Let U and V represent the respective centroid/circumcentre and centroid/Steiner point midpoints and M the Euler/Fermat line meet. Then M, U, Q, V are concyclic by reflecting a known circle across the centroid G ([2]).

Note finally that the symmdian point/Kiepert centre join on the Fermat line also lies within every triangle.

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107.43 Generalised averages of polygons

As is well known, the midpoints of the sides of a quadrilateral form a parallelogram. In [1], David Wells extends the previous result to any hexagon and, in [2], Nick Lord extends it further proving the following:

> Take any $2n$-sided polygon, and find the centres of gravity of each set of n consecutive vertices. These form a $2n$-sided polygon whose opposite sides are equal and parallel in pairs.

Retaining the notation provided by the above-cited authors, in particular, denoting by (A_k) an n-sided polygon with vertices $A_1A_2\ldots A_n$, we can generalise the previous statement to any n-sided polygons as follows:

> Take any n-sided polygon (A_k), and find the centres of gravity $(B_{h,k})$ of each set of h consecutive vertices (with $1 \leqslant h \leqslant n-1$) and the centres of gravity $(B_{n-h,k})$ of each set of $n-h$ consecutive vertices. Then, each side $B_{h.k}B_{h.k+1}$ of $(B_{h,k})$ is $\frac{n-h}{h}$-times in length and parallel to the side $B_{n-h.k+h}B_{n-h.k+h+1}$ of $(B_{n-h,k})$. Therefore, $(B_{h,k})$ and $(B_{n-h,k})$ are similar with ratio $\frac{n-h}{h}$.

Indeed, let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be the position vectors of consecutive vertices of (A_k) and continue the labelling cyclically so that $\mathbf{a}_{n+i} = \mathbf{a}_i$ for $1 \leqslant i \leqslant n-1$. The position vectors of the vertices $(B_{h,k})$ of the derived n-sided polygon are then given by $\mathbf{b}_{h,k} = \frac{1}{h}\left(\mathbf{a}_k + \mathbf{a}_{k+1} + \ldots + \mathbf{a}_{k+h-1}\right)$ for $1 \leqslant k \leqslant n$. It follows that $\mathbf{b}_{h,k+1} - \mathbf{b}_{h,k} = \frac{1}{h}\left(\mathbf{a}_{k+h} - \mathbf{a}_k\right)$, and, replacing k by $k+h$ and h by $n-h$, that $\mathbf{b}_{n-h,k+h+1} - \mathbf{b}_{n-h,k+h} = \frac{1}{n-h}\left(\mathbf{a}_{k+n} - \mathbf{a}_{k+h}\right)$. Thus, the side $B_{h,k}B_{h,k+1}$ of $(B_{h,k})$ is $\frac{n-h}{h}$–times and parallel to the side $B_{n-h,k+h}B_{n-h,k+h+1}$ of $(B_{n-h,k})$. Consequently, the angles of $(B_{h,k})$ and $(B_{n-h,k})$ are equal in pairs and all the ratios of the pairs of the corresponding sides are equal to $\frac{n-h}{h}$. Therefore, $(B_{h,k})$ and $(B_{n-h,k})$ are similar with ratio $\frac{n-h}{h}$.

In particular, we recover Nick Lord's result above when $n = 2h$. Indeed, in this case the side $B_{h,k}B_{h,k+1}$ of $(B_{h,k})$ is equal and parallel to the side $B_{n-h,k+h}B_{n-h,k+h+1} = B_{h,k+h}B_{h,k+h+1}$ of $(B_{h,k})$.

Moreover, for any n-sided polygon, replacing h by $n - 1$, shows that the side $B_{n-1,k}B_{n-1,k+1}$ of $(B_{n-1,k})$ is $\frac{1}{n-1}$–times and parallel to the side $B_{1,k+n-1}B_{1,k+n} = B_{1,k+1}B_{1,k}$ of $(B_{1,k}) = (A_k)$. Thus, we can state the following corollary, which, in the case $n = 3$, returns the well-known theorem on the segments joining the midpoints of the sides of a triangle.

> Take any n-sided polygon (A_k), and find the centres of gravity $(B_{n-1,k})$ of each set of $n - 1$ consecutive vertices. These form an n-sided polygon whose sides are parallel in pairs to the sides of (A_k). Moreover, $(B_{n-1,k})$ and (A_k) are similar with ratio $\frac{1}{n-1}$.

In Figure 1 we show that the sides of the 7-sided polygon (A_k) are parallel in pairs to the sides of $(B_{6,k})$. Moreover, each side of $(B_{4,k})$ is parallel to a side of $(B_{3,k})$; precisely, $B_{4,k}B_{4,k+1} // B_{3,k+4}B_{3,k+5}$, and their ratio is $\frac{3}{4}$.

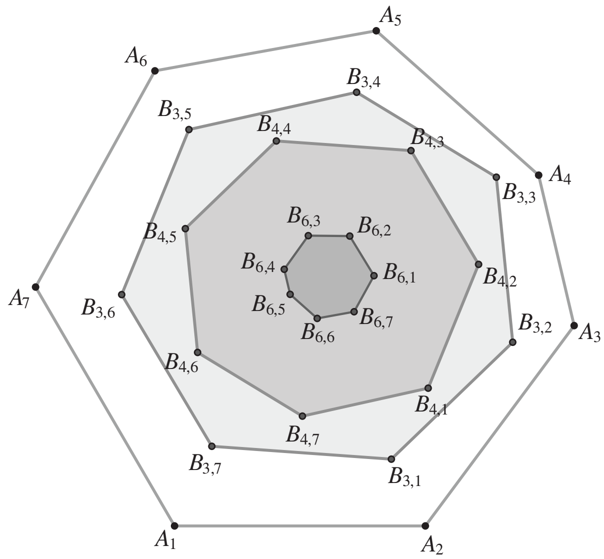

FIGURE 1

In Figure 2 we show that in the 8-sided polygon (A_k), the sides of $(B_{4,k})$ are parallel and equal in pairs; precisely, $B_{4,k}$, $B_{4,k+1} // B_{4,k+4}$, $B_{4,k+5}$. In addition, each side of $(B_{5,k})$ is parallel to a side of $(B_{3,k})$; precisely, $B_{5,k}B_{5,k+1} // B_{3,k+5}B_{3,k+6}$, moreover their ratio is $\frac{3}{5}$.

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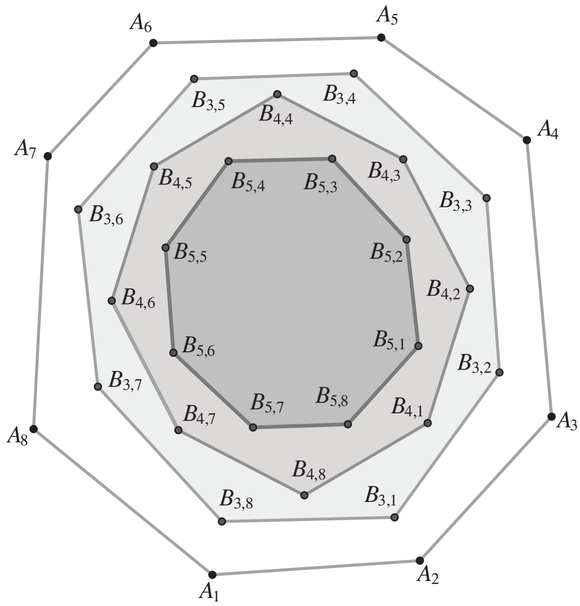

FIGURE 2

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107.44 On converses of circle theorems

It has been said that one learns more from several proofs of the same result by different methods than from proofs of several results by the same method. With this in mind, we are going to look at the well-known circle theorems about angles in the same segment, and opposite angles of a cyclic quadrilateral, Euclid III.21 and III.22, or rather at their converses, which Euclid does not bother to prove. See [1].

So, suppose we are given distinct points A, B, C and D, with $\angle ACB = \alpha$ and either (i) $\angle ADB = \alpha$ (in sign as well as magnitude) or else (ii) $\angle BDA = \pi - \alpha$ (ditto), that is, $\angle ADB = \alpha - \pi$. See Figure 1. What this amounts to, in either case, is that the line through A and D, rotated about