| \leqq D_{q\mu},$$

where $D_{q\mu}$ is a constant independent of x and s. Hence

(16) $$|s^{2q-\mu-1/2}\,\psi(s)| \leqq C_{q\delta}D_{q\mu}\int_0^\infty x^{2q+2\mu+1}\exp(x|t| - M[(a-\delta)x])dx \leqq C^{\mu'}_{q\delta}\int_0^\infty \exp(-M[(a-\delta)x] + |t|\,x)x^{2q+2\mu+1}dx.$$

Now, using the Young inequality (11), we obtain

(17) $$\gamma x(\gamma^{-1}|t|) \leqq M(\gamma x) + \Omega(\gamma^{-1}|t|) \quad \text{with } \gamma = a - 2\delta.$$

Then the exponent in (16) is transformed into

$$-M[(a-\delta)x] + M(\gamma x) + \Omega(\gamma^{-1}|t|).$$

Thus, we derive the estimate

$$|s^{2q-\mu-1/2}\psi(s)| \leqq C^{\mu''}_{q\delta}\exp(\Omega[\gamma^{-1}|t|]) \times \int_0^\infty \exp(-M([a-\delta]x) + M([a-2\delta]x))x^{2q+2\mu+1}dx \leqq C^{\mu''}_{q\delta}\exp(\Omega[(1/a+\rho)|t|]) \times \int_0^\infty \exp(-\delta x)x^{2q+2\mu+1}dx,$$

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where $\rho > 0$. Since the last integral is convergent, we have

$$|s^{2q-\mu-1/2}\psi(s)| \leqq C_{q\delta}^{\mu'''} \exp(\Omega[(1/a + \rho)]|t|).$$

This completes the proof.

THEOREM 5.2. *Let the functions M and Ω be the same as in Theorem 5.1. Then for $\mu \geqq -1/2$,*

$$H_\mu[U_\mu^{\Omega,b}] \subset U_{\mu,M,1/b}.$$

Proof. Let $\psi \in U_\mu^{\Omega,b}$. Then we can define Hankel transform of ψ by

$$\phi(u) := \int_0^\infty \psi(x)(xu)^{1/2}J_\mu(xu)dx, \quad u \geqq 0.$$

Differentiating formally under the integral sign we get

$$(-1)^q(u^{-1}D_u)^q u^{-\mu-1/2}\phi(u) = \int_0^\infty x^{2q+\mu+1/2}\psi(x)(xu)^{-\mu-q}J_{\mu+q}(xu)dx.$$

In view of the definition of the function ψ and boundedness of $(xu)^{-\mu-q}J_{\mu+q}(xu)$ it follows that the last integral converges uniformly on $0 < u < \infty$. By induction on q it can be easily seen that ϕ is smooth. Furthermore,

$$\begin{aligned}
&|(u^{-1}D_u)^q u^{-\mu-1/2}\phi(u)| \\
&\leqq \sup_{0<x<\infty} |(1 + x^2)x^{2q+\mu+1/2}\psi(x)| \sup_v |v^{-\mu-q}J_{\mu+q}(v)| \\
&\times \int_0^\infty (1 + x^2)^{-1}dx \\
&\leqq A_q \sup_{0<x<\infty} |(1 + x^2)x^{2q+2\mu+1}x^{-\mu-1/2}\psi(x)|,
\end{aligned}$$

where A_q is a constant. Let r be a positive integer such that $r \geqq q + \mu + 1/2 \geqq 0$. Then for $z = x + iy$,

$$\begin{aligned}
(18) \quad &|(u^{-1}D_u)^q u^{-\mu-1/2}\phi(u)| \\
&\leqq A_q \sup_{0<x<\infty} |(1 + x^2)^{r+1}x^{-\mu-1/2}\psi(x)| \\
&\leqq A_q \sup_{z\in\mathbf{C}} |e^{-u|y|}(1 + z^2)^{r+1}z^{-\mu-1/2}\psi(z)| \\
&\leqq A_q \sum_{n=0}^{r+1}\binom{r+1}{n}C_{n\rho}\exp(\Omega[(b + \rho)y] - u|y|).
\end{aligned}$$

Now, replacing y by $(b + \rho)|y|$ and x by $u/(b + \rho)$ in Young's inequality (11), we get

$$u|y| = M(u/(b + \rho)) + \Omega((b + \rho)|y|).$$

Then the exponent in (18) becomes

$$-u|y| + \Omega((b + \rho)y) = -M(u/(b + \rho)).$$

Next, replacing $(b + \rho)^{-1}$ by $1/b - \delta$, where δ is arbitrarily small, we obtain

$$|(u^{-1}D_u)^q u^{-\mu-1/2}\phi(u)| \leqq C_{q\delta} \exp(-M[1/b - \delta]u).$$

Thus $\phi(u) \in U_{\mu,M,1/b}$.

Combining Theorems 5.1 and 5.2 and using the fact that

$$H_\mu^{-1}(H_\mu\phi) = \phi,$$

we obtain

THEOREM 5.3. *Let $\mu \geqq -1/2$ and let $\Omega(y)$ be the dual of $M(x)$. Then*

$$H_\mu[U_{\mu,M,a}] = U_\mu^{\Omega,1/a}, \quad H_\mu[U_\mu^{\Omega,b}] = U_{\mu,M,1/b},$$

and hence

$$H_\mu[U_{\mu,M}] = U_\mu^\Omega, \quad H_\mu[U_\mu^\Omega] = U_{\mu,M},$$

and for $\mu \geqq -1/2$, H_μ is a topological mapping.

Using Theorem 5.3 and Lemma 2.1 we obtain

THEOREM 5.4. *U_μ^Ω is a subspace of $\mathscr{H}_\mu$. The topology of U_μ^Ω is stronger than the topology induced on it by $\mathscr{H}_\mu$. With the induced topology U_μ^Ω is everywhere dense in $\mathscr{H}_\mu$.*

The proofs of the following lemmas are similar to those of Lemmas 5.10 and 5.11 given later.

LEMMA 5.5. *For $\mu \geqq -1/2$, $\phi \to N_\mu\phi$ is an isomorphism from $U_\mu^{\Omega,b}$ onto $U_{\mu+1}^{\Omega,b}$, the inverse mapping being $\phi \to N_\mu^{-1}\phi$.*

LEMMA 5.6. *For $\mu \geqq -1/2$, $\phi \to M_\mu\phi$ is a continuous linear mapping of $U_{\mu+1}^{\Omega,b}$ into $U_\mu^{\Omega,b}$.*

Applying the theory of adjoint operators to Lemmas 5.5, 5.6 we derive

LEMMA 5.7. *For $\mu \geqq -1/2$, $f \to N_\mu f$ is an isomorphism from $(U_{\mu+1}^{\Omega,b})'$ onto $(U_\mu^{\Omega,b})'$, the inverse mapping being $f \to N_\mu^{-1} f$.*

LEMMA 5.8. *For $\mu \geqq -1/2$, $f \to M_\mu f$ is a continuous linear mapping of $(U_\mu^{\Omega,b})'$ into $(U_{\mu+1}^{\Omega,b})'$.*

THEOREM 5.9. *Let $\Omega_1(y)$, $M_1(x)$ be duals of $M(x)$ and $\Omega(y)$ respectively. Then for $\mu \geqq -1/2$,*

$$H_\mu[U^{\Omega,b}_{\mu,M,a}] = U^{\Omega_1,1/a}_{\mu,M_1,1/b};$$

hence

$$H_\mu[U^{\Omega}_{\mu,M}] = U^{\Omega_1}_{\mu,M_1},$$

and H_μ *is a linear topological mapping.*

Proof. Let

$$\psi(u) = \int_0^\infty \phi(x)(xu)^{1/2}J_\mu(xu)dx, \quad (u \geqq 0)$$

where $\phi \in U^{\Omega,b}_{\mu,M,a}$. Since $z^{-\mu-1/2}\phi(z)$, $z = x + iy$, is an entire analytic function of $z \in \mathbf{C}$, we can proceed as in the proof of Theorem 5.1 and write

$$\psi(u + it) = \int_0^\infty \phi(x + iy)((x + iy)(u + it))^{1/2} \times J_\mu((x + iy)(u + it))dx.$$

It is now easily seen that $s^{-(\mu+1/2)}\,\psi(s)$ is an even entire function of $s = u + it$. Moreover, using the asymptotic properties of Bessel functions and the Young inequality, we have

$$\begin{aligned}
&|s^{-\mu-1/2}\psi(s)| \\
&\leqq \int_0^\infty |z^{2\mu+1}[z^{-\mu-1/2}\,\phi(z)](zs)^{-\mu}\,J_\mu(zs)|dx \\
&\leqq C_{\delta\rho}\,D_\mu \int_0^\infty \exp(-M[(a - \delta)x] + \Omega[(b + \rho)y] + |t|\,x + |y|\,u) \\
&\times (x^2 + y^2)^{\mu+1/2}dx \\
&\leqq C_{\delta\rho}D_\mu \int_0^\infty \sum_{r=0}^{n}\binom{n}{r}\,x^{2(n-r)}y^{2r} \\
&\times \exp(-M[(a - \delta)x] + \Omega[(b + \rho)y] + |t|\,x + |y|\,u)dx,
\end{aligned}$$

where n is a positive integer such that $n \geqq \mu + 1/2$. Thus

$$\begin{aligned}
&|s^{-\mu-1/2}\psi(s)| \\
&\leqq C^{\mu}_{\delta\rho}\sum_{r=0}^{n}\binom{n}{r}(2r)!\,\exp(\Omega([b + \rho]y) + (u + 1)|y|) \\
&\times \int_0^\infty x^{2(n-r)}\exp(-M[(a - \delta)x] + |t|\,x)dx.
\end{aligned}$$

Using the Young inequality this yields

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$$|s^{-\mu-1/2}\psi(s)|$$
$$\leqq C_{\delta\rho}^{\mu'} \exp(\Omega([b+\rho]y) + M([u+1]/\rho) + (\rho|y|))$$
$$\times \sum_{r=0}^{n} \int_0^\infty x^{2(n-r)} \exp(-M([a-\delta]x)$$
$$+ M(\gamma x) + (\gamma^{-1}|t|))dx,$$

where $n \geqq r$ and $\gamma = a - 2\delta$. Therefore

$$|s^{-\mu-1/2}\psi(s)|$$
$$\leqq C_{\delta\rho}^{\mu''} \sup_{z\in\mathbf{C}} e^{-\beta u|y|} \exp(\Omega([b+\rho]y) + M([u+1]/\rho)$$
$$+ \Omega(\rho|y|) + (\gamma^{-1}|t|)),$$

where $\beta > 0$ is arbitrary.

Now, choosing $y > 0$ and invoking the Young inequality again, we have

$$-\beta uy + \Omega([b+\rho]y) + \Omega(\rho y) + M([u+1]/\rho) + (\gamma^{-1}|t|)$$
$$\leqq -\beta uy + \Omega([b+2\rho]y) + C + M([1+1/\rho]u) + \Omega(\gamma^{-1}|t|)$$
$$= C - M(\beta u/(b+2\rho)) + M([1+1/\rho]u) + (\gamma^{-1}|t|)$$
$$\leqq C - M([\beta/(b+2\rho) - 1 - 1/\rho]u) + \Omega([1/a+\rho]t), \ \rho > 0.$$

Choosing $\beta = 1 + (1 + 1/\rho)(b + 2\rho)$ we see that the last term is bounded by

$$C - M([1/b - \delta]u) + \Omega([1/a + \rho]t), \quad \rho > 0, \delta > 0.$$

Thus

$$|(u+it)^{-\mu-1/2}\psi(u+it)|$$
$$\leqq C_{\delta\rho} \exp(-M_1([1/b-\delta]u) + \Omega_1([1/a+\rho]t)),$$

where M_1 is the Young-dual of Ω_1. Hence

$$\psi \in U_{\mu,M_1,1/b}^{\Omega_1,1/a}$$

and H_μ is a topological mapping.

Similarly,

$$H_\mu[U_{\mu,M_1,1/b}^{\Omega_1,1/a}] \subset U_{\mu,M,a}^{\Omega,b}.$$

The proof of the theorem is now completed using the uniqueness property of the Hankel transform.

LEMMA 5.10. *For* $\mu \geqq -1/2$, *the operation* $\psi \to N_\mu\psi$ *is an isomorphism from* $U_{\mu,M,a}^{\Omega,b}$ *onto* $U_{\mu+1,M,a}^{\Omega,b}$.

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Proof. Let

$$\psi(u) = H_\mu[\phi(x)], \quad \text{where } \phi \in U^{\Omega,b}_{\mu,M,a}.$$

Then using analytic continuation and differentiating under the integral sign, which is a valid procedure, we have

$$D_s s^{-\mu-1/2}\psi(s) = \int_0^\infty \phi(z) z^{1/2} D_s(s^{-\mu} J_\mu(zs))dx$$

$$= -\int_0^\infty \phi(z) z^{\mu+1/2} J_{\mu+1}(zs)/(zs)^\mu dx.$$

Hence,

$$N_{\mu,s}\psi(s) = \int_0^\infty (-z\phi)(zs)^{1/2} J_{\mu+1}(zs)dx = H_{\mu+1,s}(-z\phi(z)).$$

Therefore by Theorems 4.1 and 5.9, N_μ is an isomorphism from $U^{\Omega,b}_{\mu,M,a}$ onto $U^{\Omega,b}_{\mu+1,M,a}$.

LEMMA 5.11. *For* $\mu \geqq -1/2$, *the operation* $\psi \to M_\mu \psi$ *is a continuous linear mapping from* $U^{\Omega,b}_{\mu+1,M,a}$ *into* $U^{\Omega,b}_{\mu,M,a}$.

Proof. Let

$$\psi(u) = H_{\mu+1}[\phi(x)] \quad \text{where } \phi \in U^{\Omega,b}_{\mu+1,M,a}.$$

Then, again using analytic continuation and differentiating under the integral sign, we have

$$D_s s^{\mu+1/2}\psi(s) = \int_0^\infty \phi(z) z^{1/2} D_s[s^{\mu+1} J_{\mu+1}(zs)]dx$$

$$= \int_0^\infty \phi(z) z^{3/2} s^{\mu+1} J_\mu(sz)dx.$$

Hence,

$$M_{\mu,s}\psi(s) = H_{\mu,s}(z\phi(z)).$$

Now, invoking Theorems 4.1 and 5.9, we arrive at the result.

Again, the adjoint considerations lead to the following

LEMMA 5.12. *For* $\mu \geqq -1/2$, *the operation* $f \to N_\mu f$ *is an isomorphism from* $(U^{\Omega,b}_{\mu+1,M,a})'$ *onto* $(U^{\Omega,b}_{\mu,M,a})'$.

LEMMA 5.13. *For* $\mu \geqq -1/2$, *the operation* $f \to M_\mu f$ *is a continuous linear mapping from* $(U^{\Omega,b}_{\mu,M,a})'$ *into* $(U^{\Omega,b}_{\mu+1,M,a})'$.

6. Hankel transforms of arbitrary order. Let m be a non-negative integer such that $\mu + m \geqq -1/2$ for any fixed real number μ. Define the Hankel transform $H_{\mu,m}$ for $\phi \in U_{\mu,M}$ by

(19) $\psi(y) := H_{\mu,m}[\phi(x)]$
$$= (-1)^m y^{-m} H_{\mu+m} N_{\mu+m-1} \dots N_{\mu+1} N_\mu [\phi(x)].$$

Then the inverse Hankel transform $H_{\mu,m}^{-1}$ is defined for $\psi \in U_\mu^{\Omega,b}$ by

(20) $\phi(x) := H_{\mu,m}^{-1}[\psi(y)] = (-1)^m N_\mu^{-1} N_{\mu+1}^{-1} \dots N_{\mu+m-1}^{-1}[y^m \psi(y)].$

Our fundamental result of this section is the following:

THEOREM 6.1. *For any real number μ, the extended Hankel transform $H_{\mu,m}$ as defined by* (19) *is an isomorphism from $U_{\mu,M,a}$ onto $U_\mu^{\Omega,1/a}$. Its inverse is defined by* (20). *For $\mu \geqq -1/2$, $H_{\mu,m}$ coincides with H_μ as an isomorphism from $U_{\mu,M}$ onto U_μ^Ω.*

Proof. From Section 2, property (vii) it follows that

$$\phi \rightarrow N_{\mu+m-1} \dots N_{\mu+1} N_\mu \phi$$

is an isomorphism from $U_{\mu,M,a}$ onto $U_{\mu+m,M,a}$. From Theorem 5.3,

$$\phi \rightarrow H_{\mu+m}\phi$$

is an isomorphism from $U_{\mu+m,M,a}$ onto $U_{\mu+m}^{\Omega,1/a}$ for $\mu + m \geqq -1/2$. Also, from Section 3, property (iii), it follows that

$$\psi \rightarrow z^{-m}\psi$$

is an isomorphism from $U_{\mu+m}^{\Omega,b}$ onto $U_\mu^{\Omega,b}$. All these combined together prove the first part of the theorem.

To prove that $H_{\mu,m}^{-1}$ is given by (20) we again use the above mentioned properties and Theorem 5.3, and see that for $\mu + m \geqq -1/2$,

$$H_{\mu+m}[z^m\psi(z)] = H_{\mu+m}^{-1}[z^m\psi(z)] \in U_{\mu+m,M,a}.$$

Now, let $\phi(x) \in U_{\mu,M,a}$. Then we can write

$$-z^{-1}(H_{\mu+1}N_\mu\phi)(z)$$
$$= -z^{-1} \int_0^\infty x^{\mu+1/2}[Dx^{-\mu-1/2}\phi(x)](zx)^{1/2}J_{\mu+1}(zx)dx.$$

Integrating by parts and using the estimates of $\phi(x)$ we see that the limit terms vanish and

$$(H_\mu\phi)(z) = -z^{-1}(H_{\mu+1}N_\mu\phi)(z).$$

Finally, by induction we arrive at

$$(H_\mu\phi)(z) = (-1)^m z^{-m}(H_{\mu+m}N_{\mu+m-1} \dots N_{\mu+1}N_\mu\phi)(z)$$
$$= (H_{\mu,m}\phi)(z).$$

COROLLARY 6.2. *For any real number μ, $H_{\mu,m}$ is an isomorphism from $U_{\mu,M}$ onto U_μ^Ω. For $\mu \geqq -1/2$, $H_{\mu,m}$ coincides with H_μ as an isomorphism from $U_{\mu,M}$ onto U_μ^Ω.*

Using Theorems 5.3, 6.1 and 3.1, we obtain

THEOREM 6.3. *For any real number* μ *and any two positive real numbers* a *and* b *the space* $U_{\mu,M,a}$ *is dense in* $U_{\mu,M,a+b}$.

7. Non-triviality of the U_μ-spaces. We know that $U_{\mu,M}(I)$ is a non-trivial space because it contains the Schwartz test function space $\mathscr{D}(I)$. Since

$$U_\mu^\Omega = H_\mu[U_{\mu,M}] \quad \text{for } \mu \geqq -1/2,$$

it is also non-trivial.

From [7] the space $U_{\mu,M}^\Omega$ is trivial if

$$\lim_{x\to\infty} [\Omega(bx) - M(ax)] = -\infty$$

for arbitrary a and b. This space is non-trivial when

$$M(x) = \Omega(x) = l(x)x^p,$$

where $p > 0$ and l is a slow function [7]. From [7] we also conclude that all the $U_{\mu,p}^p$-spaces are non-trivial for $1 < p < \infty$. All non-trivial U_μ-spaces are sufficiently rich in functions.

8. Hankel invariant spaces. We have already seen in Section 4 that the Hankel invariant test function spaces S_{X,A_μ} and $\tau(X, A_\mu)$ discussed in [4] are special cases of $U_{\mu,M}^\Omega$-spaces. Theorem 5.9 provides us a large class of Hankel invariant spaces.

Let the function M be self-dual in the sense of Young. Then for $\mu \geqq -1/2$,

$$H_\mu[U_{\mu,M,a}^{M,1/a}] = U_{\mu,M,a}^{M,1/a}$$

and

$$H_\mu[U_{\mu,M}^M] = U_{\mu,M}^M.$$

9. The generalized Hankel transform. Using the theory of adjoint operators [20] we can define the generalized Hankel transform H'_μ of each of the dual spaces $(U_{\mu,M,a})'$, $(U_\mu^{\Omega,b})'$ and $(U_{\mu,M,a}^{\Omega,b})'$ as follows:

$$\langle F, \Phi\rangle : = \langle f, \phi\rangle,$$

where $\Phi = H_\mu\phi$, $F = H'_\mu f$, ϕ belongs to $U_{\mu,M,a}$, $U_\mu^{\Omega,b}$ or $U_{\mu,M,a}^{\Omega,b}$, and f belongs to the corresponding dual space. Since $H_\mu = H_\mu^{-1}$ for $\mu \geqq -1/2$, from Theorems 5.3, 5.9 and [20, Theorem 1.10-2, p. 29] we have the following:

THEOREM 9.1. *For* $\mu \geqq -1/2$, *the generalized Hankel transform* H'_μ *is an isomorphism from*

$$(U_\mu^{\Omega,1/a})',\ (U_{\mu,M,1/b})',\ (U_{\mu,M_1,1/b}^{\Omega_1,1/a})'$$

onto

$$(U_{\mu,M,a})',\ (U_{\mu}^{\Omega,b})',\ (U_{\mu,M,a}^{\Omega,b})'$$

respectively.

From Theorems 5.1 and 9.1 it follows that

THEOREM 9.2. *$\mathscr{H}'_{\mu}$ is a subspace of $(U_{\mu}^{\Omega})'$. The topology of $\mathscr{H}'_{\mu}$ is stronger than the topology induced on it by $(U_{\mu}^{\Omega})'$. With the induced topology $\mathscr{H}'_{\mu}$ is everywhere dense in $(U_{\mu}^{\Omega})'$.*

Let f and ϕ belong to the spaces $(U_{\mu}^{\Omega,1/a})'$ and $U_{\mu,M,a}$ respectively. Then the generalized Hankel transform H'_{μ} for any real number μ is defined by

$$\langle H'_{\mu} f, \phi \rangle : = \langle f, H_{\mu,m}\phi \rangle,$$

where m is a non-negative integer such that $\mu + m \geqq -1/2$. We have

THEOREM 9.3. *For any real number μ, the generalized Hankel transform H'_{μ} is an isomorphism from $(U_{\mu}^{\Omega,1/a})'$ onto $(U_{\mu,M,a})'$.*

10. An operation-transform formula. The generalized Hankel transform H'_{μ} can be used to transform a differential equation of the form

$$P(S_{\mu,x})u = g,$$

where P is a polynomial, u and g possess generalized Hankel transforms and

$$S_{\mu,x} := d^2/dx^2 + (1 - 4\mu^2)/4x^2,$$

into an algebraic equation of the form

$$P(-y^2)U = G,$$

where $U = H'_{\mu}u$ and $G = H'_{\mu}g$.

THEOREM 10.1. *For each $k = 0, 1, 2, \dots, \mu \geqq -1/2$ and f belonging to one of the spaces $(U_{\mu,M,a})'$, $(U_{\mu}^{\Omega,b})'$ or $(U_{\mu,M,a}^{\Omega,b})'$,*

$$H'_{\mu}(S_{\mu,x}^{k} f) = -y^2 H'_{\mu} f.$$

Proof. Let

$$f \in (U_{\mu,M,a}^{\Omega,b})' \quad \text{and} \quad \phi \in U_{\mu,M,1/b}^{\Omega,1/a}.$$

Then, since $S_{\mu} = M_{\mu}N_{\mu}$, we have

$$\begin{aligned} \langle H'_{\mu}(S_{\mu,x}^{k} f), \phi \rangle &= \langle (M_{\mu}N_{\mu})^{k} f, H_{\mu}\phi \rangle \\ &= \langle f, (M_{\mu}N_{\mu})^{k} H_{\mu}\phi \rangle \end{aligned}$$

$$= (-1)^k y^{2k} \langle f, H_\mu \phi \rangle$$
$$= (-1)^k y^{2k} \langle H'_\mu f, \phi \rangle.$$

The above steps can be justified by using Theorem 6.4 and Lemmas 6.5-6.8. The cases of other spaces can similarly be disposed of.

Theorem 10.1 can be applied to solve a Dirichlet problem in cylindrical co-ordinates with generalized boundary values belonging to one of the above generalized function spaces, in a manner similar to [**20**, pp. 154-157].

11. Uniqueness of a Cauchy problem. In this section we apply the theory of Hankel transform developed in the preceding sections to establish a uniqueness theorem for the abstract Cauchy problem:

(21) $\partial u(x, t)/\partial t = P(S_{\mu,x})u(x, t)$

(22) $u(x, 0) = u_0(x)$

where $S_{\mu,x}$ is the Bessel differential operator studied in the preceding section and $u(x, t)$ is an $m \times 1$ column vector, and P is an $m \times m$ polynomial matrix with constant coefficients. A similar problem has been investigated by Gelfand and Shilov [**7**] and Friedman [**5**] for the operator $P(i\partial/\partial x)$.

THEOREM 11.1. *Let $\mu \geqq -1/2$. Then the Cauchy problem* (21)-(22) *possesses a unique solution $u(x, t)$ in the space*

$$(U^{q',\,1/(a-d)}_{\mu,q',1/(b+d)})', \quad q' = (2p_0 - 1)/(2p_0),$$

for the interval

$$0 \leqq t \leqq T,\ T < (4cp_0)^{-1}(d/2)^{2p_0}, \quad d < a,$$

and for any initial function $u_0(x)$ belonging to the same space, where p_0 is the reduced order of the system (21)-(22) *with $S_{\mu,x}$ replaced by $i(\partial/\partial x)$ and c is a constant depending on P. Moreover for each t,*

$$|u(x, t)| \leqq A \exp(\beta |x|^{q'}), \quad \beta > 0,$$

almost everywhere, where A is a positive constant independent of t.

Proof. According to the fundamental result [**5**, p. 177], the Cauchy problem (21)-(22) will have a solution in the space Φ'_1 for $0 < t \leqq T$ if there exists a solution of the adjoint problem

(23) $\partial \phi(x, t)/\partial t = \tilde{P}(S_{\mu,x})\phi(x, t),$

(24) $\phi(x, t_0) = \phi_0(x) \in \Phi$

in the space Φ_1 for $0 \leqq t \leqq t_0$, where t_0 is any point in the interval $0 < t \leqq T$, and $\tilde{P}$ is the adjoint of P.

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Applying the Hankel transform H_μ, $\mu \geqq -1/2$, to (23)-(24), we get

(25) $\partial\psi(y, t)/\partial t = \widetilde{P}(-y^2)\psi(y, t)$

(26) $\psi(y, t_0) = \psi_0(y)$

where

$$\psi(y, t) = H_\mu\phi(x, t).$$

A formal solution of (25)-(26) is given by

$$\psi(y, t) = \exp[\,(t - t_0)\widetilde{P}(-y^2)\,]\psi_0(y).$$

Let us write

$$Q(s, t_0, t) = \exp[\,(t - t_0)\widetilde{P}(-s^2)\,], \quad \text{where } s = u + iv.$$

Then $Q(s, t_0, t)$ is an even and entire function of s. Let p_0 be the reduced order of the system (21)-(22) with $S_{\mu,x}$ replaced by $i(\partial/\partial x)$. Then using the inequality

$$|s|^{2p_0} \leqq 2^{2p_0}(\,|u|^{2p_0} + |v|^{2p_0}),$$

from [7, p. 53] we obtain

$$\|Q(s, t_0, t)\,\| \leqq C \exp[\,(2p_0)^{-1}d^{2p_0}(\,|u|^{2p_0} + |v|^{2p_0})$$

under the assumptions

$$t_0 \leqq t \leqq t_0 + T \quad \text{and} \quad 2^{2p_0+1}cT < (2p_0)^{-1}d^{2p_0}.$$

If we set

$$M(u) = u^{2p_0}/(2p_0), \quad \Omega(v) = v^{2p_0}/(2p_0),$$

then

$$\|Q(s, t_0, t)\,\| \leqq C \exp[M(du) + \Omega(dv)\,].$$

Now, let us assume that

$$\phi_0(x) \in \Phi \equiv U^{q',1/a}_{\mu,q',1/b},$$

where $1/q' + 1/q = 1$ and $q = 2p_0$. Then

$$\psi_0(y) = H_\mu\phi_0(x) \in U^{q,b}_{\mu,q,a}.$$

By Theorem 4.2, $\psi \to Q(s, t_0, t)\psi$ is a continuous linear mapping from the space $U^{q,b}_{\mu,q,a}$ into $U^{q,b+d}_{\mu,q,a-d}$ provided $d < a$. This can be achieved by taking T sufficiently small. Thus the Cauchy problem (25)-(26) has a unique solution in $U^{q,b+d}_{\mu,q,a-d}$. Since from Theorem 5.9,

$$H_\mu^{-1}[U^{q,b+d}_{\mu,q,a-d}] = \Phi_1 = U^{q',1/(a-d)}_{\mu,q',1/(b+d)},$$

the Cauchy problem (23)-(24) has a unique solution in $U^{q',1/(a-d)}_{\mu,q',1/(b+d)}$.

Now, define the space E as the space of all measurable functions ϕ having a finite norm

$$\|g\|_r = \left\{ \int_0^\infty \exp(r\beta x^{q'}) \, |g(x)|^r dx \right\}^{1/r}, \quad 1 \leqq r < \infty,$$

where $\beta = \epsilon/(q'(b + d)^{q'})$, for some $\epsilon < 1$. Then $\Phi \subset \Phi_1 \subset E$. Since Φ is sufficiently rich in functions, (see Section 7), it follows that Φ is dense in E. The proof can now be completed by using [**5**, Theorem 6, p. 177].

12. Existence of generalized solutions. Now we are going to show that for any polynomial P there exists a generalized solution of the Cauchy problem (21)-(22) for $0 \leqq t \leqq T$.

THEOREM 12.1. *For any* $u_0 \in (U^{q',1/(a-d)}_{\mu,q',1/(b+d)})'$, *where* q' *and* d *have the same meaning as in Theorem* 11.1, *there exists a generalized solution*

$$u(x, t) \in (U^{q',1/a}_{\mu,q',1/b})'$$

of (21)-(22) *for* $0 \leqq t \leqq T$, *provided* $d < a$.

Proof. Applying formally the Hankel transform of order $\mu \geqq -1/2$ to (21)-(22), we get

$$(27) \quad \partial v(y, t)/\partial t = P(-y^2)v(y, t)$$

$$(28) \quad v(y, 0) = v_0(y)$$

where $v_0(y) = H_\mu[u_0(x)]$. A formal solution of (27)-(28) is

$$v(y, t) = Q(y, t)v_0(y),$$

where

$$Q(y, t) = \exp[tP(-y^2)].$$

Here again we take

$$\Phi = U^{q',1/a}_{\mu,q',1/b}, \quad \Phi_1 = U^{q',1/(a-d)}_{\mu,q',1/(b+d)},$$

$$\hat{\Phi} = H_\mu[\Phi] = U^{q,b}_{\mu,q,a}, \quad \hat{\Phi}_1 = H_\mu[\Phi_1] = U^{q,b+d}_{\mu,q,a-d}.$$

We also have

$$\|Q(s, t)\| \leqq C \exp[M(du) + \Omega(dv)]$$

for $0 < t < T$, $2^{p_0+1}cT < (2p_0)^{-1}d^{2p_0}$, where

$$M(u) = u^{2p_0}/(2p_0), \quad \Omega(v) = v^{2p_0}/(2p_0).$$

Then $v_0(y) \rightarrow Q(y, t)v_0(y)$ is a continuous linear mapping from $\hat{\Phi}$ into $\hat{\Phi}_1$. Therefore, by [**20**, Theorem 1.10-1, p. 20], it is also a continuous linear mapping from $\hat{\Phi}_1'$ into $\hat{\Phi}'$. Thus

$$v(y, t) \in \hat{\Phi}' \quad \text{if } u_0(x) \in \hat{\Phi}_1'.$$

Now, following [**5**, pp. 184-185] it can be shown that if $u \in \Phi_1'$ then $v(y, t)$ is a solution of (27)-(28) in the sense of $\hat{\Phi}'$. Finally, taking the inverse Hankel transform of (27)-(28) we get

$$u(x, t) = H_\mu^{-1}[v(y, t)]$$

as a generalized solution of (21)-(22).

Remark 12.2. Uniqueness and existence theorems for solutions in $(U_{\mu,M,a})'$ can also be similarly established; and using Theorem 6.1 the result may be proved for all real values of μ.

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King Saud University,
Riyad, Saudi Arabia