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The stability analysis of a 2D Keller–Segel–Navier–Stokes system in fast signal diffusion

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This paper investigates the stability of a fully parabolic–parabolic-fluid (PP-fluid) system of the Keller–Segel–Navier–Stokes type in a bounded planar domain under the natural volume-filling hypothesis. In the limit of fast signal diffusion, we first show that the global classical solutions of the PP-fluid system will converge to the solution of the corresponding parabolic–elliptic-fluid (PE-fluid) system. As a by-product, we obtain the global well-posedness of the PE-fluid system for general large initial data. We also establish some new exponential time decay estimates for suitable small initial cell mass, which in particular ensure an improvement of convergence rate on time. To further explore the stability property, we carry out three numerical examples of different types: the nontrivial and trivial equilibriums, and the rotating aggregation. The simulation results illustrate the possibility to achieve the optimal convergence and show the vanishment of the deviation between the PP-fluid system and PE-fluid system for the equilibriums and the drastic fluctuation of error for the rotating solution.

Keywords: Parabolic–parabolic-fluid system, stability, convergence rate, decay estimates

2020 Mathematics Subject Classification: 35K55, 35Q92 (Primary); 35Q30, 92C17 (Secondary)

1 Introduction

Keller–Segel system and fast signal diffusion limit. Chemotaxis, the biased movement of cells along spatial gradients of chemical cues, plays an important role in numerous biological circumstances such as bacterial aggregation, spatial pattern formation, embryonic morphogenesis, immune response and also tumour-induced angiogenesis. The most basic mathematical model for chemotaxis was originally derived in 1953 by Patlak [25] and then in 1970 by Keller and Segel [13]. The main unknowns in this so-called Keller–Segel model are the nonnegative cell density n and chemical concentration c, which satisfy the parabolic–parabolic reaction-diffusion equations:

$$\begin{cases} \partial_t n = \tau_1 \Delta n - \nabla \cdot \big(nS\nabla c\big), & x \in \Omega,\ t > 0, \\ \partial_t c = \tau_2 \Delta c - c + n, & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

where τ_1 is the positive cell diffusivity and τ_2 stands for the positive diffusivity of the chemical. In many realistic modelling situations, the chemotactic sensitivity S has to be allowed to depend on the cell density n and on the chemical concentration c.

The celebrated Keller–Segel system (1.1) has been well-studied with regard to biological implications, but beyond this, during the last decades quite a thorough comprehension of its mathematical features has grown in various directions. For instance, a striking feature of system (1.1) appears to be the occurrence of some solutions blowing up in finite time ([11]), which is commonly viewed as mathematically expressing numerous processes of spontaneous cell aggregation which can be observed in experiments. In the spatially two-dimensional framework, in particular, it was shown in [24, 11] that system (1.1) possesses some solutions which blow up in finite time provided that the initially present total mass $\int_\Omega n(x, 0)$ is large enough, whereas solutions remain bounded whenever $\int_\Omega n(x, 0)$ is small; as a precise value distinguishing the respective mass regimes either allowing for or suppressing explosions, the critical mass $m_c = 8\pi$ could be identified in the spatially radial setting or $\Omega = \mathbb{R}^2$. Such explosion phenomena can be ruled out when S is related to the prototypical assumption of volume-filling effect. Precisely, it has been shown in [12] that for the two-dimensional no-flux boundary value problem of system (1.1) with n-dependent sensitivities $S(n)$, all solutions are global and uniformly bounded provided that

$$S(n) \le \frac{C_S}{(1+n)^\alpha} \qquad \text{with} \qquad \alpha > 0 \tag{1.2}$$

for some positive constant C_S, while the solution may blow up if $\Omega \subset \mathbb{R}^2$ is a ball and

$$S(n) \ge \frac{C_S}{(1+n)^\alpha} \qquad \text{with} \qquad \alpha < 0.$$

Due to the experimental facts, the diffusion coefficient τ_2 of the chemoattractant c is usually assumed to be large and the ratio between the diffusivity of the cells and of the chemoattractant

$$\epsilon := \frac{\tau_1}{\tau_2}$$

can be regarded as a relaxation time scale such that ϵ^{-1} is the rate towards equilibrium. Then taking into account the different time scales of the two diffusion processes and replacing $\tau_1 t$ with t in the original parabolic–parabolic (PP) system (1.1), we obtain

$$\begin{cases} \partial_t n = \Delta n - \dfrac{1}{\tau_1}\nabla \cdot \big(nS(n)\nabla c\big), \\[2ex] \epsilon\partial_t c = \Delta c - \dfrac{1}{\tau_2}c + \dfrac{1}{\tau_2}n. \end{cases} \tag{1.3}$$

The formal choice $\epsilon = 0$ in (1.3) will lead to a corresponding parabolic–elliptic (PE) system:

$$\begin{cases} \partial_t n = \Delta n - \dfrac{1}{\tau_1}\nabla \cdot \big(nS(n)\nabla c\big), \\[2ex] 0 = \Delta c - \dfrac{1}{\tau_2}c + \dfrac{1}{\tau_2}n, \end{cases} \tag{1.4}$$

which describes the chemical concentration evolution in a quasi-stationary approximation. The PE system substantially differs from its fully PP system due to the circumstance that the

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former cross-diffusive interaction involves a certain memory. The theory of the PE system (1.4) is relatively well developed. For instance, a comprehensive picture was obtained in [27] for the two-dimensional PE system (1.4): the Dirac mass formation and finiteness of blow-up points were derived without substantial restrictions. Even in the 2D mass critical case, in which solutions to the Cauchy problem of the minimal PE system (1.4) exist globally but blow up in infinite time, it is known that the spatial profile near the corresponding blow-up time $T = \infty$ is essentially dictated by Dirac distributions (see [3, 9]).

Several nice analytical results in [2, 17, 18, 26] showed the stability properties of solutions to Keller–Segel system in the whole space $\mathbb{R}^2$ as $\epsilon \to 0$: solutions of the PP system (1.3) converge in some special cases (e.g. for $c(x, 0) \equiv 0$, for some finite time T or for small initial data) to those of the PE system (1.4) (see also [8, 23] for the initial-boundary value problem). These partially solved an old question raised by Biler [1]. Recently, Liu et al. [20] proposed a semi-discrete scheme based on a symmetrisation reformation and showed that their new scheme is stable as $\epsilon \to 0$ provided that the initial condition does not exceed certain threshold, and it asymptotically preserves the quasi-static limit in the transient regime.

Keller–Segel–(Navier–)Stokes system and fast signal diffusion limit. Partially motivated by the striking experiments in [28], the typical models describing the interaction between populations of chemotactically migrating individuals and viscous fluid environments have become the best-studied models in mathematical biology (see [6, 36]). In [14, 15], Kiselev and Ryzhik considered the effect chemotactic attraction on reproduction of some invertebrates, such as sea urchins, anemones and corals. In particular, they investigated the phenomenon of broadcast spawning whereby males and females release sperm and egg gametes into the surrounding flow. For the coral spawning problem, there is experimental evidence that eggs release a chemical that attracts sperm (see [4, 5]). This leads us to investigate the PP-fluid model:

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot \big(nS\nabla c\big), \\ \epsilon \partial_t c + u \cdot \nabla c \ = \Delta c - c + n, \\ \partial_t u + \kappa (u \cdot \nabla) u + \nabla P = \Delta u + n \nabla \phi, \\ \qquad \nabla \cdot u = 0 \end{cases} \tag{1.5}$$

and to consider the effect of the surrounding fluid on the chemotaxis, where the additional unknowns are the fluid velocity u and the associated pressure P. Here the given potential function $\phi = \phi(x, t)$ arose from the chemotactic boycott effect and the coefficient $\kappa \in \mathbb{R}$ measured the strength of nonlinear fluid convection.

Due to the possible singularity in the fluid-free case as mentioned before, it is natural to require a volume-filling hypothesis of the form (1.2) to establish global solvability for the PP-fluid system (1.5) and its variants (see [29, 34, 37, 41]). For $\kappa = 1$, in particular, it has been revealed in [29] that under the rotational chemotactic assumption of the form $S = S(x, n, c)$ (see [39]) satisfying the natural volume-filling hypothesis

$$|S(x, n, c)| \leq \frac{C_S}{(1+n)^{\alpha}} \tag{1.6}$$

with some positive constant C_S, there exist global bounded classical solutions to the 2D homogeneous Neumann–Neumann–Dirichlet initial-boundary value problem of system (1.5) whenever

$\alpha > 0$, which is accurately consistent with the case of fluid-free system (1.1). A corresponding 3D setting possesses a globally defined weak solution whenever $\alpha > \frac{1}{3}$ (see [16]). For $\kappa = 0$, on the other hand, Lorz [21] illustrated the behaviour of the 2D PE-fluid system

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot \big(nS\nabla c\big), \\ \qquad u \cdot \nabla c \ = \Delta c - c + n, \\ \partial_t u + \kappa(u \cdot \nabla)u + \nabla P = \Delta u + n\nabla\phi, \\ \qquad \nabla \cdot u = 0 \end{cases} \tag{1.7}$$

(i.e. $\epsilon = 0$ in (1.5)) with different numerical examples and in particular gave the numerical evidence that above the critical mass of 8π solutions still exist for PE-fluid system (1.7). Recently, Zheng [42] proved that if $\alpha > 0$, then the associated initial-boundary-value problem (1.7) possesses a global bounded classical solution for any sufficiently smooth initial data (n_0, u_0) satisfying some compatibility conditions. The Dirichlet boundary effects for the signal have also been investigated in [31, 32, 33].

In the last 2 years, two rigorous stability analyses for the chemotaxis-fluid system have been done by [30, 19]. In particular, Wang et al. [30] affirmed that under some assumptions on the model ingredients, that is,

$$\sup_{\epsilon} \|\nabla c_\epsilon\|_{L^\lambda((0,T);L^q(\Omega))} < \infty \qquad \text{and} \qquad \sup_{\epsilon} \|u_\epsilon\|_{L^\infty((0,T);L^r(\Omega))} < \infty$$

with some $\lambda \in (2, \infty]$, $q > d$ and $r > \max\big\{2, d\big\}$ fulfilling $\frac{1}{\lambda} + \frac{d}{2q} < \frac{1}{2}$, there exists a subsequence for solutions $(n_\epsilon, c_\epsilon, u_\epsilon)$ to the initial-boundary value problem of the fully PP-fluid system

$$\begin{cases} \partial_t n_\epsilon + u_\epsilon \cdot \nabla n_\epsilon = \Delta n_\epsilon - \nabla \cdot \big(n_\epsilon S(x, n_\epsilon, c_\epsilon)\nabla c_\epsilon\big) + f(x, n_\epsilon, c_\epsilon), \\ \epsilon\partial_t c_\epsilon + u_\epsilon \cdot \nabla c_\epsilon \ = \Delta c_\epsilon - c_\epsilon + n_\epsilon, \\ \partial_t u_\epsilon + \kappa(u_\epsilon \cdot \nabla)u_\epsilon + \nabla P_\epsilon = \Delta u_\epsilon + n_\epsilon\nabla\phi, \\ \qquad \nabla \cdot u_\epsilon = 0 \end{cases} \tag{1.8}$$

converging to the solution of its PE-fluid counterpart

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot \big(nS(x, n, c)\nabla c\big) + f(x, n, c), \\ \qquad u \cdot \nabla c = \Delta c - c + n, \\ \partial_t u + \kappa(u \cdot \nabla)u + \nabla P = \Delta u + n\nabla\phi, \\ \qquad \nabla \cdot u = 0 \end{cases}$$

in $\Omega \times (0, T)$ as $\epsilon \to 0$, where $\kappa \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a smoothly bounded convex domain. Then under the volume-filling assumption (1.6) with $\alpha > 0$, the first two authors [19] established an algebraic convergence rate of the fast signal diffusion limit for the PP-Stokes system (i.e., $\kappa = 0$ in (1.8)) with $f \equiv 0$ and general large initial data and removed the restriction to asserting convergence only along some subsequence in [30].

Main results. In the present work, we will further consider the stability in a full Keller–Segel–Navier–Stokes system. Precisely, we investigate the convergence of solutions of the PP-fluid system

$$\begin{cases} \partial_t n_\epsilon + u_\epsilon \cdot \nabla n_\epsilon = \Delta n_\epsilon - \nabla \cdot \big(n_\epsilon S(x, n_\epsilon, c_\epsilon)\nabla c_\epsilon\big), & x \in \Omega,\ t > 0, \\ \epsilon \partial_t c_\epsilon + u_\epsilon \cdot \nabla c_\epsilon = \Delta c_\epsilon - c_\epsilon + n_\epsilon, & x \in \Omega,\ t > 0, \\ \partial_t u_\epsilon + (u_\epsilon \cdot \nabla) u_\epsilon + \nabla P_\epsilon = \Delta u_\epsilon + n_\epsilon \nabla \phi, & x \in \Omega,\ t > 0, \\ \qquad \nabla \cdot u_\epsilon = 0, & x \in \Omega,\ t > 0, \\ \big(\nabla n_\epsilon - n_\epsilon S(x, n_\epsilon, c_\epsilon) \cdot \nabla c_\epsilon\big) \cdot \nu = \nabla c_\epsilon \cdot \nu = 0, \quad u_\epsilon = 0, & x \in \partial\Omega,\ t > 0, \\ n_\epsilon(x, 0) = n_0(x), \quad c_\epsilon(x, 0) = c_0(x), \quad u_\epsilon(x, 0) = u_0(x), & x \in \Omega \end{cases} \tag{1.9}$$

to the solution of the corresponding PE-fluid system

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot \big(nS(x, n, c)\nabla c\big), & x \in \Omega,\ t > 0, \\ \qquad u \cdot \nabla c = \Delta c - c + n, & x \in \Omega,\ t > 0, \\ \partial_t u + (u \cdot \nabla) u + \nabla P = \Delta u + n \nabla \phi, & x \in \Omega,\ t > 0, \\ \qquad \nabla \cdot u = 0, & x \in \Omega,\ t > 0, \\ \big(\nabla n - nS(x, n, c) \cdot \nabla c\big) \cdot \nu = \nabla c \cdot \nu = 0, \quad u = 0, & x \in \partial\Omega,\ t > 0, \\ n(x, 0) = n_0(x), \qquad u(x, 0) = u_0(x), & x \in \Omega \end{cases} \tag{1.10}$$

in a setting as general as possible, where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary.

Throughout this paper, we will suppose that the chemotactic sensitivity function $S = \big(S_{ij}\big)_{2\times 2}$ satisfies the requirements of regularity and the volume-filling hypothesis

$$S_{ij}(x, n_\epsilon, c_\epsilon) \in C^2\big(\overline{\Omega} \times [0, \infty) \times [0, \infty)\big) \qquad \text{and} \qquad \big|S(x, n_\epsilon, c_\epsilon)\big| \le \frac{C_S}{(1 + n_\epsilon)^\alpha} \tag{1.11}$$

for some constants $C_S > 0$ and $\alpha > 0$, and that the initial data and the potential function fulfil

$$\begin{cases} n_0 \in W^{2,\infty}(\Omega), \quad n_0 \ge 0 \quad \text{and} \quad n_0 \not\equiv 0 \quad \text{in} \quad \overline{\Omega}, \\ c_0 \in W^{1,\infty}(\Omega), \quad c_0 \ge 0 \quad \text{and} \quad c_0 \not\equiv 0 \quad \text{in} \quad \overline{\Omega}, \\ u_0 \in W^{2,\infty}\big(\Omega;\ \mathbb{R}^2\big) \quad \text{with} \quad \nabla \cdot u_0 \equiv 0 \quad \text{in} \quad \Omega \quad \text{and} \quad u_0 = 0 \quad \text{on} \quad \partial\Omega, \\ \phi \in W^{2,\infty}(\Omega). \end{cases} \tag{1.12}$$

With the above framework, it was shown in [29] that for each fixed $\epsilon > 0$, system (1.9) admits a unique global bounded classical solution $(n_\epsilon, c_\epsilon, u_\epsilon, P_\epsilon)$ satisfying $n_\epsilon \ge 0$ and $c_\epsilon \ge 0$ in $\Omega \times (0, \infty)$.

Our aim is threefold: firstly, we show the global classical solutions $(n_\epsilon, c_\epsilon, u_\epsilon, P_\epsilon)$ (not just a subsequence) of the full PP-fluid system (1.9) will converge to the solution $(n,\ c,\ u,\ P)$ of the corresponding PE-fluid system (1.10) as $\epsilon \to 0$. As a by-product, we obtain the global well-posedness of the PE-fluid system (1.10) for general large initial data. Secondly, we establish exponential time decay estimates of $(n_\epsilon, c_\epsilon, u_\epsilon)$ uniformly in ϵ for small initial cell mass, which in particular ensure an improvement of convergence rate with at most $\frac{1}{2}$-order growth on time t.

Thirdly, we further investigate the convergence behaviour on ϵ and t through the numerical simulations of three different types of solution: the nontrivial and constant equilibriums, and the rotating aggregation (see Remark 5.1 for discussion).

Without loss of generality, we only need to focus on the case of $0 < \alpha < \frac{1}{2}$. Under these assumptions, our main results are the following.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Suppose that (1.11)–(1.12) hold for $\alpha > 0$, and that $(n_\epsilon, c_\epsilon, u_\epsilon, P_\epsilon)$ solves the PP-fluid system (1.9) classically in $\Omega \times (0,\infty)$. Then there exists a unique classical solution (n, c, u, P) to the PE-fluid system (1.10) in $\Omega \times (0,\infty)$ with the property that*

$$\begin{cases} \|n_\epsilon(\cdot,t) - n(\cdot,t)\|_{L^2(\Omega)} + \|n_\epsilon(\cdot,s) - n(\cdot,s)\|_{L^2((0,t);W^{1,2}(\Omega))} \le C_1 e^{C_1 t}\epsilon^{\frac{1}{2}}, \\ \|c_\epsilon(\cdot,s) - c(\cdot,s)\|_{L^2((0,t);W^{1,2}(\Omega))} \le C_1 e^{C_1 t}\epsilon^{\frac{1}{2}}, \\ \|u_\epsilon(\cdot,t) - u(\cdot,t)\|_{L^\infty(\Omega)} + \|u_\epsilon(\cdot,s) - u(\cdot,s)\|_{L^2((0,t);W^{1,2}(\Omega))} \le C_1 e^{C_1 t}\epsilon^{\frac{1}{2}} \end{cases}$$

for all $t \in (0,\infty)$ and some uniform positive constant C_1. For each $\theta \in \left(\frac{1}{2}, \frac{3}{4}\right)$ and $p \ge 2$, we also have

$$\begin{cases} \|A^\theta u_\epsilon(\cdot,t) - A^\theta u(\cdot,t)\|_{L^2(\Omega)} \le C_2 e^{C_2 t}\epsilon^{\frac{1}{2}}, \\ \|n_\epsilon(\cdot,t) - n(\cdot,t)\|_{L^p(\Omega)} \le C_3 e^{C_3 t}\epsilon^{\frac{1}{4}} \end{cases}$$

for all $t \in (0,\infty)$ and some positive constants $C_2 := C_2(\theta)$ and $C_3 := C_3(p)$.

Our second result further reveals that the above exponential growth in time t can be improved as at most $\frac{1}{2}$-order growth for small initial cell mass based on some new exponential time decay estimates of $(n_\epsilon, c_\epsilon, u_\epsilon)$ uniformly in ϵ. For simplicity, we will set $\overline{n}_0 := \frac{1}{|\Omega|}\int_\Omega n_0(x)dx$.

Theorem 1.2. *Under the assumptions of Theorem 1.1, there exists $\delta > 0$ such that if*

$$\|n_0\|_{L^1(\Omega)} \le \delta,$$

then the solutions $(n_\epsilon, c_\epsilon, u_\epsilon)$ to the PP-fluid system (1.9) satisfy the exponential time decay estimates uniformly in ϵ

$$\|n_\epsilon(\cdot,t) - \overline{n}_0\|_{L^\infty(\Omega)} + \|c_\epsilon(\cdot,t) - \overline{n}_0\|_{W^{1,p}(\Omega)} + \|u_\epsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C_1 e^{-\mu t}$$

for any $p > 1$ and all $t \in (0,\infty)$ with some positive constants μ and C_1. Moreover, there exists some uniform positive constant C_2 with the property that

$$\begin{cases} \|n_\epsilon(\cdot,t) - n(\cdot,t)\|_{L^2(\Omega)} + \|n_\epsilon(\cdot,s) - n(\cdot,s)\|_{L^2((0,t);W^{1,2}(\Omega))} \le C_2(1+t)^{\frac{1}{2}}\epsilon^{\frac{1}{2}}, \\ \|c_\epsilon(\cdot,s) - c(\cdot,s)\|_{L^2((0,t);W^{1,2}(\Omega))} \le C_2(1+t)^{\frac{1}{2}}\epsilon^{\frac{1}{2}}, \\ \|u_\epsilon(\cdot,t) - u(\cdot,t)\|_{W^{1,2}(\Omega)} + \|u_\epsilon(\cdot,s) - u(\cdot,s)\|_{L^2((0,t);W^{1,2}(\Omega))} \le C_2(1+t)^{\frac{1}{2}}\epsilon^{\frac{1}{2}} \end{cases}$$

for all $t \in (0,\infty)$. Furthermore, for each $\theta \in \left(\frac{1}{2}, \frac{3}{4}\right)$ and $p > 2$, we have

$$\|A^\theta u_\epsilon(\cdot,t) - A^\theta u(\cdot,t)\|_{L^2(\Omega)} \le C_3(1+t)^{\frac{3}{4}}\epsilon^{\frac{1}{2}},$$

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and

$$\|n_\epsilon(\cdot,t)-n(\cdot,t)\|_{L^p(\Omega)} \le C_4(1+t)^{\frac{1}{2}}\epsilon^{\frac{2}{p^2}}$$

for all $t\in(0,\infty)$ with some positive constants $C_3 := C_3(\theta)$ and $C_4 := C_4(p)$.

Remark 1.1. *In the current two-dimensional setting, Theorem 1.2 also improved the decay estimates obtained by [40] in the sense that we removed the smallness restriction on $\|\nabla c_0\|_{L^2(\Omega)}$ and $\|u_0\|_{L^2(\Omega)}$.*

Key steps in our analysis. In Section 3, we concentrate upon the global existence of classical solution (n, c, u, P) to the PE-fluid system (1.10) as a limit of some subsequence of solutions $(n_\epsilon, c_\epsilon, u_\epsilon, P_\epsilon)$ to the PP-fluid system (1.9). Thus, we need to derive some ϵ-independent estimates for $(n_\epsilon, c_\epsilon, u_\epsilon, P_\epsilon)$.

Unlike the PP-Stokes system studied in [19], the current mass conservation property $\|n_\epsilon(\cdot,t)\|_{L^1(\Omega)} = \|n_0\|_{L^1(\Omega)}$ and the regularity of $\|c_\epsilon(\cdot,t)\|_{L^1(\Omega)}$ (Lemma 2.1) cannot immediately provide the bounds for u_ϵ due to the convective term in equation $(1.9)_3$. Instead, we will first analyse a combinational functional of the form

$$-\frac{1}{2\alpha}\|n_\epsilon^\alpha\|^2_{L^2(\Omega)} + K\epsilon\|c_\epsilon\|^2_{L^2(\Omega)}$$

for some $K>0$ to gain the uniform L^2 space-time bounds for ∇n_ϵ^α and ∇c_ϵ with respect to ϵ (Lemma 3.1), which ensures the L^2 spatial bound for u_ϵ and the L^2 space-time bound for ∇u_ϵ (Lemma 3.2). We next improve our knowledge on the space-time L^p uniform bound for c_ϵ for any $p\ge 2$ (Lemma 3.3). Based on the above conclusions, we shall further establish the key L^2 boundedness of ∇u_ϵ by an entropy-like estimate involving the combinational functional of the form

$$\int_\Omega n_\epsilon \ln n_\epsilon + K\epsilon\int_\Omega |\nabla c_\epsilon|^2 + M\int_\Omega |\nabla u_\epsilon|^2$$

for some positive constants K and M (Lemma 3.4), which guarantees the time-independent spatial L^p uniform bound for u_ϵ for any $p>1$ with respect to ϵ (Lemma 3.5). Thereafter, we will track the time evolution of the combinational functional

$$\|n_\epsilon(\cdot,t)\|^s_{L^s(\Omega)} + \epsilon\|\nabla c_\epsilon(\cdot,t)\|^2_{L^2(\Omega)}$$

for some s in Lemma 3.6. Then following from an induction argument (Corollary 3.1), we reach the L^4 regularity of n_ϵ (Corollary 3.2), which together with the damping effect of c_ϵ provides the uniform L^2 bound (Lemma 3.7) and the eventual L^q bound for ∇c_ϵ (Lemma 3.8). These bounds imply the convergence of some subsequence of $(n_\epsilon, c_\epsilon, u_\epsilon, P_\epsilon)$ (Lemma 3.9).

In Section 4, we shall first derive a linear growth estimate

$$\int_0^t\int_\Omega \partial_t c_\epsilon c \le C(1+t)$$

for the mixed components c_ϵ and c using some subtle difference quotient estimates and the maximal regularity for parabolic equations and Stokes equation (Lemma 4.1). Then the basic energy methods and the variation-of-constants representation provide the convergence rate for general

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large initial data (Lemma 4.2-Lemma 4.5). Then we show that the solution $(n_\epsilon, c_\epsilon, u_\epsilon)$ of the PP-fluid system (1.9) exponentially decays to the constant steady state $(\overline{n}_0, \overline{n}_0, 0)$ uniformly in ϵ for appropriate small initial cell mass with $\overline{n}_0 := \frac{1}{|\Omega|}\int_\Omega n_0(x)dx$ (Lemmas 4.6, 4.7, Corollary 4.1, Lemma 4.8), which ensures that we can improve the growth in time t as at most $\frac{1}{2}$-order by investigating the time evolution of the mixed functional

$$K\|n_\epsilon(\cdot,t)-n(\cdot,t)\|^2_{L^2(\Omega)}+\epsilon\|c_\epsilon(\cdot,t)\|^2_{L^2(\Omega)}+\|u_\epsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(\Omega)}$$

for some $K>0$ (Lemma 4.9). The standard smoothing effect of Stokes semigroup and some energy estimates also entail the higher convergence of u_ϵ (Corollary 4.2 and Lemma 4.10) and n_ϵ (Lemma 4.11).

Notation: In the rest of this paper, we will suppose that $(n_\epsilon, c_\epsilon, u_\epsilon, P_\epsilon)$ is a classical solution to the PP-fluid system (1.9) in $\Omega\times(0,\infty)$ with $\epsilon\in(0,1)$. The positive constants $C, C_1, C_2, \cdots$ are independent of ϵ and t.

2. Preliminaries

In this section, we collected a few preliminaries. We begin with the mass conservation of cell density.

Lemma 2.1. *Suppose that (1.11)–(1.12) hold. Then for all $\epsilon\in(0,1)$,*

$$\|n_\epsilon(\cdot,t)\|_{L^1(\Omega)}=\|n_0\|_{L^1(\Omega)} \qquad \text{for all} \quad t\in(0,\infty), \tag{2.1}$$

and

$$\|c_\epsilon(\cdot,t)\|_{L^1(\Omega)}\le \max\left\{\|n_0\|_{L^1(\Omega)},\ \|c_0\|_{L^1(\Omega)}\right\} \qquad \text{for all} \quad t\in(0,\infty). \tag{2.2}$$

Proof. The mass conversation (2.1) of n_ϵ can be obtained by taking an integration of equation $(1.9)_1$ over Ω. Similarly, integrating equation $(1.9)_2$ over Ω and using a comparison argument, we can obtain the L^1 boundedness (2.2) of c_ϵ. □

Lemma 2.2. *Suppose that (1.11)–(1.12) hold and that $\|n_\epsilon(\cdot,t)\|_{L^s(\Omega)}\le K$, $(t\in(0,\infty))$, for some $s>1$ and $K>0$. Then there exists a positive constant C depending only on s, K and c_0 such that for all $\epsilon\in(0,1)$,*

$$\|c_\epsilon(\cdot,t)\|_{L^s(\Omega)}\le C \qquad \text{for all} \quad t\in(0,\infty).$$

Proof. Multiplying equation $(1.9)_2$ by c_ϵ^{s-1} and integrating by parts over Ω, we have

$$\frac{\epsilon}{s}\frac{d}{dt}\int_\Omega c_\epsilon^s+(s-1)\int_\Omega c_\epsilon^{s-2}|\nabla c_\epsilon|^2+\int_\Omega c_\epsilon^s=\int_\Omega n_\epsilon c_\epsilon^{s-1}\le\frac{s-1}{s}\int_\Omega c_\epsilon^s+\frac{1}{s}\int_\Omega n_\epsilon^s$$

and thus

$$\epsilon\frac{d}{dt}\int_\Omega c_\epsilon^s+s(s-1)\int_\Omega c_\epsilon^{s-2}|\nabla c_\epsilon|^2+\int_\Omega c_\epsilon^s\le\int_\Omega n_\epsilon^s\le K^s$$

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for all $t \in (0, \infty)$. By a basic calculation, we deduce that

$$\int_\Omega c_\epsilon^s(\cdot, t) \leq \max\left\{\int_\Omega c_0^s,\ K^s\right\} \qquad \text{for all} \quad t \in (0, \infty).$$

This completes the proof of Lemma 2.2. □

Lemma 2.3. (Lemma 3.4 in [38]) *Let $a > 0$, $T > 0$ and $y \in C^0([0, T)) \cap C^1(0, T)$ be such that*

$$y'(t) + ay(t) \leq g(t) \qquad \text{for all} \quad t \in (0, T),$$

where the nonnegative function $g \in L^1_{loc}(\mathbb{R})$ has the property that $\frac{1}{\tau}\int_t^{t+\tau} g(s)ds \leq b$ for all $t \in (0, T)$ with some $\tau > 0$ and $b > 0$. Then

$$y(t) \leq y(0) + \frac{b\tau}{1 - e^{-a\tau}} \qquad \text{for all} \quad t \in [0, T).$$

3. Global existence of the PE-fluid system

In this section, we will establish the global existence of classical solution to the PE-fluid system (1.10) through a limit procedure in the PP-fluid system (1.9), which is highly nontrivial due to the loss of uniform parabolicity in c_ϵ equation. Our key idea is to obtain some necessary spatio-temporal estimates using a series of subtle coupled functional evolution estimates and bootstrap arguments.

3.1 The space-time L^2 bound for ∇u_ϵ and L^p bound for c_ϵ

Lemma 3.1. *Suppose that (1.11)–(1.12) hold. Then there exists some positive constant C such that for all $\epsilon \in (0, 1)$, we have*

$$\int_t^{t+1}\int_\Omega |\nabla n_\epsilon^\alpha|^2 \leq C \qquad \text{and} \qquad \int_t^{t+1}\int_\Omega |\nabla c_\epsilon|^2 \leq C \qquad \text{for all} \quad t \geq 0. \tag{3.1}$$

Proof. We first multiply equation $(1.9)_1$ by $n_\epsilon^{2\alpha-1}$, integrate by parts over Ω and use the solenoidality of u_ϵ and the Young inequality to deduce that

$$\begin{aligned}
&-\frac{1}{2\alpha}\frac{d}{dt}\int_\Omega n_\epsilon^{2\alpha} + (1-2\alpha)\int_\Omega n_\epsilon^{2\alpha-2}|\nabla n_\epsilon|^2 \\
&= -\frac{1}{2\alpha}\int_\Omega u_\epsilon \cdot \nabla n_\epsilon^{2\alpha} + (1-2\alpha)\int_\Omega n_\epsilon^{2\alpha-1}\nabla n_\epsilon \cdot \big(S(x, n_\epsilon, c_\epsilon)\cdot\nabla c_\epsilon\big) \\
&= (1-2\alpha)\int_\Omega n_\epsilon^{2\alpha-1}\nabla n_\epsilon \cdot \big(S(x, n_\epsilon, c_\epsilon)\cdot\nabla c_\epsilon\big) \\
&\leq \frac{1-2\alpha}{2}\int_\Omega n_\epsilon^{2\alpha-2}|\nabla n_\epsilon|^2 + \frac{1-2\alpha}{2}C_S^2\int_\Omega n_\epsilon^{2\alpha}(1+n_\epsilon)^{-2\alpha}|\nabla c_\epsilon|^2 \\
&\leq \frac{1-2\alpha}{2}\int_\Omega n_\epsilon^{2\alpha-2}|\nabla n_\epsilon|^2 + C_1\int_\Omega |\nabla c_\epsilon|^2
\end{aligned} \tag{3.2}$$

for all $t > 0$, where we also used the upper estimate (1.11) for S. To compensate the rightmost summand herein properly, we multiply equation $(1.9)_2$ by c_ϵ and utilise the solenoidality of u_ϵ again to find that

$$\frac{\epsilon}{2}\frac{d}{dt}\int_\Omega c_\epsilon^2 + \int_\Omega |\nabla c_\epsilon|^2 + \int_\Omega c_\epsilon^2 = \int_\Omega n_\epsilon c_\epsilon \qquad \text{for all} \quad t > 0. \tag{3.3}$$

For any fixed $\theta \in \left(1, \frac{1}{1-\alpha}\right)$, we use the Hölder inequality to obtain

$$\int_\Omega n_\epsilon c_\epsilon \le \|n_\epsilon\|_{L^\theta(\Omega)}\|c_\epsilon\|_{L^{\frac{\theta}{\theta-1}}(\Omega)} = \|n_\epsilon^\alpha\|^{\frac{1}{\alpha}}_{L^{\frac{\theta}{\alpha}}(\Omega)}\|c_\epsilon\|_{L^{\frac{\theta}{\theta-1}}(\Omega)} \qquad \text{for all} \quad t > 0.$$

Since the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{\theta}{\theta-1}}(\Omega)$ and (2.2) imply that

$$\|c_\epsilon\|^2_{L^{\frac{\theta}{\theta-1}}(\Omega)} \le C_2\|\nabla c_\epsilon\|^2_{L^2(\Omega)} + C_2\|c_\epsilon\|^2_{L^1(\Omega)} \le C_2\int_\Omega |\nabla c_\epsilon|^2 + C_3 \qquad \text{for all} \quad t > 0,$$

we make use of the Young inequality to deduce that

$$\int_\Omega n_\epsilon c_\epsilon \le \frac{1}{2C_2}\|c_\epsilon\|^2_{L^{\frac{\theta}{\theta-1}}(\Omega)} + \frac{C_2}{2}\|n_\epsilon^\alpha\|^{\frac{2}{\alpha}}_{L^{\frac{\theta}{\alpha}}(\Omega)} \le \frac{1}{2}\int_\Omega |\nabla c_\epsilon|^2 + \frac{C_3}{2C_2} + \frac{C_2}{2}\|n_\epsilon^\alpha\|^{\frac{2}{\alpha}}_{L^{\frac{\theta}{\alpha}}(\Omega)} \tag{3.4}$$

for all $t > 0$. In order to guarantee that the last summand here can be absorbed by the dissipated quantity in (3.2), we next apply the Gagliardo–Nirenberg inequality and the mass conservation (2.1) to see that

$$\frac{C_2}{2}\|n_\epsilon^\alpha\|^{\frac{2}{\alpha}}_{L^{\frac{\theta}{\alpha}}(\Omega)} \le C_4\|\nabla n_\epsilon^\alpha\|^{\frac{2(\theta-1)}{\alpha\theta}}_{L^2(\Omega)}\|n_\epsilon^\alpha\|^{\frac{2}{\alpha\theta}}_{L^{\frac{1}{\alpha}}(\Omega)} + C_4\|n_\epsilon^\alpha\|^{\frac{2}{\alpha}}_{L^{\frac{1}{\alpha}}(\Omega)} \le C_5\|\nabla n_\epsilon^\alpha\|^{\frac{2(\theta-1)}{\alpha\theta}}_{L^2(\Omega)} + C_5 \tag{3.5}$$

for all $t > 0$, whereupon substituting (3.4) and (3.5) into (3.3), we have

$$\epsilon\frac{d}{dt}\int_\Omega c_\epsilon^2 + \int_\Omega |\nabla c_\epsilon|^2 + 2\int_\Omega c_\epsilon^2 \le 2C_5\|\nabla n_\epsilon^\alpha\|^{\frac{2(\theta-1)}{\alpha\theta}}_{L^2(\Omega)} + C_6 \qquad \text{for all} \quad t > 0$$

with $C_6 := 2C_5 + \frac{C_3}{C_2}$. This together with (3.2) entails that

$$\begin{aligned}
&\frac{d}{dt}\left\{-\frac{1}{2\alpha}\int_\Omega n_\epsilon^{2\alpha} + 2C_1\epsilon\int_\Omega c_\epsilon^2\right\} + \frac{1-2\alpha}{2\alpha^2}\int_\Omega |\nabla n_\epsilon^\alpha|^2 + C_1\int_\Omega |\nabla c_\epsilon|^2 + 4C_1\int_\Omega c_\epsilon^2 \\
&\le 4C_1C_5\|\nabla n_\epsilon^\alpha\|^{\frac{2(\theta-1)}{\alpha\theta}}_{L^2(\Omega)} + 2C_1C_6 \le \frac{1-2\alpha}{4\alpha^2}\int_\Omega |\nabla n_\epsilon^\alpha|^2 + C_7 \qquad \text{for all} \quad t > 0
\end{aligned}$$

and thus that

$$\frac{d}{dt}\left\{-\frac{1}{2\alpha}\int_\Omega n_\epsilon^{2\alpha} + 2C_1\epsilon\int_\Omega c_\epsilon^2\right\} + \frac{1-2\alpha}{4\alpha^2}\int_\Omega |\nabla n_\epsilon^\alpha|^2 + C_1\int_\Omega |\nabla c_\epsilon|^2 + 4C_1\int_\Omega c_\epsilon^2 \le C_7 \tag{3.6}$$

for all $t > 0$. Here we used the Young inequality in the last inequality of (3.6) due to $\frac{\theta-1}{\alpha\theta} \in (0, 1)$, which follows from $\theta \in \left(1, \frac{1}{1-\alpha}\right)$. Then by setting

$$\begin{aligned}
y(t) &:= -\frac{1}{2\alpha}\|n_\epsilon^\alpha(\cdot, t)\|^2_{L^2(\Omega)} + 2C_1\epsilon\|c_\epsilon(\cdot, t)\|^2_{L^2(\Omega)}, \\
g(t) &:= \frac{1-2\alpha}{4\alpha^2}\|\nabla n_\epsilon^\alpha(\cdot, t)\|^2_{L^2(\Omega)} + C_1\|\nabla c_\epsilon(\cdot, t)\|^2_{L^2(\Omega)},
\end{aligned}$$

and noticing that $4C_1\epsilon\int_\Omega c_\epsilon^2 \le 4C_1\int_\Omega c_\epsilon^2$ due to the fact $\epsilon\in(0,1)$, we can conclude from (3.6) that

$$y'(t)+2y(t)+g(t)\le C_7. \tag{3.7}$$

Since $g(t)$ is nonnegative, we deduce from an ordinary differential inequality comparison argument that

$$y(t)\le C_8 := \max\left\{-\frac{1}{2\alpha}\|n_0^\alpha\|_{L^2(\Omega)}^2+2C_1\|c_0\|_{L^2(\Omega)}^2, \frac{C_7}{2}\right\} \tag{3.8}$$

for all $t>0$, which together with (3.7) yields that

$$\int_t^{t+1} g(s)ds \le y(t)-y(t+1)-2\int_t^{t+1} y(s)ds + C_7$$

for all $t\ge 0$. Due to $\alpha\in\left(0,\frac{1}{2}\right)$, we see from the Hölder inequality and the mass conservation (2.1) that

$$-y(t)\le \frac{1}{2\alpha}\|n_\epsilon^\alpha(\cdot,t)\|_{L^2(\Omega)}^2 \le \frac{|\Omega|^{1-2\alpha}}{2\alpha}\|n_\epsilon(\cdot,t)\|_{L^1(\Omega)}^{2\alpha} = C_9 := \frac{|\Omega|^{1-2\alpha}}{2\alpha}\|n_0\|_{L^1(\Omega)}^{2\alpha},$$

which together with (3.8) yields that

$$\frac{1-2\alpha}{4\alpha^2}\int_t^{t+1}\int_\Omega |\nabla n_\epsilon^\alpha|^2 + C_1\int_t^{t+1}\int_\Omega |\nabla c_\epsilon|^2 = \int_t^{t+1} g(s)ds \le C_{10} := C_8+3C_9+C_7$$

for all $t\ge 0$, which entails (3.1). □

Lemma 3.2. *Suppose that (1.11)–(1.12) hold. Then there exists some positive constant C such that for all $\epsilon\in(0,1)$ we have*

$$\|u_\epsilon(\cdot,t)\|_{L^2(\Omega)}\le C \quad \text{and} \quad \int_t^{t+1}\int_\Omega |\nabla u_\epsilon|^2 \le C \quad \text{for all} \quad t>0. \tag{3.9}$$

Proof. For any fixed $\theta\in\left(1,\frac{1}{1-\alpha}\right)$, we test equation $(1.9)_3$ by u_ϵ and employ the Hölder inequality, the Sobolev embedding $W^{1,2}(\Omega)\hookrightarrow L^{\frac{\theta}{\theta-1}}(\Omega)$, the Poincaré inequality and the Young inequality to obtain that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_\epsilon\|_{L^2(\Omega)}^2 + \|\nabla u_\epsilon\|_{L^2(\Omega)}^2 &= \int_\Omega n_\epsilon \nabla\phi\cdot u_\epsilon \\
&\le \|\nabla\phi\|_{L^\infty(\Omega)}\|n_\epsilon\|_{L^\theta(\Omega)}\|u_\epsilon\|_{L^{\frac{\theta}{\theta-1}}(\Omega)} \\
&\le C_1\|n_\epsilon\|_{L^\theta(\Omega)}\|\nabla u_\epsilon\|_{L^2(\Omega)} \\
&\le \frac{1}{2}\|\nabla u_\epsilon\|_{L^2(\Omega)}^2 + \frac{C_1^2}{2}\|n_\epsilon\|_{L^\theta(\Omega)}^2 \quad \text{for all} \quad t>0.
\end{aligned}$$

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By means of the Young inequality and (3.5), we have

$$\frac{d}{dt}\|u_\epsilon\|^2_{L^2(\Omega)} + \|\nabla u_\epsilon\|^2_{L^2(\Omega)} \leq C_1^2\|n_\epsilon\|^2_{L^\theta(\Omega)} = C_1^2\|n_\epsilon^\alpha\|^{\frac{2}{\alpha}}_{L^{\frac{\theta}{\alpha}}(\Omega)}$$
$$\leq C_2\|\nabla n_\epsilon^\alpha\|^{\frac{2(\theta-1)}{\alpha\theta}}_{L^2(\Omega)} + C_2 \leq \int_\Omega |\nabla n_\epsilon^\alpha|^2 + C_3 \qquad \text{for all} \quad t>0 \quad (3.10)$$

due to $\frac{\theta-1}{\alpha\theta} \in (0, 1)$. Noticing that

$$\int_t^{t+1}\left(\int_\Omega |\nabla n_\epsilon^\alpha(\cdot, s)|^2 + C_3\right) ds \leq C_4$$

due to Lemma 3.1, we then see from (3.10) and Lemma 2.3 that

$$\|u_\epsilon(\cdot, t)\|^2_{L^2(\Omega)} \leq C_5 := \|u_0\|^2_{L^2(\Omega)} + \frac{C_4}{1-e^{-1}} \qquad \text{for all} \quad t>0$$

and thus that

$$\int_t^{t+1}\int_\Omega |\nabla u_\epsilon(\cdot, s)|^2 ds \leq \|u_0\|^2_{L^2(\Omega)} + \int_t^{t+1}\left(\int_\Omega |\nabla n_\epsilon^\alpha(\cdot, s)|^2 + C_3\right) ds \leq C_6 := \|u_0\|^2_{L^2(\Omega)} + C_4$$

for all $t \geq 0$. This completes the proof of Lemma 3.2. □

We next intend to improve our knowledge on the space-time L^p bound for c_ϵ for any $p \geq 2$. Indeed, for $2 \leq p \leq 3$, the following lemma is a direct result of Lemma 3.1.

Lemma 3.3. *Suppose that (1.11)–(1.12) hold. Then for each $p \geq 2$, we can find $C > 0$ such that for all $\epsilon \in (0, 1)$,*

$$\int_t^{t+1}\int_\Omega c_\epsilon^p(\cdot, s) ds \leq C \qquad \text{for all} \quad t \geq 0.$$

Proof. Testing equation $(1.9)_2$ by c_ϵ^{p-1}, we obtain using the Hölder inequality that

$$\frac{\epsilon}{p}\frac{d}{dt}\|c_\epsilon^{\frac{p}{2}}\|^2_{L^2(\Omega)} + \frac{4(p-1)}{p^2}\|\nabla c_\epsilon^{\frac{p}{2}}\|^2_{L^2(\Omega)} + \|c_\epsilon^{\frac{p}{2}}\|^2_{L^2(\Omega)} = \int_\Omega n_\epsilon c_\epsilon^{p-1} \leq \|n_\epsilon\|_{L^r(\Omega)}\|c_\epsilon^{\frac{p}{2}}\|^{\frac{2(p-1)}{p}}_{L^{\frac{2(p-1)r}{p(r-1)}}(\Omega)}$$

for any $r > 1$ and all $t > 0$. It then follows from the Gagliardo–Nirenberg inequality, the Young inequality and (2.2) that

$$\frac{\epsilon}{p}\frac{d}{dt}\|c_\epsilon^{\frac{p}{2}}\|^2_{L^2(\Omega)} + \frac{4(p-1)}{p^2}\|\nabla c_\epsilon^{\frac{p}{2}}\|^2_{L^2(\Omega)} + \|c_\epsilon^{\frac{p}{2}}\|^2_{L^2(\Omega)}$$
$$\leq C_1\|n_\epsilon\|_{L^r(\Omega)}\left(\|\nabla c_\epsilon^{\frac{p}{2}}\|^{\frac{2(pr-2r+1)}{pr}}_{L^2(\Omega)}\|c_\epsilon^{\frac{p}{2}}\|^{\frac{2(r-1)}{pr}}_{L^{\frac{2}{p}}(\Omega)} + \|c_\epsilon^{\frac{p}{2}}\|^{\frac{2(p-1)}{p}}_{L^{\frac{2}{p}}(\Omega)}\right)$$
$$= C_1\|n_\epsilon\|_{L^r(\Omega)}\left(\|\nabla c_\epsilon^{\frac{p}{2}}\|^{\frac{2(pr-2r+1)}{pr}}_{L^2(\Omega)}\|c_0\|^{\frac{r-1}{r}}_{L^1(\Omega)} + \|c_0\|^{p-1}_{L^1(\Omega)}\right)$$
$$\leq C_2\|n_\epsilon\|_{L^r(\Omega)}\left(\|\nabla c_\epsilon^{\frac{p}{2}}\|^{\frac{2(pr-2r+1)}{pr}}_{L^2(\Omega)} + 1\right)$$
$$\leq \frac{4(p-1)}{p^2}\|\nabla c_\epsilon^{\frac{p}{2}}\|^2_{L^2(\Omega)} + C_3\|n_\epsilon\|^{\frac{pr}{2r-1}}_{L^r(\Omega)} + C_3 \qquad \text{for all} \quad t>0,$$

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and thus that

$$\epsilon\frac{d}{dt}\|c_\epsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^2+p\|c_\epsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^2\le pC_3\|n_\epsilon\|_{L^r(\Omega)}^{\frac{pr}{2r-1}}+pC_3 \qquad \text{for all} \quad t>0. \tag{3.11}$$

Taking $r=\frac{p-2\alpha}{p-4\alpha}$ and applying the Gagliardo–Nirenberg inequality again, Lemma 3.1 and the mass conservation (2.1), we have

$$\begin{aligned}\int_t^{t+1}\|n_\epsilon\|_{L^r(\Omega)}^{\frac{pr}{2r-1}}&=\int_t^{t+1}\|n_\epsilon^\alpha\|_{L^{\frac{r}{\alpha}}(\Omega)}^{\frac{2r}{r-1}}\\&\le C_4\int_t^{t+1}\left(\|\nabla n_\epsilon^\alpha\|_{L^2(\Omega)}^2\|n_\epsilon^\alpha\|_{L^{\frac{1}{\alpha}}(\Omega)}^{\frac{2}{r-1}}+\|n_\epsilon^\alpha\|_{L^{\frac{1}{\alpha}}(\Omega)}^{\frac{2r}{r-1}}\right)\\&=C_4\int_t^{t+1}\left(\|\nabla n_\epsilon^\alpha\|_{L^2(\Omega)}^2\|n_\epsilon\|_{L^1(\Omega)}^{\frac{2\alpha}{r-1}}+\|n_\epsilon\|_{L^1(\Omega)}^{\frac{2r\alpha}{r-1}}\right)\\&\le C_5 \qquad \text{for all} \quad t>0.\end{aligned}\tag{3.12}$$

Consequently, setting

$$y(t):=\epsilon\|c_\epsilon^{\frac{p}{2}}(\cdot,t)\|_{L^2(\Omega)}^2,\qquad g(t)=p\|c_\epsilon^{\frac{p}{2}}(\cdot,t)\|_{L^2(\Omega)}^2,\qquad h(t):=pC_3\|n_\epsilon(\cdot,t)\|_{L^r(\Omega)}^{\frac{pr}{2r-1}}+pC_3,$$

we can use (3.11) and the fact $p\|c_\epsilon^{\frac{p}{2}}(\cdot,t)\|_{L^2(\Omega)}^2>py(t)$ for any $\epsilon\in(0,1)$ to deduce that

$$y'(t)+py(t)\le y'(t)+g(t)\le h(t)$$

for all $t>0$ and thus from Lemma 2.3 and (3.12) that

$$y(t)\le y(0)+\frac{C_6}{1-e^{-p}}=\epsilon\|c_0^{\frac{p}{2}}\|_{L^2(\Omega)}^2+\frac{C_6}{1-e^{-p}}<C_7:=\|c_0^{\frac{p}{2}}\|_{L^2(\Omega)}^2+\frac{C_6}{1-e^{-p}}$$

for all $t>0$ with $C_6:=pC_3C_5+pC_3$, which also implies that

$$\int_t^{t+1}g(s)ds\le y(t)-y(t+1)+\int_t^{t+1}h(s)ds<C_7+C_6 \qquad \text{for all} \quad t\ge 0.$$

This completes the proof of Lemma 3.3. □

3.2 A time-independent spatial L^p bound for u_ϵ.

In this subsection, we derive further regularity of u_ϵ. Precisely, we will show the boundedness of $\|u_\epsilon(\cdot,t)\|_{L^p(\Omega)}$ for all $p\ge 1$, which is based on the boundedness of $\|\nabla u_\epsilon(\cdot,t)\|_{L^2(\Omega)}$ obtained by a key entropy-like estimate.

Lemma 3.4. *Suppose that (1.11)–(1.12) hold. Then there exists some positive constant C such that for all* $\epsilon\in(0,1)$ *we have*

$$\|\nabla u_\epsilon(\cdot,t)\|_{L^2(\Omega)}\le C \qquad \text{for all} \quad t>0.$$

Proof. We will deduce our desired result by investigating the combinational functional of the form

$$\int_\Omega n_\epsilon\ln n_\epsilon+K\epsilon\int_\Omega|\nabla c_\epsilon|^2+M\int_\Omega|\nabla u_\epsilon|^2$$

with positive constants K and M to be determined.

For this purpose, since $\|A^{\frac{1}{2}}u_\epsilon\|_{L^2(\Omega)} = \|\nabla u_\epsilon\|_{L^2(\Omega)}$, we first apply the Helmholtz projection $\mathcal{P}$ to both sides of equation (1.9)$_3$, multiply the result with Au_ϵ, integrate by parts over Ω and use the Hölder inequality, the L^2 boundedness of $\mathcal{P}$, the Gagliardo–Nirenberg inequality and the Young inequality to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u_\epsilon|^2 + \int_\Omega |Au_\epsilon|^2 \\
&\le \|\mathcal{P}(u_\epsilon\cdot\nabla u_\epsilon)\|_{L^2(\Omega)}\|Au_\epsilon\|_{L^2(\Omega)} + \|\mathcal{P}(n_\epsilon\nabla\phi)\|_{L^2(\Omega)}\|Au_\epsilon\|_{L^2(\Omega)} \\
&\le C_1\|u_\epsilon\|_{L^4(\Omega)}\|\nabla u_\epsilon\|_{L^4(\Omega)}\|Au_\epsilon\|_{L^2(\Omega)} + C_1\|\nabla\phi\|_{L^\infty(\Omega)}\|n_\epsilon\|_{L^2(\Omega)}\|Au_\epsilon\|_{L^2(\Omega)} \\
&\le C_2\left(\|u_\epsilon\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla u_\epsilon\|_{L^2(\Omega)}^{\frac{1}{2}} + \|u_\epsilon\|_{L^2(\Omega)}\right)\left(\|Au_\epsilon\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla u_\epsilon\|_{L^2(\Omega)}^{\frac{1}{2}} + \|\nabla u_\epsilon\|_{L^2(\Omega)}\right) \\
&\qquad \cdot\|Au_\epsilon\|_{L^2(\Omega)} + C_2\|n_\epsilon\|_{L^2(\Omega)}\|Au_\epsilon\|_{L^2(\Omega)} \\
&\le \|Au_\epsilon\|_{L^2(\Omega)}^2 + C_3\|\nabla u_\epsilon\|_{L^2(\Omega)}^4 + C_3\|\nabla u_\epsilon\|_{L^2(\Omega)}^2 + C_3\|n_\epsilon\|_{L^2(\Omega)}^2
\end{aligned}$$

for all $t > 0$. Here we used (3.9) in the last inequality. Then, we have

$$\frac{d}{dt}\|\nabla u_\epsilon\|_{L^2(\Omega)}^2 \le 2C_3\|\nabla u_\epsilon\|_{L^2(\Omega)}^4 + 2C_3\|\nabla u_\epsilon\|_{L^2(\Omega)}^2 + 2C_3\|n_\epsilon\|_{L^2(\Omega)}^2 \qquad \text{for all} \quad t>0. \tag{3.13}$$

We next estimate the last term on the right-hand side of (3.13). Due to $n_\epsilon > 0$ in $\bar{\Omega}\times(0,\infty)$, we may test (1.9)$_1$ by $\ln n_\epsilon + 1$ to see from the integration by parts over Ω and the Young inequality, as well as the upper estimate (1.11) for S, that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega n_\epsilon\ln n_\epsilon + \int_\Omega \frac{|\nabla n_\epsilon|^2}{n_\epsilon} &= \int_\Omega \nabla n_\epsilon\cdot\Big(S(x,n_\epsilon,c_\epsilon)\cdot\nabla c_\epsilon\Big) \\
&\le \frac{1}{2}\int_\Omega\frac{|\nabla n_\epsilon|^2}{n_\epsilon} + \frac{1}{2}\int_\Omega n_\epsilon\Big|S(x,n_\epsilon,c_\epsilon)\Big|^2\cdot|\nabla c_\epsilon|^2 \\
&\le \frac{1}{2}\int_\Omega\frac{|\nabla n_\epsilon|^2}{n_\epsilon} + \frac{C_S^2}{2}\int_\Omega n_\epsilon(1+n_\epsilon)^{-2\alpha}|\nabla c_\epsilon|^2 \qquad \text{for all} \quad t>0
\end{aligned}$$

and thus that

$$\frac{d}{dt}\int_\Omega n_\epsilon\ln n_\epsilon + \frac{1}{2}\int_\Omega\frac{|\nabla n_\epsilon|^2}{n_\epsilon} \le \frac{C_S^2}{2}\int_\Omega n_\epsilon^{1-2\alpha}|\nabla c_\epsilon|^2 \qquad \text{for all} \quad t>0. \tag{3.14}$$

Noticing that

$$\int_\Omega n_\epsilon^2 = \|\sqrt{n_\epsilon}\|_{L^4(\Omega)}^4 \le C_4\|\nabla\sqrt{n_\epsilon}\|_{L^2(\Omega)}^2\|\sqrt{n_\epsilon}\|_{L^2(\Omega)}^2 + C_4\|\sqrt{n_\epsilon}\|_{L^2(\Omega)}^4 \le C_5\int_\Omega\frac{|\nabla n_\epsilon|^2}{n_\epsilon} + C_5$$

for all $t > 0$ by the Gagliardo–Nirenberg inequality and the mass conservation (2.1), we deduce from the Young inequality that

$$\frac{C_S^2}{2}\int_\Omega n_\epsilon^{1-2\alpha}|\nabla c_\epsilon|^2 \le \frac{1}{4C_5}\int_\Omega n_\epsilon^2 + C_6\int_\Omega|\nabla c_\epsilon|^{\frac{4}{1+2\alpha}} \qquad \text{for all} \quad t>0,$$

which together with (3.14) implies that

$$\frac{d}{dt}\int_\Omega n_\epsilon\ln n_\epsilon + \frac{1}{4C_5}\int_\Omega n_\epsilon^2 \le \frac{1}{2} + C_6\int_\Omega|\nabla c_\epsilon|^{\frac{4}{1+2\alpha}} \qquad \text{for all} \quad t>0. \tag{3.15}$$

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For the right-hand side of (3.15), we see from the Gagliardo–Nirenberg inequality and the Young inequality again that

$$\begin{aligned}\frac{1}{2}+C_6\int_\Omega |\nabla c_\epsilon|^{\frac{4}{1+2\alpha}} &= \frac{1}{2}+C_6\|\nabla c_\epsilon\|^{\frac{4}{1+2\alpha}}_{L^{\frac{4}{1+2\alpha}}(\Omega)}\\ &\le \frac{1}{2}+C_7\|\Delta c_\epsilon\|^{\frac{2}{1+2\alpha}}_{L^2(\Omega)}\|c_\epsilon\|^{\frac{2}{1+2\alpha}}_{L^{\frac{1}{\alpha}}(\Omega)}+C_7\|c_\epsilon\|^{\frac{4}{1+2\alpha}}_{L^{\frac{1}{\alpha}}(\Omega)}\\ &\le \frac{1}{16C_5}\int_\Omega|\Delta c_\epsilon|^2+C_8\|c_\epsilon\|^{\frac{1}{\alpha}}_{L^{\frac{1}{\alpha}}(\Omega)}+C_8 \qquad \text{for all} \quad t>0\end{aligned}$$

due to $\alpha\in\left(0,\frac{1}{2}\right)$, and thus that

$$\frac{d}{dt}\int_\Omega n_\epsilon\ln n_\epsilon+\frac{1}{4C_5}\int_\Omega n_\epsilon^2\le\frac{1}{16C_5}\int_\Omega|\Delta c_\epsilon|^2+C_8\|c_\epsilon\|^{\frac{1}{\alpha}}_{L^{\frac{1}{\alpha}}(\Omega)}+C_8 \qquad \text{for all} \quad t>0. \tag{3.16}$$

To deal with the first integral on the right-hand side of (3.16), we test $(1.9)_2$ by $-\Delta c_\epsilon$ and integrate by parts over Ω to get

$$\frac{\epsilon}{2}\frac{d}{dt}\int_\Omega|\nabla c_\epsilon|^2+\int_\Omega|\Delta c_\epsilon|^2+\int_\Omega|\nabla c_\epsilon|^2=-\int_\Omega n_\epsilon\Delta c_\epsilon+\int_\Omega(u_\epsilon\cdot\nabla c_\epsilon)\Delta c_\epsilon \qquad \text{for all} \quad t>0. \tag{3.17}$$

For the first integral on the right-hand side of (3.17), it is clear that

$$-\int_\Omega n_\epsilon\Delta c_\epsilon\le\frac{1}{4}\int_\Omega|\Delta c_\epsilon|^2+\int_\Omega n_\epsilon^2 \qquad \text{for all} \quad t>0. \tag{3.18}$$

On the other hand, for the second integral on the right-hand side of (3.17), we apply the Hölder inequality to obtain

$$\int_\Omega(u_\epsilon\cdot\nabla c_\epsilon)\Delta c_\epsilon\le\|\Delta c_\epsilon\|_{L^2(\Omega)}\|u_\epsilon\|_{L^4(\Omega)}\|\nabla c_\epsilon\|_{L^4(\Omega)} \qquad \text{for all} \quad t>0. \tag{3.19}$$

Since

$$\int_\Omega|D^2c_\epsilon|^2=\frac{1}{2}\int_{\partial\Omega}\nabla|\nabla c_\epsilon|^2\cdot\nu-\int_\Omega\nabla\Delta c_\epsilon\cdot\nabla c_\epsilon=\frac{1}{2}\int_{\partial\Omega}\nabla|\nabla c_\epsilon|^2\cdot\nu+\int_\Omega(\Delta c_\epsilon)^2$$

by the integration by parts and $\nabla c_\epsilon\cdot\nu=0$, we can use the geometric property

$$\nabla|\nabla c_\epsilon|^2\cdot\nu\le 2C_\Omega|\nabla c_\epsilon|^2 \tag{3.20}$$

with C_Ω an upper bound for the curvatures of $\partial\Omega$ (see Lemma 4.2 in [22]), the trace theorem and the Gagliardo–Nirenberg inequality to see that

$$\begin{aligned}\|D^2c_\epsilon\|^2_{L^2(\Omega)} &\le C_\Omega\|\nabla c_\epsilon\|^2_{L^2(\partial\Omega)}+\|\Delta c_\epsilon\|^2_{L^2(\Omega)}\\ &\le C_\Omega\|\nabla c_\epsilon\|^2_{W^{\frac{3}{4},2}(\Omega)}+\|\Delta c_\epsilon\|^2_{L^2(\Omega)}\\ &\le C_9\left(\|D^2c_\epsilon\|^{\frac{11}{6}}_{L^2(\Omega)}\|c_\epsilon\|^{\frac{1}{6}}_{L^1(\Omega)}+\|c_\epsilon\|^2_{L^1(\Omega)}\right)+\|\Delta c_\epsilon\|^2_{L^2(\Omega)}\\ &\le\frac{1}{2}\|D^2c_\epsilon\|^2_{L^2(\Omega)}+\frac{C_{10}}{2}\left(\|c_\epsilon\|^2_{L^1(\Omega)}+\|\Delta c_\epsilon\|^2_{L^2(\Omega)}\right) \qquad \text{for all} \quad t\in(0,\infty)\end{aligned}$$

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and thus that

$$\|D^2 c_\epsilon\|^2_{L^2(\Omega)} \le C_{10}\Big(\|c_\epsilon\|^2_{L^1(\Omega)} + \|\Delta c_\epsilon\|^2_{L^2(\Omega)}\Big) \qquad \text{for all} \quad t \in (0,\infty). \tag{3.21}$$

It then follows from the Gagliardo–Nirenberg inequality that

$$\begin{aligned}
\|\nabla c_\epsilon\|^2_{L^4(\Omega)} &\le C_{11}\|D^2 c_\epsilon\|_{L^2(\Omega)}\|\nabla c_\epsilon\|_{L^2(\Omega)} + C_{11}\|\nabla c_\epsilon\|^2_{L^2(\Omega)} \\
&\le C_{11}C_{10}^{\frac{1}{2}}\Big(\|c_\epsilon\|^2_{L^1(\Omega)} + \|\Delta c_\epsilon\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}\|\nabla c_\epsilon\|_{L^2(\Omega)} + C_{11}\|\nabla c_\epsilon\|^2_{L^2(\Omega)} \\
&\le C_{12}\|\Delta c_\epsilon\|_{L^2(\Omega)}\|\nabla c_\epsilon\|_{L^2(\Omega)} + C_{12}\|\nabla c_\epsilon\|_{L^2(\Omega)} + C_{12}\|\nabla c_\epsilon\|^2_{L^2(\Omega)} \qquad \text{for all} \quad t>0,
\end{aligned}$$

which together with (3.19) by the Gagliardo–Nirenberg inequality, the Poincaré inequality and Lemma 3.2 entails that

$$\begin{aligned}
&\int_\Omega (u_\epsilon\cdot\nabla c_\epsilon)\Delta c_\epsilon \\
&\le C_{13}\left(\|\Delta c_\epsilon\|^{\frac{3}{2}}_{L^2(\Omega)}\|\nabla c_\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)} + \|\Delta c_\epsilon\|_{L^2(\Omega)}\|\nabla c_\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)} + \|\Delta c_\epsilon\|_{L^2(\Omega)}\|\nabla c_\epsilon\|_{L^2(\Omega)}\right)\|u_\epsilon\|_{L^4(\Omega)} \\
&\le C_{14}\left(\|\Delta c_\epsilon\|^{\frac{3}{2}}_{L^2(\Omega)}\|\nabla c_\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)} + \|\Delta c_\epsilon\|_{L^2(\Omega)}\|\nabla c_\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)}\right)\|\nabla u_\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)}\|u_\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)} \\
&\qquad + C_{14}\|\Delta c_\epsilon\|_{L^2(\Omega)}\|\nabla c_\epsilon\|_{L^2(\Omega)}\|\nabla u_\epsilon\|_{L^2(\Omega)} \\
&\le \frac{1}{4}\|\Delta c_\epsilon\|^2_{L^2(\Omega)} + C_{15}\|\nabla u_\epsilon\|^2_{L^2(\Omega)}\|\nabla c_\epsilon\|^2_{L^2(\Omega)} + C_{15}\|\nabla u_\epsilon\|_{L^2(\Omega)}\|\nabla c_\epsilon\|_{L^2(\Omega)} \\
&\le \frac{1}{4}\|\Delta c_\epsilon\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla c_\epsilon\|^2_{L^2(\Omega)} + C_{16}\|\nabla u_\epsilon\|^2_{L^2(\Omega)}\|\nabla c_\epsilon\|^2_{L^2(\Omega)} + C_{16}\|\nabla u_\epsilon\|^2_{L^2(\Omega)}
\end{aligned} \tag{3.22}$$

for all $t>0$. Combining (3.18), (3.22) and (3.17), we can deduce that

$$\begin{aligned}
\epsilon\frac{d}{dt}\int_\Omega |\nabla c_\epsilon|^2 + \int_\Omega |\Delta c_\epsilon|^2 + \int_\Omega |\nabla c_\epsilon|^2 &\le 2\int_\Omega n_\epsilon^2 \\
&+ 2C_{16}\left(\int_\Omega |\nabla u_\epsilon|^2\right)\cdot\left(\int_\Omega |\nabla c_\epsilon|^2\right) + 2C_{16}\int_\Omega |\nabla u_\epsilon|^2
\end{aligned}$$

for all $t>0$, which together with (3.16) and (3.13) yields that

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega n_\epsilon \ln n_\epsilon + \frac{\epsilon}{16C_5}\int_\Omega |\nabla c_\epsilon|^2 + \frac{1}{32C_3C_5}\int_\Omega |\nabla u_\epsilon|^2\right) + \frac{1}{16C_5}\int_\Omega n_\epsilon^2 + \frac{1}{16C_5}\int_\Omega |\nabla c_\epsilon|^2 \\
&\le \frac{C_{16}}{8C_5}\left(\int_\Omega |\nabla u_\epsilon|^2\right)\left(\int_\Omega |\nabla c_\epsilon|^2 + 1\right) + \frac{1}{16C_5}\left(\int_\Omega |\nabla u_\epsilon|^2\right)\left(\int_\Omega |\nabla u_\epsilon|^2 + 1\right) \\
&\quad + C_8\|c_\epsilon\|^{\frac{1}{\alpha}}_{L^{\frac{1}{\alpha}}(\Omega)} + C_8 \\
&= \frac{1}{32C_3C_5}\int_\Omega |\nabla u_\epsilon|^2\left(4C_3C_{16}\int_\Omega |\nabla c_\epsilon|^2 + 2C_3\int_\Omega |\nabla u_\epsilon|^2 + 4C_3C_{16} + 2C_3\right) \\
&\quad + C_8\|c_\epsilon\|^{\frac{1}{\alpha}}_{L^{\frac{1}{\alpha}}(\Omega)} + C_8
\end{aligned}$$

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for all $t > 0$. Thus, by taking $K := \frac{1}{16C_5}$ with $M := \frac{1}{32C_3C_5}$ and setting

$$y(t) := \int_\Omega n_\epsilon \ln n_\epsilon + K\epsilon \int_\Omega |\nabla c_\epsilon|^2 + M \int_\Omega |\nabla u_\epsilon|^2, \qquad g(t) := K \int_\Omega n_\epsilon^2 + K \int_\Omega |\nabla c_\epsilon|^2$$

and

$$h(t) := 4C_3C_{16} \int_\Omega |\nabla c_\epsilon|^2 + 2C_3 \int_\Omega |\nabla u_\epsilon|^2 + 4C_3C_{16} + 2C_3, \qquad m(t) := C_8 \|c_\epsilon\|^{\frac{1}{\alpha}}_{L^{\frac{1}{\alpha}}(\Omega)} + C_8,$$

we have

$$y'(t) + g(t) \le h(t) \left\{ y(t) + \frac{|\Omega|}{e} \right\} + m(t) \qquad \text{for all} \quad t > 0. \tag{3.23}$$

Here we used the fact that

$$-\int_\Omega n_\epsilon \ln n_\epsilon \le \frac{|\Omega|}{e}, \qquad \text{for all} \quad t > 0 \tag{3.24}$$

due to $\xi \ln \xi \ge -\frac{1}{e}$ for all $\xi > 0$.

According to the Gagliardo–Nirenberg inequality, the mass conservation (2.1) and Lemma 3.1, we can achieve that

$$\begin{aligned}
\int_{t-1}^t \|n_\epsilon(\cdot,s)\|^2_{L^{\frac{1}{1-\alpha}}(\Omega)} ds &= \int_{t-1}^t \|n_\epsilon^\alpha(\cdot,s)\|^{\frac{2}{\alpha}}_{L^{\frac{1}{\alpha(1-\alpha)}}(\Omega)} ds \\
&\le C_{17} \int_{t-1}^t \|\nabla n_\epsilon^\alpha(\cdot,s)\|^2_{L^2(\Omega)} \|n_\epsilon^\alpha(\cdot,s)\|^{\frac{2(1-\alpha)}{\alpha}}_{L^{\frac{1}{\alpha}}(\Omega)} ds + C_{17} \int_{t-1}^t \|n_\epsilon^\alpha(\cdot,s)\|^{\frac{2}{\alpha}}_{L^{\frac{1}{\alpha}}(\Omega)} ds \\
&\le C_{18} \int_{t-1}^t \|\nabla n_\epsilon^\alpha(\cdot,s)\|^2_{L^2(\Omega)} ds + C_{18} \\
&\le C_{19} \qquad \text{for all} \quad t \ge 1.
\end{aligned}$$

This together with Lemmas 3.1 and 3.2 implies that

$$\int_{t-1}^t \left(\|n_\epsilon(\cdot,s)\|^2_{L^{\frac{1}{1-\alpha}}(\Omega)} + \|\nabla c_\epsilon(\cdot,s)\|^2_{L^2(\Omega)} + \|\nabla u_\epsilon(\cdot,s)\|^2_{L^2(\Omega)} \right) ds \le C_{20} \qquad \text{for all} \quad t \ge 1,$$

and thus that for each fixed $t > 0$, we can find $t_\star \equiv t_\star(t;\epsilon) \ge 0$ such that $t_\star \in \big((t-1)_+, t\big)$ and

$$\begin{aligned}
&\|n_\epsilon(\cdot,t_\star)\|^2_{L^{\frac{1}{1-\alpha}}(\Omega)} + \|\nabla c_\epsilon(\cdot,t_\star)\|^2_{L^2(\Omega)} + \|\nabla u_\epsilon(\cdot,t_\star)\|^2_{L^2(\Omega)} \\
&\le C_{21} := \max\left\{ C_{20}, \|n_0\|^2_{L^{\frac{1}{1-\alpha}}(\Omega)} + \|\nabla c_0\|^2_{L^2(\Omega)} + \|\nabla u_0\|^2_{L^2(\Omega)} \right\}.
\end{aligned}$$

By means of the elementary inequality $\xi \ln \xi \le \frac{1-\alpha}{\alpha e} \xi^{\frac{1}{1-\alpha}}$ for all $\xi > 0$, we infer that

$$\int_\Omega n_\epsilon(\cdot,t_\star) \ln n_\epsilon(\cdot,t_\star) \le \frac{1-\alpha}{\alpha e} \int_\Omega n_\epsilon^{\frac{1}{1-\alpha}}(\cdot,t_\star) \le C_{22} := \frac{1-\alpha}{\alpha e} C_{21}^{\frac{1}{2(1-\alpha)}}$$

and thus that

$$y(t_\star) \le C_{23} := C_{22} + KC_{21} + MC_{21}.$$

On the other hand, we can see from Lemmas 3.1, 3.2 and 3.3 that

$$\int_{t-1}^{t} h(s)ds = \int_{t-1}^{t} \left(4C_3C_{16}\int_\Omega |\nabla c_\epsilon(\cdot,s)|^2 + 2C_3\int_\Omega |\nabla u_\epsilon(\cdot,s)|^2 + 4C_3C_{16} + 2C_3\right) ds \le C_{24}$$

and

$$\int_{t-1}^{t} m(s)ds = \int_{t-1}^{t} \left(C_8\|c_\epsilon(\cdot,s)\|^{\frac{1}{\alpha}}_{L^{\frac{1}{\alpha}}(\Omega)} + C_8\right) ds \le C_{25}$$

for all $t \ge 1$. Thus integrating (3.23) from $t_\star$ to t, we can deduce that

$$\begin{aligned} y(t) &\le \left(y(t_\star) + \frac{|\Omega|}{e}\right) e^{\int_{t_\star}^{t} h(s)ds} + \int_{t_\star}^{t} e^{\int_s^t h(\sigma)d\sigma} m(s)ds \\ &\le \left(C_{23} + \frac{|\Omega|}{e}\right) e^{C_{24}} + \int_{t_\star}^{t} e^{C_{24}} m(s)ds \le C_{26} \qquad \text{for all} \quad t > 0. \end{aligned}$$

Whereupon, this together with (3.24) yields our desired conclusion. □

Lemma 3.5. *Suppose that (1.11)–(1.12) hold. Then for all $p > 1$, we can find some positive constant C such that for all $\epsilon \in (0, 1)$,*

$$\|u_\epsilon(\cdot,t)\|_{L^p(\Omega)} \le C \qquad \text{for all} \quad t \in (0,\infty).$$

Proof. This is a direct consequence of Lemma 3.4, the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ and the Poincaré inequality. □

3.3 Time-independent spatial L^4 bounds for n_ϵ and ∇c_ϵ.

We now improve our knowledge on the spatial regularity of n_ϵ by utilising a very subtle induction argument for n_ϵ, which together with the damping effect of c_ϵ will provide the key uniform L^2 bound for ∇c_ϵ.

Lemma 3.6. *Suppose that (1.11)–(1.12) hold. For any fixed $\widehat{r} \in \left(1, \frac{2}{2-\alpha}\right)$ and $p \ge 1$, if it holds that for all $\epsilon \in (0, 1)$,*

$$\|n_\epsilon(\cdot,t)\|_{L^p(\Omega)} \le K \qquad \text{and} \qquad \|c_\epsilon(\cdot,t)\|_{L^p(\Omega)} \le K \tag{3.25}$$

with some positive constant K, then for any

$$s \in \left(\max\left\{1,\ 2\alpha + \frac{p}{\widehat{r}}\right\},\ 2\alpha + \frac{p(p + 2\alpha\widehat{r} + 2 - 2\alpha)}{p + 2\widehat{r}}\right),$$

we have

$$\|n_\epsilon(\cdot,t)\|_{L^s(\Omega)} \le C \qquad \text{for all} \quad t \in (0,\infty)$$

with some positive constant C.

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Proof. Firstly, testing equation (1.9)$_1$ by n_ϵ^{s-1}, integrating by parts over Ω and making use of the Young inequality, the Hölder inequality and the upper estimate (1.11) for S, we have

$$\frac{1}{s}\frac{d}{dt}\int_\Omega n_\epsilon^s+(s-1)\int_\Omega n_\epsilon^{s-2}|\nabla n_\epsilon|^2=(s-1)\int_\Omega n_\epsilon^{s-1}\nabla n_\epsilon\cdot\big(S(x,n_\epsilon,c_\epsilon)\nabla c_\epsilon\big)$$
$$\leq\frac{(s-1)}{4}\int_\Omega n_\epsilon^{s-2}|\nabla n_\epsilon|^2+(s-1)C_S^2\int_\Omega n_\epsilon^{s-2\alpha}|\nabla c_\epsilon|^2$$

and thus that

$$\frac{d}{dt}\int_\Omega n_\epsilon^s+\frac{3(s-1)}{s}\int_\Omega|\nabla n_\epsilon^{\frac{s}{2}}|^2\leq s(s-1)C_S^2\int_\Omega n_\epsilon^{s-2\alpha}|\nabla c_\epsilon|^2$$
$$\leq s(s-1)C_S^2\left(\int_\Omega n_\epsilon^{\widehat{r}(s-2\alpha)}\right)^{\frac{1}{\widehat{r}}}\left(\int_\Omega|\nabla c_\epsilon|^{\frac{2\widehat{r}}{\widehat{r}-1}}\right)^{\frac{\widehat{r}-1}{\widehat{r}}}\tag{3.26}$$

for all $t\in(0,\infty)$. For the first integral on the right-hand side of (3.26), we can deduce that

$$\left(\int_\Omega n_\epsilon^{\widehat{r}(s-2\alpha)}\right)^{\frac{1}{\widehat{r}}}=\|n_\epsilon^{\frac{s}{2}}\|^{\frac{2(s-2\alpha)}{s}}_{L^{\frac{2\widehat{r}(s-2\alpha)}{s}}(\Omega)}\leq C_1\left(\|\nabla n_\epsilon^{\frac{s}{2}}\|^{\frac{\widehat{r}(s-2\alpha)-p}{\widehat{r}(s-2\alpha)}}_{L^2(\Omega)}\|n_\epsilon^{\frac{s}{2}}\|^{\frac{p}{\widehat{r}(s-2\alpha)}}_{L^{\frac{2p}{s}}(\Omega)}+\|n_\epsilon^{\frac{s}{2}}\|_{L^{\frac{2p}{s}}(\Omega)}\right)^{\frac{2(s-2\alpha)}{s}}$$
$$\leq C_2\left(\int_\Omega|\nabla n_\epsilon^{\frac{s}{2}}|^2\right)^{\frac{\widehat{r}(s-2\alpha)-p}{\widehat{r}s}}+C_2\tag{3.27}$$

for all $t\in(0,\infty)$ by (3.25) and $s>2\alpha+\frac{p}{\widehat{r}}$, while for the second one, we have

$$\left(\int_\Omega|\nabla c_\epsilon|^{\frac{2\widehat{r}}{\widehat{r}-1}}\right)^{\frac{\widehat{r}-1}{\widehat{r}}}=\|\nabla c_\epsilon\|^2_{L^{\frac{2\widehat{r}}{\widehat{r}-1}}(\Omega)}$$
$$\leq C_3\left(\|D^2c_\epsilon\|^{\frac{p+2\widehat{r}}{\widehat{r}(p+2)}}_{L^2(\Omega)}\|c_\epsilon\|^{\frac{(\widehat{r}-1)p}{\widehat{r}(p+2)}}_{L^p(\Omega)}+\|c_\epsilon\|_{L^p(\Omega)}\right)^2$$
$$\leq C_4\left(\left(\|c_\epsilon\|^2_{L^1(\Omega)}+\|\Delta c_\epsilon\|^2_{L^2(\Omega)}\right)^{\frac{p+2\widehat{r}}{2\widehat{r}(p+2)}}\|c_\epsilon\|^{\frac{(\widehat{r}-1)p}{\widehat{r}(p+2)}}_{L^p(\Omega)}+\|c_\epsilon\|_{L^p(\Omega)}\right)^2$$
$$\leq C_5\left(\int_\Omega|\Delta c_\epsilon|^2\right)^{\frac{p+2\widehat{r}}{\widehat{r}(p+2)}}+C_6\tag{3.28}$$

for all $t\in(0,\infty)$ by (3.21) and (3.25). Thus, by substituting (3.27) and (3.28) into (3.26), we can obtain

$$\frac{d}{dt}\int_\Omega n_\epsilon^s+\frac{3(s-1)}{s}\int_\Omega|\nabla n_\epsilon^{\frac{s}{2}}|^2$$
$$\leq C_7\left(\int_\Omega|\nabla n_\epsilon^{\frac{s}{2}}|^2\right)^{\frac{\widehat{r}(s-2\alpha)-p}{\widehat{r}s}}\left(\int_\Omega|\Delta c_\epsilon|^2\right)^{\frac{p+2\widehat{r}}{\widehat{r}(p+2)}}+C_7\left(\int_\Omega|\nabla n_\epsilon^{\frac{s}{2}}|^2\right)^{\frac{\widehat{r}(s-2\alpha)-p}{\widehat{r}s}}$$
$$+C_7\left(\int_\Omega|\Delta c_\epsilon|^2\right)^{\frac{p+2\widehat{r}}{\widehat{r}(p+2)}}+C_7$$

for all $t\in(0,\infty)$. Noticing that

$$0<\frac{\widehat{r}(s-2\alpha)-p}{\widehat{r}s}<1\qquad\text{and}\qquad 0<\frac{s(p+2\widehat{r})}{(p+2)(p+2\alpha\widehat{r})}<1$$

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due to $1 < s < 2\alpha + \frac{p(p+2\alpha\widehat{r}+2-2\alpha)}{p+2\widehat{r}}$, we can use the Young inequality twice to obtain that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega n_\epsilon^s + \frac{3(s-1)}{s}\int_\Omega |\nabla n_\epsilon^{\frac{s}{2}}|^2 \\
&\le \frac{s-1}{s}\int_\Omega |\nabla n_\epsilon^{\frac{s}{2}}|^2 + C_8\left(\int_\Omega |\Delta c_\epsilon|^2\right)^{\frac{s(p+2\widehat{r})}{(p+2)(2\alpha\widehat{r}+p)}} + C_8\left(\int_\Omega |\Delta c_\epsilon|^2\right)^{\frac{p+2\widehat{r}}{\widehat{r}(p+2)}} + C_8 \\
&\le \frac{s-1}{s}\int_\Omega |\nabla n_\epsilon^{\frac{s}{2}}|^2 + \int_\Omega |\Delta c_\epsilon|^2 + C_9
\end{aligned}$$

for all $t \in (0,\infty)$ and thus that

$$\frac{d}{dt}\int_\Omega n_\epsilon^s + \frac{2(s-1)}{s}\int_\Omega |\nabla n_\epsilon^{\frac{s}{2}}|^2 \le \int_\Omega |\Delta c_\epsilon|^2 + C_9 \tag{3.29}$$

for all $t \in (0,\infty)$.

On the other hand, in order to absorb the integral on the right-hand side of (3.29), we multiply equation $(1.9)_2$ by $-\Delta c_\epsilon$ and integrate on Ω to obtain that

$$\frac{\epsilon}{2}\frac{d}{dt}\int_\Omega |\nabla c_\epsilon|^2 + \int_\Omega |\Delta c_\epsilon|^2 + \int_\Omega |\nabla c_\epsilon|^2 = -\int_\Omega n_\epsilon \Delta c_\epsilon + \int_\Omega (u_\epsilon\cdot\nabla c_\epsilon)\Delta c_\epsilon \tag{3.30}$$

for all $t \in (0,\infty)$. Since the interpolation and the mass conservation (2.1) imply that

$$\begin{aligned}
\int_\Omega n_\epsilon^2 = \|n_\epsilon^{\frac{s}{2}}\|^{\frac{4}{s}}_{L^{\frac{4}{s}}(\Omega)} &\le C_{10}\|\nabla n_\epsilon^{\frac{s}{2}}\|^{\frac{2}{s}}_{L^2(\Omega)}\|n_\epsilon^{\frac{s}{2}}\|^{\frac{2}{s}}_{L^{\frac{2}{s}}(\Omega)} + C_{10}\|n_\epsilon^{\frac{s}{2}}\|^{\frac{4}{s}}_{L^{\frac{2}{s}}(\Omega)} \\
&\le C_{11}\|\nabla n_\epsilon^{\frac{s}{2}}\|^{\frac{2}{s}}_{L^2(\Omega)} + C_{11} \le \frac{s-1}{2s}\int_\Omega |\nabla n_\epsilon^{\frac{s}{2}}|^2 + C_{12} \qquad \text{for all} \quad t\in(0,\infty),
\end{aligned}$$

we have

$$-\int_\Omega n_\epsilon\Delta c_\epsilon \le \frac{1}{4}\int_\Omega |\Delta c_\epsilon|^2 + \int_\Omega n_\epsilon^2 \le \frac{1}{4}\int_\Omega |\Delta c_\epsilon|^2 + \frac{s-1}{2s}\int_\Omega |\nabla n_\epsilon^{\frac{s}{2}}|^2 + C_{12} \tag{3.31}$$

for all $t \in (0,\infty)$. Similarly, it follows from

$$\|\nabla c_\epsilon\|_{L^4(\Omega)} \le C_{13}\|\Delta c_\epsilon\|^{\frac{5}{6}}_{L^2(\Omega)}\|c_\epsilon\|^{\frac{1}{6}}_{L^1(\Omega)} + C_{13}\|c_\epsilon\|_{L^1(\Omega)} \le C_{14}\|\Delta c_\epsilon\|^{\frac{5}{6}}_{L^2(\Omega)} + C_{14}$$

and the boundedness of $\|u_\epsilon(\cdot,t)\|_{L^4(\Omega)}$ obtained in Lemma 3.5 that

$$\begin{aligned}
\int_\Omega (u_\epsilon\cdot\nabla c_\epsilon)\Delta c_\epsilon &\le \|u_\epsilon\|_{L^4(\Omega)}\|\nabla c_\epsilon\|_{L^4(\Omega)}\|\Delta c_\epsilon\|_{L^2(\Omega)} \\
&\le C_{14}\|u_\epsilon\|_{L^4(\Omega)}\left(\|\Delta c_\epsilon\|^{\frac{11}{6}}_{L^2(\Omega)} + \|\Delta c_\epsilon\|_{L^2(\Omega)}\right) \le \frac{1}{4}\int_\Omega |\Delta c_\epsilon|^2 + C_{15}
\end{aligned} \tag{3.32}$$

for all $t \in (0,\infty)$. Substituting (3.31) and (3.32) into (3.30), we can see

$$\epsilon\frac{d}{dt}\int_\Omega |\nabla c_\epsilon|^2 + \int_\Omega |\Delta c_\epsilon|^2 + 2\int_\Omega |\nabla c_\epsilon|^2 \le \frac{s-1}{s}\int_\Omega |\nabla n_\epsilon^{\frac{s}{2}}|^2 + 2(C_{12}+C_{15}) \tag{3.33}$$

for all $t \in (0,\infty)$.

Then combining (3.29) and (3.33), we obtain

$$\frac{d}{dt}\left\{\int_\Omega n_\epsilon^s + \epsilon\int_\Omega |\nabla c_\epsilon|^2\right\} + \frac{s-1}{s}\int_\Omega |\nabla n_\epsilon^{\frac{s}{2}}|^2 + 2\int_\Omega |\nabla c_\epsilon|^2 \le C_{16} \qquad \text{for all} \quad t\in(0,\infty)$$

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with $C_{16} := C_9 + 2(C_{12} + C_{15})$. To establish the uniform bound for the functional $\int_\Omega n_\epsilon^s + \epsilon \int_\Omega |\nabla c_\epsilon|^2$, we apply the Gagliardo–Nirenberg inequality, the Young inequality and the mass conservation (2.1) to gain

$$\int_\Omega n_\epsilon^s = \|n_\epsilon^{\frac{s}{2}}\|_{L^2(\Omega)}^2 \le C_{17}\left(\|\nabla n_\epsilon^{\frac{s}{2}}\|_{L^2(\Omega)}^{\frac{2(s-1)}{s}} \|n_\epsilon^{\frac{s}{2}}\|_{L^{\frac{2}{s}}(\Omega)}^{\frac{2}{s}} + \|n_\epsilon^{\frac{s}{2}}\|_{L^{\frac{2}{s}}(\Omega)}^2\right) \le \frac{s-1}{s}\int_\Omega |\nabla n_\epsilon^{\frac{s}{2}}|^2 + C_{18}$$

for all $t \in (0, \infty)$. Also noting $\epsilon \in (0, 1)$, we have

$$\frac{d}{dt}\left\{\int_\Omega n_\epsilon^s + \epsilon\int_\Omega |\nabla c_\epsilon|^2\right\} + \left\{\int_\Omega n_\epsilon^s + \epsilon\int_\Omega |\nabla c_\epsilon|^2\right\} \le C_{16} + C_{18}$$

for all $t \in (0, \infty)$. A direct calculation yields the desired result. □

Corollary 3.1. *Suppose that (1.11)–(1.12) hold. For any fixed* $\widehat{r} \in \left(1, \frac{2}{2-\alpha}\right)$, *we define the sequence* $\{s_k\}_{k=0}^\infty$ *by taking*

$$s_0 := 1, \quad s_k \in \left(\max\left\{1,\ 2\alpha + \frac{s_{k-1}}{\widehat{r}}\right\},\ 2\alpha + \frac{s_{k-1}(s_{k-1} + 2\alpha\widehat{r} + 2 - 2\alpha)}{s_{k-1} + 2\widehat{r}}\right), \quad (k = 1, 2, 3, \cdots).$$

Then for every $k = 1, 2, 3, \cdots$, *there exists a positive constant* C_k *such that for all* $\epsilon \in (0, 1)$,

$$\|n_\epsilon(\cdot, t)\|_{L^{s_k}(\Omega)} \le C_k \qquad \text{for all} \quad t \in (0, \infty).$$

Proof. We can show the conclusion by an induction argument on k. Indeed, due to Lemmas 2.1 and 3.6, we see that the case $k = 1$ is true. Then assuming that the conclusion is valid for some positive constant k, we can use Lemma 2.2 to obtain

$$\|c_\epsilon(\cdot, t)\|_{L^{s_k}(\Omega)} \le C_1 \qquad \text{for all} \quad t \in (0, \infty)$$

with some $C_1 > 0$, and thus have

$$\|n_\epsilon(\cdot, t)\|_{L^{s_{k+1}}(\Omega)} \le C_2 \qquad \text{for all} \quad t \in (0, \infty)$$

with some positive constant C_2 by Lemma 3.6. This implies Corollary 3.1. □

Corollary 3.2. *Suppose that (1.11)–(1.12) hold. Then there exists* $C > 0$ *such that for all* $\epsilon \in (0, 1)$,

$$\|n_\epsilon(\cdot, t)\|_{L^4(\Omega)} \le C \qquad \text{for all} \quad t \in (0, \infty).$$

Proof. Let $\{s_k\}_{k=0}^\infty$ be defined by Corollary 3.1. Then $s_k > 2\alpha + \frac{1}{\widehat{r}}s_{k-1}$, $(k = 1, 2, 3, \cdots)$, implies that

$$s_k > 2\alpha + \frac{1}{\widehat{r}}\left(2\alpha + \frac{1}{\widehat{r}}s_{k-2}\right) > \cdots > 2\alpha\left(1 + \frac{1}{\widehat{r}} + \frac{1}{\widehat{r}^2} + \cdots + \frac{1}{\widehat{r}^{k-1}}\right)$$
$$+ \frac{1}{\widehat{r}^k}s_0 = \frac{2\alpha\widehat{r}}{\widehat{r}-1} + \frac{1}{\widehat{r}^k}\left(s_0 - \frac{2\alpha\widehat{r}}{\widehat{r}-1}\right)$$

for $k = 1, 2, 3, \cdots$. Noticing that $\frac{2\alpha\widehat{r}}{\widehat{r}-1} > 4$ due to $\widehat{r} \in \left(1, \frac{2}{2-\alpha}\right)$, we have $s_k \ge 4$ for k large enough. It then follows from Corollary 3.1 that $\|n_\epsilon(\cdot, t)\|_{L^4(\Omega)}$ is bounded in $(0, \infty)$. □

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With the boundedness of $\|n_\epsilon(\cdot,t)\|_{L^4(\Omega)}$ at hand, we now turn back the proof of Lemma 3.6 to achieve the key boundedness of $\|\nabla c_\epsilon(\cdot,t)\|_{L^2(\Omega)}$.

Lemma 3.7. *Suppose that (1.11)–(1.12) hold. Then there exists $C>0$ such that for all $\epsilon\in(0,1)$,*

$$\|\nabla c_\epsilon(\cdot,t)\|_{L^2(\Omega)}\le C \qquad \text{for all} \quad t\in(0,\infty).$$

Proof. By repeating the proof of (3.30), (3.31) and (3.32) in Lemma 3.6 and using Corollary 3.2, we can deduce that

$$\frac{\epsilon}{2}\frac{d}{dt}\int_\Omega|\nabla c_\epsilon|^2+\int_\Omega|\Delta c_\epsilon|^2+\int_\Omega|\nabla c_\epsilon|^2\le\frac{1}{2}\int_\Omega|\Delta c_\epsilon|^2+\int_\Omega n_\epsilon^2+C_1\le\frac{1}{2}\int_\Omega|\Delta c_\epsilon|^2+C_2$$

for all $t\in(0,\infty)$. Then a direct calculation can complete the proof of Lemma 3.7. □

Lemma 3.8. *Suppose that (1.11)–(1.12) hold. Then for any $q>1$, there exists $C>0$ such that for all $\epsilon\in(0,1)$,*

$$\|\nabla c_\epsilon(\cdot,t)\|_{L^q(\Omega)}\le C \qquad \text{for all} \quad t\in(0,\infty).$$

Proof. We apply ∇ to equation $(1.9)_2$ and test the resulting equation by $|\nabla c_\epsilon|^{2(p-1)}\nabla c_\epsilon$ with $p>\frac{3}{2}$ to get

$$\frac{\epsilon}{2p}\frac{d}{dt}\int_\Omega|\nabla c_\epsilon|^{2p}-\int_\Omega|\nabla c_\epsilon|^{2(p-1)}\nabla c_\epsilon\cdot\Delta\nabla c_\epsilon+\int_\Omega|\nabla c_\epsilon|^{2p}=\int_\Omega|\nabla c_\epsilon|^{2(p-1)}\nabla c_\epsilon\cdot\nabla n_\epsilon$$
$$-\int_\Omega|\nabla c_\epsilon|^{2(p-1)}\nabla c_\epsilon\cdot\nabla\big(u_\epsilon\cdot\nabla c_\epsilon\big),$$

which together with the pointwise identity $2\nabla c_\epsilon\cdot\nabla\Delta c_\epsilon=\Delta|\nabla c_\epsilon|^2-2|D^2c_\epsilon|^2$ and the integration by parts yields that

$$\begin{aligned}&\frac{\epsilon}{2p}\frac{d}{dt}\int_\Omega|\nabla c_\epsilon|^{2p}+\frac{p-1}{2}\int_\Omega|\nabla c_\epsilon|^{2(p-2)}\left|\nabla|\nabla c_\epsilon|^2\right|^2+\int_\Omega|\nabla c_\epsilon|^{2(p-1)}|D^2c_\epsilon|^2+\int_\Omega|\nabla c_\epsilon|^{2p}\\&=\int_\Omega|\nabla c_\epsilon|^{2(p-1)}\nabla n_\epsilon\cdot\nabla c_\epsilon+(p-1)\int_\Omega(u_\epsilon\cdot\nabla c_\epsilon)|\nabla c_\epsilon|^{2(p-2)}\nabla c_\epsilon\cdot\nabla|\nabla c_\epsilon|^2\\&\quad+\int_\Omega(u_\epsilon\cdot\nabla c_\epsilon)|\nabla c_\epsilon|^{2(p-1)}\Delta c_\epsilon+\frac{1}{2}\int_{\partial\Omega}|\nabla c_\epsilon|^{2(p-1)}\frac{\partial|\nabla c_\epsilon|^2}{\partial\nu}\end{aligned}\tag{3.34}$$

for all $t\in(0,\infty)$. For the first term on the right-hand side of (3.34), it follows from the pointwise inequality $|\Delta c_\epsilon|^2\le2|D^2c_\epsilon|^2$, the integration by parts and the Young inequality that

$$\begin{aligned}&\int_\Omega|\nabla c_\epsilon|^{2(p-1)}\nabla n_\epsilon\cdot\nabla c_\epsilon\\&=-\int_\Omega|\nabla c_\epsilon|^{2(p-1)}n_\epsilon\Delta c_\epsilon-(p-1)\int_\Omega|\nabla c_\epsilon|^{2(p-2)}n_\epsilon\nabla c_\epsilon\cdot\nabla|\nabla c_\epsilon|^2\\&\le\sqrt{2}\int_\Omega|\nabla c_\epsilon|^{2(p-1)}n_\epsilon|D^2c_\epsilon|+(p-1)\int_\Omega n_\epsilon|\nabla c_\epsilon|^{2p-3}\left|\nabla|\nabla c_\epsilon|^2\right|\\&\le\frac{1}{2}\int_\Omega|\nabla c_\epsilon|^{2(p-1)}|D^2c_\epsilon|^2+\frac{p-1}{4}\int_\Omega|\nabla c_\epsilon|^{2(p-2)}\left|\nabla|\nabla c_\epsilon|^2\right|^2+p\int_\Omega n_\epsilon^2|\nabla c_\epsilon|^{2(p-1)},\end{aligned}\tag{3.35}$$

while for the second and third terms, we have

$$(p-1)\int_\Omega (u_\epsilon\cdot\nabla c_\epsilon)|\nabla c_\epsilon|^{2(p-2)}\nabla c_\epsilon\cdot\nabla|\nabla c_\epsilon|^2$$
$$\le \frac{p-1}{8}\int_\Omega |\nabla c_\epsilon|^{2(p-2)}\left|\nabla|\nabla c_\epsilon|^2\right|^2 + 2(p-1)\int_\Omega |u_\epsilon|^2|\nabla c_\epsilon|^{2p} \tag{3.36}$$

and

$$\int_\Omega (u_\epsilon\cdot\nabla c_\epsilon)|\nabla c_\epsilon|^{2(p-1)}\Delta c_\epsilon \le \sqrt{2}\int_\Omega |u_\epsilon||\nabla c_\epsilon|^{2p-1}|D^2c_\epsilon| \le \frac{1}{2}\int_\Omega |\nabla c_\epsilon|^{2(p-1)}|D^2c_\epsilon|^2$$
$$+\int_\Omega |u_\epsilon|^2|\nabla c_\epsilon|^{2p}. \tag{3.37}$$

For the last term on the right-hand side of (3.34), we know from the geometry property (3.20), the trace theorem, the Gagliardo–Nirenberg inequality, the Young inequality and Lemma 3.7 that

$$\frac{1}{2}\int_{\partial\Omega}|\nabla c_\epsilon|^{2(p-1)}\frac{\partial|\nabla c_\epsilon|^2}{\partial\nu} \le C_1\int_{\partial\Omega}|\nabla c_\epsilon|^{2p} = C_1\left\||\nabla c_\epsilon|^p\right\|^2_{L^2(\partial\Omega)}$$
$$\le C_2\left\||\nabla c_\epsilon|^p\right\|^2_{W^{\frac{3}{4},2}(\Omega)}$$
$$\le C_3\left\|\nabla|\nabla c_\epsilon|^p\right\|^{\frac{4p-1}{2p}}_{L^2(\Omega)}\left\||\nabla c_\epsilon|^p\right\|^{\frac{1}{2p}}_{L^{\frac{2}{p}}(\Omega)} + C_3\left\||\nabla c_\epsilon|^p\right\|^2_{L^{\frac{2}{p}}(\Omega)}$$
$$\le \frac{p-1}{4p^2}\int_\Omega |\nabla|\nabla c_\epsilon|^p|^2 + C_4 \tag{3.38}$$

for all $t\in(0,\infty)$. Inserting (3.35)–(3.38) into (3.34), we deduce that

$$\epsilon\frac{d}{dt}\int_\Omega|\nabla c_\epsilon|^{2p} + \frac{p-1}{2p}\int_\Omega\left|\nabla|\nabla c_\epsilon|^p\right|^2 + 2p\int_\Omega|\nabla c_\epsilon|^{2p} \le C_5\int_\Omega n_\epsilon^2|\nabla c_\epsilon|^{2(p-1)}$$
$$+C_5\int_\Omega|u_\epsilon|^2|\nabla c_\epsilon|^{2p} + C_5$$

for all $t\in(0,\infty)$. Noticing that

$$C_5\int_\Omega n_\epsilon^2|\nabla c_\epsilon|^{2(p-1)} \le C_5\|n_\epsilon\|^2_{L^4(\Omega)}\|\nabla c_\epsilon\|^{2(p-1)}_{L^{4(p-1)}(\Omega)}$$
$$\le C_6\|\nabla c_\epsilon\|^{2(p-1)}_{L^{4(p-1)}(\Omega)} = C_6\left\||\nabla c_\epsilon|^p\right\|^{\frac{2(p-1)}{p}}_{L^{\frac{4(p-1)}{p}}(\Omega)}$$
$$\le C_7\left(\left\|\nabla|\nabla c_\epsilon|^p\right\|^{\frac{2p-3}{p}}_{L^2(\Omega)}\left\||\nabla c_\epsilon|^p\right\|^{\frac{1}{p}}_{L^{\frac{2}{p}}(\Omega)} + \left\||\nabla c_\epsilon|^p\right\|^{\frac{2(p-1)}{p}}_{L^{\frac{2}{p}}(\Omega)}\right)$$
$$\le \frac{p-1}{4p}\left\|\nabla|\nabla c_\epsilon|^p\right\|^2_{L^2(\Omega)} + C_8$$

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by Corollary 3.2 and Lemma 3.7, and that

$$\begin{aligned} C_5\int_\Omega |u_\epsilon|^2|\nabla c_\epsilon|^{2p} &\le C_5\||u_\epsilon|^2\|_{L^2(\Omega)}\left\||\nabla c_\epsilon|^{2p}\right\|_{L^2(\Omega)} \\ &\le C_9\left\||\nabla c_\epsilon|^{2p}\right\|_{L^2(\Omega)} \\ &\le C_{10}\left(\left\|\nabla|\nabla c_\epsilon|^p\right\|_{L^2(\Omega)}^{\frac{2p-1}{p}}\left\||\nabla c_\epsilon|^p\right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{1}{p}}+\left\||\nabla c_\epsilon|^p\right\|_{L^{\frac{2}{p}}(\Omega)}^2\right) \\ &\le \frac{p-1}{4p}\left\|\nabla|\nabla c_\epsilon|^p\right\|_{L^2(\Omega)}^2+C_{11} \end{aligned}$$

by Lemmas 3.5 and 3.7 again, we have

$$\epsilon\frac{d}{dt}\int_\Omega|\nabla c_\epsilon|^{2p}+2p\int_\Omega|\nabla c_\epsilon|^{2p}\le C_{12}\qquad\text{for all}\quad t\in(0,\infty).$$

Thus, a direct calculation yields the boundedness of $\|\nabla c_\epsilon(\cdot,t)\|_{L^{2p}(\Omega)}$ for any $p>\frac{3}{2}$. Then in light of the Hölder inequality we can complete the proof of Lemma 3.8. □

Lemma 3.9. *Suppose that (1.11)–(1.12) hold. Then there exist a sequence* $\{\epsilon_j\}_{j=1}^\infty$ *and a unique global classical solution (n, c, u, P) to the PE-fluid system (1.10) with the properties that*

$$\begin{aligned} n_{\epsilon_j}&\to n &&\text{in}\quad C^0\big(\bar\Omega\times[0,\infty)\big),\\ n_{\epsilon_j}&\rightharpoonup n &&\text{in}\quad L^2_{loc}\big((0,\infty);W^{1,2}(\Omega)\big),\\ c_{\epsilon_j}&\to c &&\text{in}\quad L^\infty_{loc}\big((0,\infty);C^0(\bar\Omega)\big)\bigcap L^2_{loc}\big((0,\infty);W^{1,2}(\Omega)\big),\\ \nabla c_{\epsilon_j}&\overset{*}{\rightharpoonup}\nabla c &&\text{in}\quad \bigcap_{\hat q>2}L^\infty_{loc}\big((0,\infty);W^{1,\hat q}(\Omega)\big)\bigcap L^\infty_{loc}\big(\Omega\times(0,\infty)\big)\quad\text{and}\\ u_{\epsilon_j}&\to u &&\text{in}\quad C^0\big(\bar\Omega\times[0,\infty);\mathbb{R}^2\big)\bigcap C^{2,1}_{loc}\big(\bar\Omega\times(0,\infty);\mathbb{R}^2\big) \end{aligned}$$

as $j\to\infty$.

Proof. We know from Lemmas 3.8 and 3.5 that

$$\sup_{\epsilon\in(0,1)}\|\nabla c_\epsilon\|_{L^\infty((0,\infty);L^4(\Omega))}\le C\qquad\text{and}\qquad\sup_{\epsilon\in(0,1)}\|u_\epsilon\|_{L^\infty((0,\infty);L^4(\Omega))}\le C.$$

Therefore, in light of taking $d=2$, $T=\infty$, $\lambda=\infty$, $q=4$ and $r=4$ in Theorem 1.1 of [30], we can complete the proof of Lemma 3.9. Here the convexity of Ω can actually be removed as pointed out by Remark (i) in [30] and the uniqueness of solution $(n,c,u.P)$ to the PE-fluid system (1.10) can be derived from a standard energy method and a bootstrap argument, and thus we omit the details. □

4 Convergence rate

In the section, we will first derive the convergence rate for general large initial data and in particular obtain the convergence of the whole sequence $(n_\epsilon,c_\epsilon,u_\epsilon)$ (not just a subsequence as

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in Lemma 3.9). Then we derive some new exponential time decay estimates of $(n_\epsilon, c_\epsilon, u_\epsilon)$ with suitable small initial cell mass uniformly in ϵ. As a by-product, we improve the growth in time t as at most $\frac{1}{2}$-order.

For simplicity, letting (n, c, u, P) be the classical solution of the PE-fluid system (1.10) obtained in Lemma 3.9 and setting

$$\widehat{n} := n_\epsilon - n, \qquad \widehat{c} := c_\epsilon - c, \qquad \widehat{u} := u_\epsilon - u, \qquad \text{and} \qquad \widehat{P} := P_\epsilon - P,$$

we see that $(\widehat{n}, \widehat{c}, \widehat{u})$ is a solution to the following system:

$$\begin{cases} \partial_t \widehat{n} = \Delta\widehat{n} - u_\epsilon \cdot \nabla\widehat{n} - \widehat{u} \cdot \nabla n - \nabla \cdot \Big(\widehat{n} S(x, n_\epsilon, c_\epsilon) \cdot \nabla c_\epsilon & \\ \qquad + nS(x, n_\epsilon, c_\epsilon) \cdot \nabla\widehat{c} + n\big(S(x, n_\epsilon, c_\epsilon) - S(x, n, c)\big) \cdot \nabla c\Big), & x \in \Omega, \ t > 0, \\ \epsilon \partial_t c_\epsilon = \Delta\widehat{c} - u_\epsilon \cdot \nabla\widehat{c} - \widehat{u} \cdot \nabla c - \widehat{c} + \widehat{n}, & x \in \Omega, \ t > 0, \\ \partial_t \widehat{u} = \Delta\widehat{u} - (u_\epsilon \cdot \nabla)\widehat{u} - (\widehat{u} \cdot \nabla)u - \nabla\widehat{P} + \widehat{n}\nabla\phi, & x \in \Omega, \ t > 0, \\ \nabla \cdot \widehat{u} = 0, & x \in \Omega, \ t > 0, \\ \Big(\nabla\widehat{n} - \widehat{n} S(x, n_\epsilon, c_\epsilon) \cdot \nabla c_\epsilon - nS(x, n_\epsilon, c_\epsilon) \cdot \nabla\widehat{c} & \\ \qquad - n\big(S(x, n_\epsilon, c_\epsilon) - S(x, n, c)\big) \cdot \nabla c\Big) \cdot \nu = 0, \quad \nabla\widehat{c} \cdot \nu = 0, \quad \widehat{u} = 0, & x \in \partial\Omega, \ t > 0, \\ \widehat{n}(x, 0) = 0, \quad \widehat{u}(x, 0) = 0, & x \in \Omega. \end{cases} \tag{4.1}$$

4.1 The general large initial data case

Lemma 4.1. *Suppose that (1.11)–(1.12) hold. There exists a positive constant C such that for each $\epsilon \in (0, 1)$,*

$$\int_0^t \int_\Omega \partial_t c_\epsilon c \le C(1 + t) \qquad \textit{for all} \quad t \in (0, \infty).$$

Proof. Since

$$\int_\Omega \partial_t c_\epsilon c = \frac{d}{dt}\int_\Omega c_\epsilon c - \int_\Omega c_\epsilon \partial_t c \le \frac{d}{dt}\int_\Omega c_\epsilon c + \frac{1}{2}\left(\int_\Omega c_\epsilon^2 + \int_\Omega (\partial_t c)^2\right) \qquad \text{for all} \quad t \in (0, \infty),$$

we can see from the boundedness of c_ϵ and c, which follows from Lemma 3.8 and the Sobolev embedding, that

$$\begin{aligned} \int_0^t \int_\Omega \partial_t c_\epsilon c &\le \int_\Omega c_\epsilon(\cdot, t)c(\cdot, t) - \int_\Omega c_0(\cdot)c(\cdot, 0) + \frac{1}{2}\left(\int_0^t \int_\Omega c_\epsilon^2 + \int_0^t \int_\Omega (\partial_t c)^2\right) \\ &\le \|c_\epsilon\|_{L^\infty(\Omega\times(0,\infty))}\|c\|_{L^\infty(\Omega\times(0,\infty))}|\Omega| + \frac{1}{2}\left(\|c_\epsilon\|^2_{L^\infty(\Omega\times(0,\infty))}|\Omega|t + \int_0^t \int_\Omega (\partial_t c)^2\right) \\ &\le C_1(1 + t) + \frac{1}{2}\int_0^t \int_\Omega (\partial_t c)^2 \qquad \text{for all} \quad t \in (0, \infty). \end{aligned} \tag{4.2}$$

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To establish the L^2 space-time estimate of $\partial_t c$, we first derive the uniform estimates for the difference quotient

$$c_h(x,t) := \frac{c(x,t+h)-c(x,t)}{h} \qquad \text{for all} \quad t \in (\tau,\infty)$$

with $\tau \in (0,\infty)$ and $h \in (-\tau,\infty)$. It is easy to show that for each $t \in (\tau,\infty)$, $c_h(\cdot,t) \in C^2(\overline{\Omega})$ is a classical solution of the homogeneous Neumann boundary-value problem for

$$-\Delta c_h(\cdot,t) + c_h(\cdot,t) = -u_h(\cdot,t)\cdot\nabla c(\cdot,t+h) - u(\cdot,t)\cdot\nabla c_h(\cdot,t) + n_h(\cdot,t)$$

in Ω, where

$$u_h(\cdot,t) := \frac{u(x,t+h)-u(x,t)}{h} \qquad \text{and} \qquad n_h(\cdot,t) := \frac{n(x,t+h)-n(x,t)}{h}.$$

Then testing the above equation by $c_h(\cdot,t)$, we obtain from integrating by parts over Ω, the solenoidality of u and the Hölder inequality that

$$\begin{aligned}
&\|\nabla c_h(\cdot,t)\|^2_{L^2(\Omega)} + \|c_h(\cdot,t)\|^2_{L^2(\Omega)} \\
&= -\int_\Omega c_h(\cdot,t)u_h(\cdot,t)\cdot\nabla c(\cdot,t+h) - \frac{1}{2}\int_\Omega u(\cdot,t)\cdot\nabla c_h^2(\cdot,t) + \int_\Omega c_h(\cdot,t)n_h(\cdot,t) \\
&= -\int_\Omega c_h(\cdot,t)u_h(\cdot,t)\cdot\nabla c(\cdot,t+h) + \int_\Omega c_h(\cdot,t)n_h(\cdot,t) \\
&\le \|c_h(\cdot,t)\|_{L^4(\Omega)}\|u_h(\cdot,t)\|_{L^2(\Omega)}\|\nabla c(\cdot,t+h)\|_{L^4(\Omega)} + \|c_h(\cdot,t)\|_{L^2(\Omega)}\|n_h(\cdot,t)\|_{L^2(\Omega)}
\end{aligned}$$

for all $t \in (\tau,\infty)$, which together with the uniform boundedness of $\|\nabla c(\cdot,t+h)\|_{L^4(\Omega)}$ entails that

$$\begin{aligned}
&\|\nabla c_h(\cdot,t)\|^2_{L^2(\Omega)} + \|c_h(\cdot,t)\|^2_{L^2(\Omega)} \\
&\le C_2\|c_h(\cdot,t)\|_{L^4(\Omega)}\|u_h(\cdot,t)\|_{L^2(\Omega)} + \|c_h(\cdot,t)\|_{L^2(\Omega)}\|n_h(\cdot,t)\|_{L^2(\Omega)} \\
&\le C_3\Big(\|\nabla c_h(\cdot,t)\|^{\frac12}_{L^2(\Omega)}\|c_h(\cdot,t)\|^{\frac12}_{L^2(\Omega)} + \|c_h(\cdot,t)\|_{L^2(\Omega)}\Big)\|u_h(\cdot,t)\|_{L^2(\Omega)} + \|c_h(\cdot,t)\|_{L^2(\Omega)}\|n_h(\cdot,t)\|_{L^2(\Omega)} \\
&\le \frac12\Big(\|\nabla c_h(\cdot,t)\|^2_{L^2(\Omega)} + \|c_h(\cdot,t)\|^2_{L^2(\Omega)}\Big) + C_4\Big(\|u_h(\cdot,t)\|^2_{L^2(\Omega)} + \|n_h(\cdot,t)\|^2_{L^2(\Omega)}\Big)
\end{aligned}$$

for all $t \in (\tau,\infty)$, and thus that

$$\begin{aligned}
\int_\tau^t\int_\Omega |\nabla c_h|^2 + \int_\tau^t\int_\Omega c_h^2 &\le 2C_4\left(\int_\tau^t\int_\Omega n_h^2 + \int_\tau^t\int_\Omega |u_h|^2\right) \\
&\le C_5\left(\int_0^{t+1}\int_\Omega (\partial_t n)^2 + \int_0^{t+1}\int_\Omega |\partial_t u|^2\right), \qquad (4.3)
\end{aligned}$$

where in the last inequality we used the temporal version of Theorem 3(i) in Section 5.8.2, [7].

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For the first integral on the right-hand side of (4.3), we can use the maximal regularity of parabolic equations (see Theorem 2.3 in [10]) and the trace theorem to obtain

$$\begin{aligned}
\|\partial_t n\|_{L^2((0,t+1);L^2(\Omega))} &\le C_6\Big(\|u\cdot\nabla n\|_{L^2((0,t+1);L^2(\Omega))} + \|\nabla\cdot(nS(x,n,c)\cdot\nabla c)\|_{L^2((0,t+1);L^2(\Omega))} \\
&\qquad + \|nS(x,n,c)\cdot\nabla c\|_{L^2((0,t+1);W^{\frac{1}{2},2}(\partial\Omega))} + \|n_0\|_{W^{1,2}(\Omega)}\Big) \\
&\le C_7\Big(\|u\cdot\nabla n\|_{L^2((0,t+1);L^2(\Omega))} + \|nS(x,n,c)\cdot\nabla c\|_{L^2((0,t+1);W^{1,2}(\Omega))} + \|n_0\|_{W^{1,2}(\Omega)}\Big) \\
&\le C_8\Big(\|u\|_{L^\infty(\Omega\times(0,\infty))}\|\nabla n\|_{L^2((0,t+1);L^2(\Omega))} + \|\nabla n\|_{L^2((0,t+1);L^2(\Omega))}\|\nabla c\|_{L^\infty(\Omega\times(0,\infty))} \\
&\qquad + \|n\|_{L^\infty(\Omega\times(0,\infty))}\|\nabla c\|_{L^2((0,t+1);W^{1,2}(\Omega))} + \|n_0\|_{W^{1,2}(\Omega)}\Big) \\
&\le C_9(1+t)^{\frac{1}{2}} \qquad \text{for all} \quad t\in(0,\infty),
\end{aligned} \tag{4.4}$$

where in the last inequality, we used the bounds for n, u and ∇c derived from the proofs of Lemmas 2.2, 2.3 and 5.3 in [30], respectively, and the growth estimates $\|\nabla n\|^2_{L^2((0,t+1);L^2(\Omega))} \le C(1+t)$ and $\|D^2 c\|^2_{L^2((0,t+1);L^2(\Omega))} \le C(1+t)$ obtained by following the proof of Lemma 3.4 in [30] and by a direct integral in equation $(1.10)_2$, respectively. On the other hand, for the second integral on the right-hand side of (4.3), we can utilise the Sobolev maximal regularity of Stokes equation (see Theorem 2.1 in [10]) and $\|n\|^2_{L^2((0,t+1);L^2(\Omega))} \le C(1+t)$ to obtain that

$$\begin{aligned}
&\int_0^{t+1} \|\partial_t u(\cdot,s)\|^2_{L^2(\Omega)} ds + \int_0^{t+1} \|u(\cdot,s)\|^2_{H^2(\Omega)} ds \\
&\quad \le C_{10}\int_0^{t+1}\Big(\big\|u\cdot\nabla u(\cdot,s)\big\|^2_{L^2(\Omega)} + \big\|n\nabla\phi(\cdot,s)\big\|^2_{L^2(\Omega)}\Big) ds \\
&\quad \le C_{10}\|u\|^2_{L^\infty(\Omega\times(0,\infty))}\int_0^{t+1}\|\nabla u(\cdot,s)\|^2_{L^2(\Omega)} ds + C_{10}\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_0^{t+1}\|n(\cdot,s)\|^2_{L^2(\Omega)} ds \\
&\quad \le C_{11}\int_0^{t+1}\Big(\|u(\cdot,s)\|_{H^2(\Omega)}\|u(\cdot,s)\|_{L^2(\Omega)} + \|u(\cdot,s)\|^2_{L^2(\Omega)}\Big) ds + C_{11}(1+t) \\
&\quad \le \frac{1}{2}\int_0^{t+1}\|u(\cdot,s)\|^2_{H^2(\Omega)} ds + C_{12}\int_0^{t+1}\|u(\cdot,s)\|^2_{L^2(\Omega)} ds + C_{12}(1+t) \\
&\quad \le \frac{1}{2}\int_0^{t+1}\|u(\cdot,s)\|^2_{H^2(\Omega)} ds + C_{13}(1+t) \qquad \text{for all} \quad t\in(0,\infty)
\end{aligned}$$

and thus that

$$\int_0^{t+1} \|\partial_t u(\cdot,s)\|^2_{L^2(\Omega)} ds + \frac{1}{2}\int_0^{t+1} \|u(\cdot,s)\|^2_{H^2(\Omega)} ds \le C_{13}(1+t) \qquad \text{for all} \quad t\in(0,\infty). \tag{4.5}$$

Consequently, inserting (4.4) and (4.5) into (4.3), we see that

$$\int_\tau^t\int_\Omega |\nabla c_h|^2 + \int_\tau^t\int_\Omega c_h^2 \le C_{14}(1+t) := C_5\big(C_9^2(1+t) + C_{13}(1+t)\big) \qquad \text{for all} \quad t\in(\tau,\infty)$$

and that there exists $(h_i)_{i\in\mathbb{N}}$ satisfying $h_i\to 0$ and $c_{h_i}\to\partial_t c$ in $L^2(\Omega\times(\tau,t))$ as $i\to\infty$ with the same bound $\int_\tau^t\int_\Omega(\partial_t c)^2 \le C_{14}(1+t)$. Since C_{14} is independent of τ, we may take $\tau\searrow 0$ to obtain $\int_0^t\int_\Omega(\partial_t c)^2 \le C_{14}(1+t)$, which together with (4.2) completes the proof of Lemma 4.1. □

Lemma 4.2. *Suppose that (1.11)–(1.12) hold. Then there exists $C > 0$ such that for each $\epsilon \in (0, 1)$,*

$$\|\widehat{n}(\cdot, t)\|_{L^2(\Omega)} + \|\widehat{u}(\cdot, t)\|_{L^2(\Omega)} \le Ce^{Ct}\epsilon^{\frac{1}{2}} \qquad \text{for all} \quad t \in (0, \infty)$$

and

$$\|\widehat{n}\|_{L^2((0,t);W^{1,2}(\Omega))} + \|\widehat{c}\|_{L^2((0,t);W^{1,2}(\Omega))} + \|\widehat{u}\|_{L^2((0,t);W^{1,2}(\Omega))} \le Ce^{Ct}\epsilon^{\frac{1}{2}} \qquad \text{for all} \quad t \in (0, \infty).$$

Proof. The proof is parallel to that of Lemma 5.2 in [19] and thus we omit the details here. □

Lemma 4.3. *Suppose that (1.11)–(1.12) hold. Then there exists $C > 0$ such that for each $\epsilon \in (0, 1)$,*

$$\|\nabla\widehat{u}(\cdot, t)\|_{L^2(\Omega)} \le Ce^{Ct}\epsilon^{\frac{1}{2}} \qquad \text{for all} \quad t \in (0, \infty).$$

Proof. Applying the Helmholtz projection $\mathcal{P}$ to both sides of equation $(4.1)_3$ and testing the resulting equation with $A\widehat{u}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\widehat{u}\|^2_{L^2(\Omega)} + \|A\widehat{u}\|^2_{L^2(\Omega)} \\
&\quad = -\int_\Omega \mathcal{P}\big(u_\epsilon \cdot \nabla\widehat{u} + \widehat{u} \cdot \nabla u - \widehat{n}\nabla\phi\big) \cdot A\widehat{u} \\
&\quad \le C_1\Big(\|u_\epsilon\|_{L^4(\Omega)}\|\nabla\widehat{u}\|_{L^4(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\|\widehat{u}\|_{L^\infty(\Omega)} + \|\widehat{n}\|_{L^2(\Omega)}\|\nabla\phi\|_{L^\infty(\Omega)}\Big)\|A\widehat{u}\|_{L^2(\Omega)} \\
&\quad \le C_2\Big(\|A\widehat{u}\|^{\frac{3}{4}}_{L^2(\Omega)}\|\widehat{u}\|^{\frac{1}{4}}_{L^2(\Omega)} + \|A\widehat{u}\|^{\frac{1}{2}}_{L^2(\Omega)}\|\widehat{u}\|^{\frac{1}{2}}_{L^2(\Omega)} + \|\widehat{u}\|_{L^2(\Omega)} + \|\widehat{n}\|_{L^2(\Omega)}\Big)\|A\widehat{u}\|_{L^2(\Omega)} \\
&\quad \le \|A\widehat{u}\|^2_{L^2(\Omega)} + C_3\|\widehat{u}\|^2_{L^2(\Omega)} + C_3\|\widehat{n}\|^2_{L^2(\Omega)} \qquad \text{for all} \quad t \in (0, \infty),
\end{aligned}$$

where we used the uniform boundedness of $\|u_\epsilon\|_{L^4(\Omega)}$ and the boundedness of $\|\nabla u\|_{L^2(\Omega)}$. It then follows from Lemma 4.2 and $\widehat{u}(\cdot, 0) = 0$ that

$$\|\nabla\widehat{u}(\cdot, t)\|^2_{L^2(\Omega)} \le 2C_3\int_0^t \Big(\|\widehat{n}(\cdot, s)\|^2_{L^2(\Omega)} + \|\widehat{u}(\cdot, s)\|^2_{L^2(\Omega)}\Big)ds \le C_4e^{C_4t}\epsilon \tag{4.6}$$

for all $t \in (0, \infty)$. This completes the proof of Lemma 4.3. □

Lemma 4.4. *Suppose that (1.11)–(1.12) hold. Then for any given $\theta \in \left(\frac{1}{2}, \frac{3}{4}\right)$, there exists $C(\theta) > 0$ such that for each $\epsilon \in (0, 1)$,*

$$\|A^\theta\widehat{u}(\cdot, t)\|_{L^2(\Omega)} \le C(\theta)e^{C(\theta)t}\epsilon^{\frac{1}{2}} \qquad \text{for all} \quad t \in (0, \infty).$$

In particular, there exists a positive constant C such that

$$\|\widehat{u}(\cdot, t)\|_{L^\infty(\Omega)} \le Ce^{Ct}\epsilon^{\frac{1}{2}} \qquad \text{for all} \quad t \in (0, \infty).$$

Proof. Using the variation-of-constants representation of $\widehat{u}$, we have

$$A^\theta \widehat{u}(\cdot,t) := -\int_0^t A^\theta e^{-(t-s)A}\mathcal{P}\big(u_\epsilon\cdot\nabla\widehat{u}+\widehat{u}\cdot\nabla u-\widehat{n}\nabla\phi\big)(\cdot,s)ds. \tag{4.7}$$

Setting $f(\cdot,s) := \mathcal{P}\big(u_\epsilon\cdot\nabla\widehat{u}+\widehat{u}\cdot\nabla u-\widehat{n}\nabla\phi\big)(\cdot,s)$, we can use the Hölder inequality, the Gagliardo–Nirenberg inequality, the boundedness of u_ϵ, Lemmas 4.2 and 4.3 to obtain

$$\begin{aligned}
\|f(\cdot,s)\|_{L^2(\Omega)} &\le \|u_\epsilon\|_{L^\infty(\Omega)}\|\nabla\widehat{u}\|_{L^2(\Omega)} + \|\widehat{u}\|_{L^4(\Omega)}\|\nabla u\|_{L^4(\Omega)} + \|\widehat{n}\|_{L^2(\Omega)}\|\nabla\phi\|_{L^\infty(\Omega)}\\
&\le C_1\|\nabla\widehat{u}\|_{L^2(\Omega)} + C_1\|\nabla u\|_{L^4(\Omega)}\left(\|\nabla\widehat{u}\|^{\frac12}_{L^2(\Omega)}\|\widehat{u}\|^{\frac12}_{L^2(\Omega)} + \|\widehat{u}\|_{L^2(\Omega)}\right) + C_1\|\widehat{n}\|_{L^2(\Omega)}\\
&\le C_2 e^{C_2 s}\epsilon^{\frac12} + C_2\|\nabla u(\cdot,s)\|_{L^4(\Omega)}e^{C_2 s}\epsilon^{\frac12}
\end{aligned}$$

and thus have

$$\begin{aligned}
\|A^\theta\widehat{u}(\cdot,t)\|_{L^2(\Omega)} &\le C_3\int_0^t (t-s)^{-\theta}e^{-\lambda(t-s)}\|f(\cdot,s)\|_{L^2(\Omega)}ds\\
&\le C_4\epsilon^{\frac12}\int_0^t (t-s)^{-\theta}e^{-\lambda(t-s)+C_2 s}ds\\
&\quad + C_4\epsilon^{\frac12}\int_0^t \|\nabla u(\cdot,s)\|_{L^4(\Omega)}(t-s)^{-\theta}e^{-\lambda(t-s)+C_2 s}ds\\
&\le C_5 e^{C_5 t}\epsilon^{\frac12} + C_4\epsilon^{\frac12}\left(\int_0^t \left((t-s)^{-\theta}e^{-\lambda(t-s)+C_2 s}\right)^{\frac43}ds\right)^{\frac34}\left(\int_0^t \|\nabla u(\cdot,s)\|^4_{L^4(\Omega)}ds\right)^{\frac14}\\
&\le C_5 e^{C_5 t}\epsilon^{\frac12} + C_6 e^{C_6 t}\epsilon^{\frac12}\left(\int_0^t \|\nabla u(\cdot,s)\|^4_{L^4(\Omega)}ds\right)^{\frac14}
\end{aligned}\tag{4.8}$$

for some $\lambda > 0$ due to $\widehat{u}(\cdot,0) = 0$ and $\theta \in \left(\frac12,\frac34\right)$. For the last integral on the right-hand side of (4.8), we apply the Gagliardo–Nirenberg inequality, (4.5) and the boundedness of $\|u\|_{L^\infty(\Omega\times(0,\infty))}$ to obtain

$$\begin{aligned}
\int_0^t \|\nabla u(\cdot,s)\|^4_{L^4(\Omega)}ds &\le \int_0^t \left(\|u(\cdot,s)\|^2_{H^2(\Omega)}\|u(\cdot,s)\|^2_{L^\infty(\Omega)} + \|u(\cdot,s)\|^4_{L^\infty(\Omega)}\right)ds\\
&\le \|u\|^2_{L^\infty(\Omega\times(0,\infty))}\int_0^t \|u(\cdot,s)\|^2_{H^2(\Omega)}ds + \|u\|^4_{L^\infty(\Omega\times(0,\infty))}|\Omega|t\\
&\le C_7(1+t)
\end{aligned}\tag{4.9}$$

for all $t \in (0,\infty)$. This together with (4.8) and the fact $1+t \le e^t$ yields that

$$\|A^\theta\widehat{u}(\cdot,t)\|_{L^2(\Omega)} \le C_8 e^{C_8 t}\epsilon^{\frac12} \qquad \text{for all} \quad t\in(0,\infty).$$

Meanwhile, the estimate of $\|\widehat{u}(\cdot,t)\|_{L^\infty(\Omega)}$ will follow from $D(A^\theta) \hookrightarrow L^\infty(\Omega)$ due to $\theta \in \left(\frac12,\frac34\right)$. This completes the proof of Lemma 4.4. □

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Lemma 4.5. *Suppose that (1.11)–(1.12) hold. Then for any $p \geq 2$, there exists $C(p) > 0$ such that for each $\epsilon \in (0, 1)$,*

$$\|\widehat{n}(\cdot, t)\|_{L^p(\Omega)} \leq C(p) e^{C(p)t} \epsilon^{\frac{1}{4}} \qquad \text{for all} \quad t \in (0, \infty).$$

Proof. The proof is parallel to that of Lemma 5.4 in [19] and thus we omit the details here. □

Proof of Theorem 1.1. We can conclude the desired result using Lemmas 3.9, 4.2, 4.4 and 4.5. □

4.2 The small initial cell mass case

In this subsection, we first show the time decay of solutions to the PP-fluid system (1.9) with suitable small initial cell mass, which will be used to improve the exponential growth on time t in the fast signal diffusion limit procedure.

Lemma 4.6. *Suppose that (1.11)–(1.12) hold. Then if*

$$\|n_0\|_{L^1(\Omega)} \leq \delta$$

holds for some suitable small $\delta > 0$, then there exist two positive constants C and μ such that

$$\|n_\epsilon(\cdot, t) - \overline{n}_0\|_{L^2(\Omega)} \leq Ce^{-\mu t} \qquad \text{for all} \quad t \in (0, \infty) \tag{4.10}$$

provided that ϵ is suitable small.

Proof. Firstly, we multiply equation $(1.9)_1$ by $n_\epsilon - \overline{n}_0$ and integrate by parts over Ω to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|n_\epsilon - \overline{n}_0\|^2_{L^2(\Omega)} + \|\nabla n_\epsilon\|^2_{L^2(\Omega)} &= \int_\Omega \nabla n_\epsilon \cdot \big(n_\epsilon S(x, n_\epsilon, c_\epsilon)\nabla c_\epsilon\big) \\ &\leq \frac{1}{2}\|\nabla n_\epsilon\|^2_{L^2(\Omega)} + \frac{C_S^2}{2}\int_\Omega n_\epsilon^2 |\nabla c_\epsilon|^2\end{aligned}$$

for all $t \in (0, \infty)$, and thus have

$$\begin{aligned}\frac{d}{dt}\|n_\epsilon - \overline{n}_0\|^2_{L^2(\Omega)} + \|\nabla n_\epsilon\|^2_{L^2(\Omega)} &\leq C_S^2 \int_\Omega n_\epsilon^2 |\nabla c_\epsilon|^2 \\ &\leq 2C_S^2 \int_\Omega (n_\epsilon - \overline{n}_0)^2 |\nabla c_\epsilon|^2 + 2C_S^2 \int_\Omega \overline{n}_0^2 |\nabla c_\epsilon|^2 \\ &\leq 2C_S^2 \|n_\epsilon - \overline{n}_0\|^2_{L^4(\Omega)} \|\nabla c_\epsilon\|^2_{L^4(\Omega)} + 2C_S^2 \overline{n}_0^2 \|\nabla c_\epsilon\|^2_{L^2(\Omega)}\end{aligned}$$

for all $t \in (0, \infty)$. Since the Poincaré inequality and the interpolation entail that

$$\|n_\epsilon - \overline{n}_0\|^2_{L^2(\Omega)} = \|n_\epsilon - \overline{n_\epsilon}\|^2_{L^2(\Omega)} \leq C_1 \|\nabla n_\epsilon\|^2_{L^2(\Omega)}$$

and

$$\|n_\epsilon - \overline{n}_0\|^2_{L^{\frac{10}{9}}(\Omega)} \leq 2\left(\|n_\epsilon\|^{\frac{9}{5}}_{L^1(\Omega)} \|n_\epsilon\|^{\frac{1}{5}}_{L^\infty(\Omega)} + \|\overline{n}_0\|^2_{L^{\frac{10}{9}}(\Omega)}\right) \leq C_2 \left(\|n_0\|^{\frac{9}{5}}_{L^1(\Omega)} + \|n_0\|^2_{L^1(\Omega)}\right),$$

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we deduce from the Gagliardo–Nirenberg inequality, the Young inequality, Lemma 3.8 and the interpolation that

$$\begin{aligned}
&2C_S^2\|n_\epsilon-\overline{n}_0\|^2_{L^4(\Omega)}\|\nabla c_\epsilon\|^2_{L^4(\Omega)}\\
&\le C_3\left(\|\nabla(n_\epsilon-\overline{n}_0)\|^{\frac{5}{4}}_{L^2(\Omega)}\|n_\epsilon-\overline{n}_0\|^{\frac{3}{4}}_{L^{\frac{3}{2}}(\Omega)}+\|n_\epsilon-\overline{n}_0\|^2_{L^{\frac{3}{2}}(\Omega)}\right)\left(\|\nabla c_\epsilon\|^{\frac{7}{6}}_{L^{14}(\Omega)}\|\nabla c_\epsilon\|^{\frac{5}{6}}_{L^2(\Omega)}\right)\\
&\le \frac{1}{2}\|\nabla n_\epsilon\|^2_{L^2(\Omega)}+C_4\|n_\epsilon-\overline{n}_0\|^2_{L^{\frac{3}{2}}(\Omega)}\|\nabla c_\epsilon\|^{\frac{20}{9}}_{L^2(\Omega)}+C_4\|n_\epsilon-\overline{n}_0\|^2_{L^{\frac{3}{2}}(\Omega)}\|\nabla c_\epsilon\|^{\frac{5}{6}}_{L^2(\Omega)}\\
&\le \frac{1}{2}\|\nabla n_\epsilon\|^2_{L^2(\Omega)}+C_5\|n_\epsilon-\overline{n}_0\|^2_{L^{\frac{3}{2}}(\Omega)}\|\nabla c_\epsilon\|^{\frac{5}{6}}_{L^2(\Omega)}\\
&\le \frac{1}{2}\|\nabla n_\epsilon\|^2_{L^2(\Omega)}+C_6\|n_\epsilon-\overline{n}_0\|^{\frac{7}{6}}_{L^2(\Omega)}\|n_\epsilon-\overline{n}_0\|^{\frac{5}{6}}_{L^{\frac{10}{9}}(\Omega)}\|\nabla c_\epsilon\|^{\frac{5}{6}}_{L^2(\Omega)}\\
&\le \frac{1}{2}\|\nabla n_\epsilon\|^2_{L^2(\Omega)}+\frac{1}{4C_1}\|n_\epsilon-\overline{n}_0\|^2_{L^2(\Omega)}+C_7\|n_\epsilon-\overline{n}_0\|^2_{L^{\frac{10}{9}}(\Omega)}\|\nabla c_\epsilon\|^2_{L^2(\Omega)}\\
&\le \frac{1}{2}\|\nabla n_\epsilon\|^2_{L^2(\Omega)}+\frac{1}{4C_1}\|n_\epsilon-\overline{n}_0\|^2_{L^2(\Omega)}+C_8\left(\|n_0\|^{\frac{9}{5}}_{L^1(\Omega)}+\|n_0\|^2_{L^1(\Omega)}\right)\|\nabla c_\epsilon\|^2_{L^2(\Omega)}
\end{aligned}$$

and thus that

$$\begin{aligned}
\frac{d}{dt}\|n_\epsilon-\overline{n}_0\|^2_{L^2(\Omega)}+\frac{1}{4C_1}\|n_\epsilon-\overline{n}_0\|^2_{L^2(\Omega)}&\le C_9\left(\|n_0\|^{\frac{9}{5}}_{L^1(\Omega)}+\|n_0\|^2_{L^1(\Omega)}\right)\|\nabla c_\epsilon\|^2_{L^2(\Omega)}\\
&\le C_9\left(\delta^{\frac{9}{5}}+\delta^2\right)\|\nabla c_\epsilon\|^2_{L^2(\Omega)}\le\frac{1}{4C_1}\|\nabla c_\epsilon\|^2_{L^2(\Omega)} \qquad (4.11)
\end{aligned}$$

for all $t\in(0,\infty)$ provided that δ is suitable small.

To absorb the right-hand side of (4.11), we test equation $(1.9)_2$ by $c_\epsilon-\overline{n}_0$, integrate by parts over Ω and utilise the Hölder inequality and the Young inequality to obtain

$$\frac{\epsilon}{2}\frac{d}{dt}\|c_\epsilon-\overline{n}_0\|^2_{L^2(\Omega)}+\|\nabla c_\epsilon\|^2_{L^2(\Omega)}+\|c_\epsilon-\overline{n}_0\|^2_{L^2(\Omega)}\le\frac{1}{2}\|c_\epsilon-\overline{n}_0\|^2_{L^2(\Omega)}+\frac{1}{2}\|n_\epsilon-\overline{n}_0\|^2_{L^2(\Omega)}$$

for all $t\in(0,\infty)$, and then have

$$\epsilon\frac{d}{dt}\|c_\epsilon-\overline{n}_0\|^2_{L^2(\Omega)}+2\|\nabla c_\epsilon\|^2_{L^2(\Omega)}+\|c_\epsilon-\overline{n}_0\|^2_{L^2(\Omega)}\le\|n_\epsilon-\overline{n}_0\|^2_{L^2(\Omega)} \qquad (4.12)$$

for all $t\in(0,\infty)$. This together with (4.11) yields that

$$\frac{d}{dt}\left(\|n_\epsilon-\overline{n}_0\|^2_{L^2(\Omega)}+\frac{\epsilon}{8C_1}\|c_\epsilon-\overline{n}_0\|^2_{L^2(\Omega)}\right)+\frac{1}{8C_1}\|n_\epsilon-\overline{n}_0\|^2_{L^2(\Omega)}+\frac{1}{8C_1}\|c_\epsilon-\overline{n}_0\|^2_{L^2(\Omega)}\le 0$$

for all $t\in(0,\infty)$. Without loss of generality, we can assume $0<\epsilon<\min\{1,8C_1\}$ and set

$$y(t):=\|n_\epsilon-\overline{n}_0\|^2_{L^2(\Omega)}+\frac{\epsilon}{8C_1}\|c_\epsilon-\overline{n}_0\|^2_{L^2(\Omega)}$$

to obtain

$$y'(t)+\frac{1}{8C_1}y(t)\le 0 \qquad \text{for all} \quad t\in(0,\infty). \qquad (4.13)$$

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Integrating (4.13) from 0 to t, we conclude that

$$y(t) \le e^{-\frac{t}{8C_1}} y(0) \le \left(\|n_0 - \overline{n}_0\|^2_{L^2(\Omega)} + \frac{1}{8C_1} \|c_0 - \overline{n}_0\|^2_{L^2(\Omega)} \right) e^{-\frac{t}{8C_1}} \le C_{10} e^{-\frac{t}{8C_1}}$$

for all $t \in (0, \infty)$, which entails (4.10). □

Lemma 4.7. *Under the assumption of Lemma 4.6, there exist two positive constants C and μ such that*

$$\|c_\epsilon(\cdot, t) - \overline{n}_0\|_{L^2(\Omega)} \le Ce^{-\mu t} \qquad \text{for all} \quad t \in (0, \infty)$$

provided that ϵ is suitable small.

Proof. By repeating the proof of (4.12), we can see from Lemma 4.6 that

$$\epsilon \frac{d}{dt} \|c_\epsilon - \overline{n}_0\|^2_{L^2(\Omega)} + 2\|\nabla c_\epsilon\|^2_{L^2(\Omega)} + \|c_\epsilon - \overline{n}_0\|^2_{L^2(\Omega)} \le \|n_\epsilon - \overline{n}_0\|^2_{L^2(\Omega)} \le C_1 e^{-\mu_1 t}$$

for all $t \in (0, \infty)$ and some positive constants C_1 and μ_1. Then by setting

$$y(t) := \|c_\epsilon(\cdot, t) - \overline{n}_0\|^2_{L^2(\Omega)}$$

and letting $0 < \epsilon < \min\left\{1, \frac{1}{2\mu_1}\right\}$, we have

$$\epsilon y'(t) + y(t) \le C_1 e^{-\mu_1 t}$$

and thus

$$y(t) \le y(0)e^{-\frac{t}{\epsilon}} + \frac{C_1 e^{-\frac{t}{\epsilon}}}{\epsilon} \int_0^t e^{(\frac{1}{\epsilon} - \mu_1)s} ds \le y(0)e^{-\frac{t}{\epsilon}} + 2C_1 e^{-\mu_1 t} \le C_2 e^{-\min\{\mu_1, 1\}t}$$

for all $t \in (0, \infty)$. This completes the proof of Lemma 4.7. □

Corollary 4.1. *Under the assumptions of Lemma 4.6, there exist two positive constants C and μ such that for any $p > 1$,*

$$\|c_\epsilon(\cdot, t) - \overline{n}_0\|_{W^{1,p}(\Omega)} \le Ce^{-\mu t} \qquad \text{for all} \quad t \in (0, \infty)$$

provided that ϵ is suitable small.

Proof. For any fixed $q > p > 1$, we use the Gagliardo–Nirenberg inequality, the $W^{1,q}$ boundedness of c_ϵ and Lemma 4.7 to get

$$\begin{aligned}
\|c_\epsilon - \overline{n}_0\|_{W^{1,p}(\Omega)} &\le C_1 \|c_\epsilon - \overline{n}_0\|^{\theta_1}_{W^{1,q}(\Omega)} \|c_\epsilon - \overline{n}_0\|^{1-\theta_1}_{L^2(\Omega)} + C_1 \|c_\epsilon - \overline{n}_0\|_{L^2(\Omega)} \\
&\le C_1 \left(\|c_\epsilon\|_{W^{1,q}(\Omega)} + \|\overline{n}_0\|_{W^{1,q}(\Omega)} \right)^{\theta_1} \|c_\epsilon - \overline{n}_0\|^{1-\theta_1}_{L^2(\Omega)} + C_1 \|c_\epsilon - \overline{n}_0\|_{L^2(\Omega)} \\
&\le C_2 \|c_\epsilon - \overline{n}_0\|^{1-\theta_1}_{L^2(\Omega)} + C_1 \|c_\epsilon - \overline{n}_0\|_{L^2(\Omega)} \\
&\le C_3 e^{-\mu t}
\end{aligned}$$

for some $\mu > 0$ and all $t \in (0, \infty)$, where $\theta_1 = \frac{q(p-1)}{(q-1)p} \in (0, 1)$. □

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Lemma 4.8. *Under the assumptions of Lemma 4.6, there exist two positive constants C and μ such that*

$$\|n_\epsilon(\cdot,t)-\overline{n}_0\|_{L^\infty(\Omega)}\le Ce^{-\mu t}\qquad \text{for all}\quad t\in(0,\infty) \tag{4.14}$$

and

$$\|u_\epsilon(\cdot,t)\|_{L^\infty(\Omega)}\le Ce^{-\mu t}\qquad \text{for all}\quad t\in(0,\infty) \tag{4.15}$$

provided that ϵ is suitable small.

Proof. Let $\left(e^{t\Delta}\right)_{t\ge0}$ be the homogeneous Neumann heat semigroup in Ω. Since

$$n_\epsilon(\cdot,t)=e^{t\Delta}n_0-\int_0^t e^{(t-s)\Delta}\Big(\nabla\cdot\big(n_\epsilon S(\cdot,n_\epsilon,c_\epsilon)\nabla c_\epsilon\big)+u_\epsilon\cdot\nabla n_\epsilon\Big)(\cdot,s)ds$$

for all $t\in(0,\infty)$, we can use the fact $e^{t\Delta}\overline{n}_0=\overline{n}_0$ to obtain

$$\begin{aligned}\|n_\epsilon(\cdot,t)-\overline{n}_0\|_{L^\infty(\Omega)}&\le\|e^{t\Delta}(n_0-\overline{n}_0)\|_{L^\infty(\Omega)}+\int_0^t\|e^{(t-s)\Delta}\nabla\cdot\big(n_\epsilon S(\cdot,n_\epsilon,c_\epsilon)\nabla c_\epsilon\big)(\cdot,s)\|_{L^\infty(\Omega)}ds\\&\quad+\int_0^t\|e^{(t-s)\Delta}u_\epsilon(\cdot,s)\cdot\nabla n_\epsilon(\cdot,s)\|_{L^\infty(\Omega)}ds\\&:=J_1+J_2+J_3,\qquad \text{for all}\quad t\in(0,\infty).\end{aligned} \tag{4.16}$$

To estimate J_1, J_2 and J_3, we first deduce from the asymptotics of the Neumann heat semigroup (see Lemma 1.3 in [35]) that

$$J_1\le k_1e^{-\lambda_1 t}\|n_0-\overline{n}_0\|_{L^\infty(\Omega)}\le C_1e^{-\lambda_1 t}\qquad \text{for all}\quad t\in(0,\infty)$$

due to $\int_\Omega(n_0-\overline{n}_0)dx=0$,

$$J_2\le k_4C_S\int_0^t\left(1+(t-s)^{-\frac12-\frac13}\right)e^{-\lambda_1(t-s)}\|n_\epsilon(\cdot,s)\cdot\nabla c_\epsilon(\cdot,s)\|_{L^3(\Omega)}ds\qquad \text{for all}\quad t\in(0,\infty)$$

and

$$\begin{aligned}J_3&=\int_0^t\|e^{(t-s)\Delta}\nabla\cdot\big((n_\epsilon-\overline{n}_0)u_\epsilon\big)(\cdot,s)\|_{L^\infty(\Omega)}ds\\&\le k_4\int_0^t\left(1+(t-s)^{-\frac12-\frac1q}\right)e^{-\lambda_1(t-s)}\|(n_\epsilon(\cdot,s)-\overline{n}_0)u_\epsilon(\cdot,s)\|_{L^q(\Omega)}ds\qquad \text{for all}\quad t\in(0,\infty)\end{aligned}$$

for any fixed $2<q<3$ with some positive constants k_1,k_4 and λ_1. Then since Corollaries 3.2 and 4.1 entail that

$$\begin{aligned}\|n_\epsilon(\cdot,s)\cdot\nabla c_\epsilon(\cdot,s)\|_{L^3(\Omega)}&\le\|n_\epsilon(\cdot,s)\|_{L^4(\Omega)}\|\nabla c_\epsilon(\cdot,s)\|_{L^{12}(\Omega)}\\&=\|n_\epsilon(\cdot,s)\|_{L^4(\Omega)}\|\nabla\big(c_\epsilon(\cdot,s)-\overline{n}_0\big)\|_{L^{12}(\Omega)}\le C_2e^{-\mu_1 s}\end{aligned}$$

for some $\mu_1>0$, we can further estimate J_2 as

$$J_2\le C_3\int_0^t\left(1+(t-s)^{-\frac56}\right)e^{-\lambda_1(t-s)}e^{-\mu_1 s}ds\le C_4e^{-\min\{\lambda_1,\mu_1\}t}$$

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for all $t \in (0, \infty)$. Next, we can take a similar procedure to further estimate J_3. Indeed, it follows from the interpolation, Lemmas 4.6, 3.5 and Corollary 3.2 that

$$
\begin{aligned}
&\|(n_\epsilon(\cdot,s)-\overline{n}_0)u_\epsilon(\cdot,s)\|_{L^q(\Omega)} \\
&\le \|(n_\epsilon(\cdot,s)-\overline{n}_0)u_\epsilon(\cdot,s)\|_{L^{\frac{3}{2}}(\Omega)}^{\frac{3-q}{q}} \|(n_\epsilon(\cdot,s)-\overline{n}_0)u_\epsilon(\cdot,s)\|_{L^3(\Omega)}^{\frac{2q-3}{q}} \\
&\le C_5\Big(\|n_\epsilon(\cdot,s)-\overline{n}_0\|_{L^2(\Omega)}\|u_\epsilon(\cdot,s)\|_{L^6(\Omega)}\Big)^{\frac{3-q}{q}}\Big(\|n_\epsilon(\cdot,s)-\overline{n}_0\|_{L^4(\Omega)}\|u_\epsilon(\cdot,s)\|_{L^{12}(\Omega)}\Big)^{\frac{2q-3}{q}} \\
&\le C_6 e^{-\mu_2 s}
\end{aligned}
$$

for some $\mu_2 > 0$, which entails that

$$
J_3 \le C_7 \int_0^t \Big(1+(t-s)^{-\frac{1}{2}-\frac{1}{q}}\Big)e^{-\lambda_1(t-s)}e^{-\mu_2 s} \le C_8 e^{-\min\{\lambda_1,\mu_2\}t}
$$

for all $t \in (0, \infty)$. Inserting J_1, J_2 and J_3 into (4.16), we can achieve (4.14).

In order to get (4.15), it is sufficient to show that

$$
\|A^\beta u_\epsilon(\cdot,t)\|_{L^2(\Omega)} \le Ce^{-\mu t} \qquad \text{for all} \quad t\in(0,\infty)
$$

with $\beta \in (\frac{1}{2}, 1)$ and $\mu > 0$ due to the embedding $D(A^\beta) \hookrightarrow L^\infty(\Omega)$. For this purpose, we first test equation $(1.9)_3$ by u_ϵ, integrate by parts over Ω and employ the solenoidality of u_ϵ, the Hölder inequality, Lemma 4.6 and Lemma 3.5 to obtain that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_\epsilon\|_{L^2(\Omega)}^2 + \|\nabla u_\epsilon\|_{L^2(\Omega)}^2 &= \int_\Omega n_\epsilon \nabla\phi\cdot u_\epsilon \\
&= \int_\Omega (n_\epsilon-\overline{n}_0)\nabla\phi\cdot u_\epsilon \\
&\le \|\nabla\phi\|_{L^\infty(\Omega)}\|n_\epsilon-\overline{n}_0\|_{L^2(\Omega)}\|u_\epsilon\|_{L^2(\Omega)} \le C_9 e^{-\mu_3 t}
\end{aligned}
$$

for all $t \in (0, \infty)$ with some $\mu_3 > 0$. Due to $u_\epsilon = 0$ on $\partial\Omega$, it then follows from the Poincaré inequality that

$$
\frac{d}{dt}\|u_\epsilon(\cdot,t)\|_{L^2(\Omega)}^2 + \frac{1}{C_{10}}\|u_\epsilon(\cdot,t)\|_{L^2(\Omega)}^2 \le 2C_9 e^{-\mu_3 t} \qquad \text{for all} \quad t\in(0,\infty).
$$

Thus, without loss of generality, we may assume $\mu_3 < \frac{1}{C_{10}}$ to deduce that

$$
\|u_\epsilon(\cdot,t)\|_{L^2(\Omega)}^2 \le e^{-\frac{t}{C_{10}}}\|u_0\|_{L^2(\Omega)}^2 + C_{11}e^{-\mu_3 t} \le C_{12}e^{-\mu_3 t} \qquad \text{for all} \quad t\in(0,\infty). \tag{4.17}
$$

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Following the proof of (3.13), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u_\epsilon\|^2_{L^2(\Omega)} + \|Au_\epsilon\|^2_{L^2(\Omega)}\\
&= -\int_\Omega \mathcal{P}(u_\epsilon\cdot\nabla u_\epsilon)\cdot Au_\epsilon + \int_\Omega \mathcal{P}(n_\epsilon\nabla\phi)\cdot Au_\epsilon\\
&= -\int_\Omega \mathcal{P}(u_\epsilon\cdot\nabla u_\epsilon)\cdot Au_\epsilon + \int_\Omega \mathcal{P}\big((n_\epsilon-\overline{n}_0)\nabla\phi\big)\cdot Au_\epsilon\\
&\le C_{13}\Big(\|u_\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla u_\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)} + \|u_\epsilon\|_{L^2(\Omega)}\Big)\Big(\|Au_\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla u_\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)} + \|\nabla u_\epsilon\|_{L^2(\Omega)}\Big)\|Au_\epsilon\|_{L^2(\Omega)}\\
&\quad + C_{13}\|n_\epsilon-\overline{n}_0\|_{L^2(\Omega)}\|Au_\epsilon\|_{L^2(\Omega)}\\
&\le \frac{1}{2}\|Au_\epsilon\|^2_{L^2(\Omega)} + C_{14}\|\nabla u_\epsilon\|^2_{L^2(\Omega)}\left(\left(\|u_\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla u_\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)} + \|u_\epsilon\|_{L^2(\Omega)}\right)^4\right.\\
&\quad \left. + \left(\|u_\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla u_\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)} + \|u_\epsilon\|_{L^2(\Omega)}\right)^2\right) + C_{14}\|n_\epsilon-\overline{n}_0\|^2_{L^2(\Omega)}
\end{aligned}$$

for all $t\in(0,\infty)$, which together with (4.17), Lemmas 4.6 and 3.4 yields that

$$\frac{d}{dt}\|\nabla u_\epsilon\|^2_{L^2(\Omega)} + \|Au_\epsilon\|^2_{L^2(\Omega)} \le C_{15}e^{-\mu_4 t} \qquad \text{for all} \quad t\in(0,\infty)$$

with some $\mu_4>0$. Since

$$\|\nabla u_\epsilon\|^2_{L^2(\Omega)} \le C_{16}\|Au_\epsilon\|_{L^2(\Omega)}\|u_\epsilon\|_{L^2(\Omega)} + C_{16}\|u_\epsilon\|^2_{L^2(\Omega)} \le \|Au_\epsilon\|^2_{L^2(\Omega)} + C_{17}\|u_\epsilon\|^2_{L^2(\Omega)}$$

for all $t\in(0,\infty)$, we have

$$\frac{d}{dt}\|\nabla u_\epsilon\|^2_{L^2(\Omega)} + \|\nabla u_\epsilon\|^2_{L^2(\Omega)} \le C_{15}e^{-\mu_4 t} + C_{17}\|u_\epsilon\|^2_{L^2(\Omega)} \qquad \text{for all} \quad t\in(0,\infty).$$

This together with (4.17) warrants that

$$\|\nabla u_\epsilon\|_{L^2(\Omega)} \le C_{18}e^{-\mu_5 t} \qquad \text{for all} \quad t\in(0,\infty) \tag{4.18}$$

with some $\mu_5>0$. Noticing that $\mathcal{P}\big(\overline{n}_0\nabla\phi\big)=0$, we can use the variation-of-constants representation

$$u_\epsilon(\cdot,t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\mathcal{P}\big(u_\epsilon\cdot\nabla u_\epsilon - (n_\epsilon-\overline{n}_0)\nabla\phi\big)(\cdot,s)ds$$

of u_ϵ to get

$$\begin{aligned}
\|A^\beta u_\epsilon\|_{L^2(\Omega)} &\le \|A^\beta e^{-tA}u_0\|_{L^2(\Omega)} + \int_0^t \|A^\beta e^{-(t-s)A}\mathcal{P}\big(u_\epsilon\cdot\nabla u_\epsilon - (n_\epsilon-\overline{n}_0)\nabla\phi\big)(\cdot,s)\|_{L^2(\Omega)}ds\\
&\le K_1 t^{-\beta}e^{-\mu_6 t}\|u_0\|_{L^2(\Omega)}\\
&\quad + K_1\int_0^t (t-s)^{-\beta}e^{-\mu_6(t-s)}\|\mathcal{P}\big(u_\epsilon\cdot\nabla u_\epsilon - (n_\epsilon-\overline{n}_0)\nabla\phi\big)(\cdot,s)\|_{L^2(\Omega)}ds
\end{aligned}$$

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for all $t \in (0, \infty)$ with some $K_1 > 0$ and $\mu_6 > 0$ by Lemma 2.3(i) in [40]. Since Lemma 3.9, (4.18) and Lemma 4.6 entail that

$$\begin{aligned}
&\|\mathcal{P}\big(u_\epsilon \cdot \nabla u_\epsilon - (n_\epsilon - \overline{n}_0)\nabla\phi\big)(\cdot, s)\|_{L^2(\Omega)}\\
&\le \|(u_\epsilon \cdot \nabla u_\epsilon)(\cdot, s)\|_{L^2(\Omega)} + \|(n_\epsilon - \overline{n}_0)\nabla\phi\big)(\cdot, s)\|_{L^2(\Omega)}\\
&\le C_{19}\|u_\epsilon\|_{L^\infty(\Omega)}\|\nabla u_\epsilon\|_{L^2(\Omega)} + C_{19}\|n_\epsilon - \overline{n}_0\|_{L^2(\Omega)}\|\nabla\phi\|_{L^\infty(\Omega)}\\
&\le C_{20}e^{-\mu_7 s} \qquad \text{for all} \quad t \in (0, \infty)
\end{aligned}$$

with some $\mu_7 > 0$, we have

$$\begin{aligned}
\|A^\beta u_\epsilon\|_{L^2(\Omega)} &\le K_1 t^{-\beta} e^{-\mu_6 t}\|u_0\|_{L^2(\Omega)} + K_1 C_{20}\int_0^t (t-s)^{-\beta} e^{-\mu_6(t-s)} e^{-\mu_7 s} ds\\
&\le C_{21}e^{-\mu_6 t} + C_{22}e^{-\mu_7 t}\\
&\le C_{23}e^{-\mu_8 t} \qquad \text{for all} \quad t \in (1, \infty)
\end{aligned} \tag{4.19}$$

with some $\mu_8 > 0$. On the other hand, it is clear that

$$\|A^\beta u_\epsilon\|_{L^2(\Omega)} \le \|A^\beta u_0\|_{L^2(\Omega)} + C_{22}e^{-\mu_7 t} \le C_{24}e^{-\mu_8 t} \qquad \text{for all} \quad t \in (0, 1],$$

which together with (4.19) implies (4.15). This completes the proof of Lemma 4.8. □

We now apply the exponential decay estimate of $(n_\epsilon, c_\epsilon, u_\epsilon)$ to establish an at most $\frac{1}{2}$-order growth on time t in the convergence of the PP-fluid system (1.9) to the PE-fluid system (1.10).

Lemma 4.9. *Under the assumptions of Lemma 4.6, there exists $C > 0$ such that for each suitable small $\epsilon \in (0, 1)$,*

$$\|\widehat{n}(\cdot, t)\|_{L^2(\Omega)} + \|\widehat{u}(\cdot, t)\|_{L^2(\Omega)} \le C(1+t)^{\frac{1}{2}}\epsilon^{\frac{1}{2}} \qquad \text{for all} \quad t \in (0, \infty)$$

and

$$\|\widehat{n}\|_{L^2((0,t);\, W^{1,2}(\Omega))} + \|\widehat{c}\|_{L^2((0,t);\, W^{1,2}(\Omega))} + \|\widehat{u}\|_{L^2((0,t);\, W^{1,2}(\Omega))} \le C(1+t)^{\frac{1}{2}}\epsilon^{\frac{1}{2}} \quad \text{for all} \quad t \in (0, \infty).$$

Proof. We will deduce our desired result by analysing an entropy-like evolution estimate involving $\widehat{n}$, $\widehat{u}$ and c_ϵ of the form

$$K\|\widehat{n}(\cdot, t)\|^2_{L^2(\Omega)} + \epsilon\|c_\epsilon(\cdot, t)\|^2_{L^2(\Omega)} + \|\widehat{u}(\cdot, t)\|^2_{L^2(\Omega)}$$

with some K to be determined. For this purpose, testing equation $(4.1)_1$ by $\widehat{n}$ and integrating by parts over Ω, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\widehat{n}\|^2_{L^2(\Omega)} + \|\nabla\widehat{n}\|^2_{L^2(\Omega)}\\
&= \int_\Omega \Big((n - \overline{n}_0)\widehat{u} + \widehat{n}S(x, n_\epsilon, c_\epsilon)\nabla(c_\epsilon - \overline{n}_0) + (n - \overline{n}_0)S(x, n_\epsilon, c_\epsilon)\nabla\widehat{c} + \overline{n}_0 S(x, n_\epsilon, c_\epsilon)\nabla\widehat{c}\Big) \cdot \nabla\widehat{n}\\
&\quad + \int_\Omega n\big(S(x, n_\epsilon, c_\epsilon) - S(x, n, c)\big) \cdot \nabla(c - \overline{n}_0) \cdot \nabla\widehat{n}
\end{aligned}$$

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$$\leq \frac{1}{4}\|\nabla\widehat{n}\|_{L^2(\Omega)}^2 + C_1\|n-\overline{n}_0\|_{L^\infty(\Omega)}^2\|\widehat{u}\|_{L^2(\Omega)}^2 + C_1\|\nabla(c_\epsilon-\overline{n}_0)\|_{L^4(\Omega)}^2\|\widehat{n}\|_{L^4(\Omega)}^2$$
$$+ C_1\|n-\overline{n}_0\|_{L^\infty(\Omega)}^2\|\nabla\widehat{c}\|_{L^2(\Omega)}^2 + C_1\overline{n}_0^2\|\nabla\widehat{c}\|_{L^2(\Omega)}^2$$
$$+ C_1\int_\Omega n^2\left|S(x,n_\epsilon,c_\epsilon)-S(x,n,c)\right|^2|\nabla(c-\overline{n}_0)|^2.$$

Noticing that the Gagliardo–Nirenberg inequality entails that

$$C_1\|\nabla(c_\epsilon-\overline{n}_0)\|_{L^4(\Omega)}^2\|\widehat{n}\|_{L^4(\Omega)}^2$$
$$\leq C_2\|\nabla(c_\epsilon-\overline{n}_0)\|_{L^4(\Omega)}^2\left(\|\nabla\widehat{n}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\widehat{n}\|_{L^2(\Omega)}^{\frac{1}{2}} + \|\widehat{n}\|_{L^2(\Omega)}\right)^2$$
$$\leq \frac{1}{8}\|\nabla\widehat{n}\|_{L^2(\Omega)}^2 + C_3\left(\|\nabla(c_\epsilon-\overline{n}_0)\|_{L^4(\Omega)}^4 + \|\nabla(c_\epsilon-\overline{n}_0)\|_{L^4(\Omega)}^2\right)\|\widehat{n}\|_{L^2(\Omega)}^2$$

and

$$C_1\int_\Omega n^2\left|S(x,n_\epsilon,c_\epsilon)-S(x,n,c)\right|^2|\nabla(c-\overline{n}_0)|^2$$
$$\leq 2C_1\int_\Omega n^2\left(\left|S(x,n_\epsilon,c_\epsilon)-S(x,n,c_\epsilon)\right|^2 + \left|S(x,n,c_\epsilon)-S(x,n,c)\right|^2\right)|\nabla(c-\overline{n}_0)|^2$$
$$\leq 2C_1\int_\Omega n^2\left(\left|\nabla S(x,\xi,c_\epsilon)\right|^2|\widehat{n}|^2 + \left|\nabla S(x,n,\eta)\right|^2|\widehat{c}|^2\right)|\nabla(c-\overline{n}_0)|^2$$
$$\leq C_4\|\nabla(c-\overline{n}_0)\|_{L^4(\Omega)}^2\left(\|\widehat{n}\|_{L^4(\Omega)}^2 + \|\widehat{c}\|_{L^4(\Omega)}^2\right)$$
$$\leq \frac{1}{8}\|\nabla\widehat{n}\|_{L^2(\Omega)}^2 + \gamma\|\nabla\widehat{c}\|_{L^2(\Omega)}^2 + C_5\left(\|\nabla(c-\overline{n}_0)\|_{L^4(\Omega)}^4 + \|\nabla(c-\overline{n}_0)\|_{L^4(\Omega)}^2\right)\left(\|\widehat{n}\|_{L^2(\Omega)}^2 + \|\widehat{c}\|_{L^2(\Omega)}^2\right),$$

where ξ lies between n_ϵ and n, and η lies between c_ϵ and c, respectively, while $\gamma > 0$ is to be determined, we can use Corollary 4.1 and Lemma 4.8 to deduce that

$$\frac{d}{dt}\|\widehat{n}\|_{L^2(\Omega)}^2 + \|\nabla\widehat{n}\|_{L^2(\Omega)}^2$$
$$\leq 2\gamma\|\nabla\widehat{c}\|_{L^2(\Omega)}^2 + C_6\left(e^{-\mu_1 t}+\overline{n}_0^2\right)\|\nabla\widehat{c}\|_{L^2(\Omega)}^2 + C_6e^{-\mu_1 t}\|\widehat{c}\|_{L^2(\Omega)}^2 + C_6e^{-\mu_1 t}\left(\|\widehat{n}\|_{L^2(\Omega)}^2 + \|\widehat{u}\|_{L^2(\Omega)}^2\right) \tag{4.20}$$

for all $t \in (0,\infty)$ with some $\mu_1 > 0$.

We next test equation $(4.1)_2$ by $\widehat{c}$ and use the integration by parts over Ω to obtain

$$\frac{\epsilon}{2}\frac{d}{dt}\|c_\epsilon\|_{L^2(\Omega)}^2 + \|\nabla\widehat{c}\|_{L^2(\Omega)}^2 + \|\widehat{c}\|_{L^2(\Omega)}^2$$
$$= \epsilon\int_\Omega \partial_t c_\epsilon c + \int_\Omega\left(\widehat{n} - \widehat{u}\cdot\nabla(c-\overline{n}_0)\right)\widehat{c}$$
$$\leq \epsilon\int_\Omega \partial_t c_\epsilon c + \frac{1}{2}\|\widehat{c}\|_{L^2(\Omega)}^2 + \|\widehat{n}\|_{L^2(\Omega)}^2 + \|\nabla(c-\overline{n}_0)\|_{L^4(\Omega)}^2\|\widehat{u}\|_{L^4(\Omega)}^2$$
$$\leq \epsilon\int_\Omega \partial_t c_\epsilon c + \frac{1}{2}\|\widehat{c}\|_{L^2(\Omega)}^2 + \|\widehat{n}\|_{L^2(\Omega)}^2 + \frac{1}{4}\|\nabla\widehat{u}\|_{L^2(\Omega)}^2 + C_7\Big(\|\nabla(c-\overline{n}_0)\|_{L^4(\Omega)}^4$$
$$+ \|\nabla(c-\overline{n}_0)\|_{L^4(\Omega)}^2\Big)\|\widehat{u}\|_{L^2(\Omega)}^2.$$

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Due to $\int_\Omega \widehat{n}=0$, the Poincaré inequality and Corollary 4.1 imply that

$$\begin{aligned}&\epsilon\frac{d}{dt}\|c_\epsilon\|^2_{L^2(\Omega)}+2\|\nabla\widehat{c}\|^2_{L^2(\Omega)}+\|\widehat{c}\|^2_{L^2(\Omega)}\\&\le 2\epsilon\int_\Omega \partial_t c_\epsilon c+\frac{1}{2}\|\nabla\widehat{u}\|^2_{L^2(\Omega)}+C_8\|\nabla\widehat{n}\|^2_{L^2(\Omega)}+C_8e^{-\mu_2 t}\|\widehat{u}\|^2_{L^2(\Omega)}\end{aligned}\tag{4.21}$$

for all $t\in(0,\infty)$ with some $\mu_2>0$.

Finally, multiplying equation $(4.1)_3$ by $\widehat{u}$ and using the Poincaré inequality again since $\widehat{u}=0$ on $\partial\Omega$ and $\int_\Omega \widehat{n}=0$, we see that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\widehat{u}\|^2_{L^2(\Omega)}+\|\nabla\widehat{u}\|^2_{L^2(\Omega)}&=\int_\Omega \widehat{n}\nabla\phi\cdot\widehat{u}\le\|\nabla\phi\|_{L^\infty(\Omega)}\|\widehat{u}\|_{L^2(\Omega)}\|\widehat{n}\|_{L^2(\Omega)}\\&\le C_9\|\nabla\widehat{u}\|_{L^2(\Omega)}\|\nabla\widehat{n}\|_{L^2(\Omega)}\le\frac{1}{2}\|\nabla\widehat{u}\|^2_{L^2(\Omega)}+C_{10}\|\nabla\widehat{n}\|^2_{L^2(\Omega)}\end{aligned}$$

and thus that

$$\frac{d}{dt}\|\widehat{u}\|^2_{L^2(\Omega)}+\|\nabla\widehat{u}\|^2_{L^2(\Omega)}\le C_{11}\|\nabla\widehat{n}\|^2_{L^2(\Omega)}\qquad\text{for all}\quad t\in(0,\infty).\tag{4.22}$$

We now take T_1 large enough such that

$$C_6e^{-\mu_1 t}<\frac{1}{6(C_8+C_{11})}\qquad\text{for all}\quad t\in(T_1,\infty)$$

and δ suitable small such that

$$C_6\overline{n}_0^2<\frac{1}{6(C_8+C_{11})}.$$

Then by setting

$$\gamma=\frac{1}{12(C_8+C_{11})}\qquad\text{and}\qquad K=2(C_8+C_{11}),$$

we deduce from (4.20)–(4.22) that

$$\begin{aligned}&\frac{d}{dt}\Big(K\|\widehat{n}\|^2_{L^2(\Omega)}+\epsilon\|c_\epsilon\|^2_{L^2(\Omega)}+\|\widehat{u}\|^2_{L^2(\Omega)}\Big)+\frac{K}{2}\|\nabla\widehat{n}\|^2_{L^2(\Omega)}+\|\nabla\widehat{c}\|^2_{L^2(\Omega)}+\frac{1}{2}\|\nabla\widehat{u}\|^2_{L^2(\Omega)}+\frac{2}{3}\|\widehat{c}\|^2_{L^2(\Omega)}\\&\le 2\epsilon\int_\Omega \partial_t c_\epsilon c+C_{12}e^{-\mu t}\Big(\|\widehat{n}\|^2_{L^2(\Omega)}+\|\widehat{u}\|^2_{L^2(\Omega)}\Big)\qquad\text{for all}\quad t\in(T_1,\infty)\end{aligned}$$

with $\mu=\min\{\mu_1,\mu_2\}$, which together with the Poincaré inequality yields that

$$\begin{aligned}&\frac{d}{dt}\Big(K\|\widehat{n}\|^2_{L^2(\Omega)}+\epsilon\|c_\epsilon\|^2_{L^2(\Omega)}+\|\widehat{u}\|^2_{L^2(\Omega)}\Big)+C_{13}\Big(K\|\widehat{n}\|^2_{L^2(\Omega)}+\epsilon\|c_\epsilon\|^2_{L^2(\Omega)}+\|\widehat{u}\|^2_{L^2(\Omega)}\Big)\\&\quad+\frac{1}{4}\Big(K\|\nabla\widehat{n}\|^2_{L^2(\Omega)}+\|\nabla\widehat{c}\|^2_{L^2(\Omega)}+\|\nabla\widehat{u}\|^2_{L^2(\Omega)}+\|\widehat{c}\|^2_{L^2(\Omega)}\Big)\\&\le C_{14}e^{-\mu t}\Big(K\|\widehat{n}\|^2_{L^2(\Omega)}+\epsilon\|c_\epsilon\|^2_{L^2(\Omega)}+\|\widehat{u}\|^2_{L^2(\Omega)}\Big)+C_{13}\epsilon\|c_\epsilon\|^2_{L^2(\Omega)}+2\epsilon\int_\Omega \partial_t c_\epsilon c.\end{aligned}$$

For simplicity, we set

$$y(t):=K\|\widehat{n}\|^2_{L^2(\Omega)}+\epsilon\|c_\epsilon\|^2_{L^2(\Omega)}+\|\widehat{u}\|^2_{L^2(\Omega)}$$

and

$$g(t) := \frac{1}{4}\left(K\|\nabla\widehat{n}\|^2_{L^2(\Omega)} + \|\nabla\widehat{c}\|^2_{L^2(\Omega)} + \|\nabla\widehat{u}\|^2_{L^2(\Omega)} + \|\widehat{c}\|^2_{L^2(\Omega)}\right)$$

as well as

$$h(t) := C_{13}\|c_\epsilon\|^2_{L^2(\Omega)} + 2\int_\Omega \partial_t c_\epsilon c,$$

we have

$$y'(t) + C_{13}y(t) + g(t) \le C_{14}e^{-\mu t}y(t) + \epsilon h(t) \qquad \text{for all} \quad t \in (T_1, \infty).$$

If we further take T_2 large enough such that $2C_{14}e^{-\mu t} < C_{13}$ for all $t > T_2$, we can conclude that

$$y'(t) + \frac{C_{13}}{2}y(t) + g(t) \le \epsilon h(t) \qquad \text{for all} \quad t \in (T_0, \infty)$$

with $T_0 := \max\left\{T_1, T_2\right\}$. It then follows from Lemmas 4.1, 4.2 and the uniform boundedness of c_ϵ that

$$\begin{aligned} y(t) &\le y(T_0) + \epsilon\int_{T_0}^t h(s)ds = y(T_0) + \epsilon\left(C_{13}\int_{T_0}^t \|c_\epsilon\|^2_{L^2(\Omega)}ds + 2\int_{T_0}^t\int_\Omega \partial_t c_\epsilon c ds\right) \\ &\le \left((K+1)C_{15}e^{C_{15}T_0}\epsilon + \epsilon\|c_\epsilon(\cdot, T_0)\|^2_{L^2(\Omega)}\right) + \epsilon C_{15}(1+t) := C_{16}(1+t)\epsilon \end{aligned}$$

for all $t \in (T_0, \infty)$, which also implies that

$$\int_{T_0}^t g(s)ds \le y(T_0) + \epsilon\int_{T_0}^t h(s)ds \le C_{16}(1+t)\epsilon \qquad \text{for all} \quad t \in (T_0, \infty).$$

This together with the Poincaré inequality yields the desired results on (T_0, ∞), while on $(0, T_0)$, the conclusions follow from Lemma 4.2 directly. This completes the proof of Lemma 4.9. □

Applying the second estimate in Lemma 4.9 to the first inequality in (4.6), we can also obtain the growth estimate of $\nabla\widehat{u}$.

Corollary 4.2. *Under the assumptions of Lemma 4.6, there exists $C > 0$ such that for each suitable small $\epsilon \in (0, 1)$,*

$$\|\nabla\widehat{u}(\cdot, t)\|_{L^2(\Omega)} \le C(1+t)^{\frac{1}{2}}\epsilon^{\frac{1}{2}} \qquad \text{for all} \quad t \in (0, \infty).$$

Lemma 4.10. *Under the assumptions of Lemma 4.6, for any given $\theta \in \left(\frac{1}{2}, \frac{3}{4}\right)$, there exists a positive constant $C(\theta)$ such that for each suitable small $\epsilon \in (0, 1)$,*

$$\|A^\theta\widehat{u}(\cdot, t)\|_{L^2(\Omega)} \le C(\theta)(1+t)^{\frac{3}{4}}\epsilon^{\frac{1}{2}} \qquad \text{for all} \quad t \in (0, \infty).$$

Proof. Similar to the proof of Lemma 4.4, we can see from Lemma 4.9, Corollary 4.2 and (4.9) that

$$\begin{aligned}
\|A^\theta\widehat{u}(\cdot,t)\|_{L^2(\Omega)} &\le C_1\int_0^t (t-s)^{-\theta}e^{-\lambda(t-s)}\Big(\|\nabla\widehat{u}\|_{L^2(\Omega)}+\|\widehat{u}\|_{L^4(\Omega)}\|\nabla u\|_{L^4(\Omega)}+\|\widehat{n}\|_{L^2(\Omega)}\Big)ds\\
&\le C_2\epsilon^{\frac12}\int_0^t (1+s)^{\frac12}(t-s)^{-\theta}e^{-\lambda(t-s)}\big(1+\|\nabla u\|_{L^4(\Omega)}\big)ds\\
&\le C_3(1+t)^{\frac12}\epsilon^{\frac12}+C_3(1+t)^{\frac12}\epsilon^{\frac12}\left(\int_0^t\big((t-s)^{-\theta}e^{-\lambda(t-s)}\big)^{\frac43}ds\right)^{\frac34}\\
&\quad\times\left(\int_0^t\|\nabla u(\cdot,s)\|^4_{L^4(\Omega)}ds\right)^{\frac14}\\
&\le C_4(1+t)^{\frac12}\epsilon^{\frac12}+C_4(1+t)^{\frac12}\epsilon^{\frac12}\left(\int_0^t\|\nabla u(\cdot,s)\|^4_{L^4(\Omega)}ds\right)^{\frac14}\\
&\le C_5(1+t)^{\frac34}\epsilon^{\frac12}\qquad\text{for all}\quad t\in(0,\infty).
\end{aligned}$$

□

Lemma 4.11. *Under the assumptions of Lemma 4.6, for any given $p>2$, there exists a positive constant $C(p)$ such that for each suitable small $\epsilon\in(0,1)$,*

$$\|\widehat{n}(\cdot,t)\|_{L^p(\Omega)}\le C(p)(1+t)^{\frac12}\epsilon^{\frac{2}{p^2}}\qquad\text{for all}\quad t\in(0,\infty).$$

Proof. We first multiply equation $(4.1)_1$ by $\widehat{n}^{p-1}$ with $p>2$ and integrate by parts over Ω to obtain

$$\begin{aligned}
&\frac1p\frac{d}{dt}\|\widehat{n}^{\frac p2}\|^2_{L^2(\Omega)}+\frac{4(p-1)}{p^2}\|\nabla\widehat{n}^{\frac p2}\|^2_{L^2(\Omega)}\\
&=(p-1)\int_\Omega\Big((n-\overline{n}_0)\widehat{n}^{p-2}\widehat{u}+\widehat{n}^{p-1}S(x,n_\epsilon,c_\epsilon)\nabla(c_\epsilon-\overline{n}_0)+(n-\overline{n}_0)\widehat{n}^{p-2}S(x,n_\epsilon,c_\epsilon)\nabla\widehat{c}\\
&\qquad+\overline{n}_0\widehat{n}^{p-2}S(x,n_\epsilon,c_\epsilon)\nabla\widehat{c}\Big)\cdot\nabla\widehat{n}+(p-1)\int_\Omega n\widehat{n}^{p-2}\big(S(x,n_\epsilon,c_\epsilon)-S(x,n,c)\big)\nabla(c-\overline{n}_0)\cdot\nabla\widehat{n}\\
&\le\frac{p-1}{4}\int_\Omega\widehat{n}^{p-2}|\nabla\widehat{n}|^2+C_1\|n-\overline{n}_0\|^2_{L^\infty(\Omega)}\int_\Omega\widehat{n}^{p-2}|\widehat{u}|^2+C_1\int_\Omega\widehat{n}^p|\nabla(c_\epsilon-\overline{n}_0)|^2\\
&\qquad+C_1\|n-\overline{n}_0\|^2_{L^\infty(\Omega)}\int_\Omega\widehat{n}^{p-2}|\nabla\widehat{c}|^2+C_1\overline{n}_0\int_\Omega\widehat{n}^{p-2}|\nabla\widehat{c}|^2\\
&\qquad+C_1\|n\|^2_{L^\infty(\Omega)}\int_\Omega\widehat{n}^{p-2}\big(\widehat{n}^2+\widehat{c}^2\big)|\nabla(c-\overline{n}_0)|^2.
\end{aligned}$$

Noting that Lemmas 4.8, 3.9 and Corollary 4.1 entail that

$$\|n-\overline{n}_0\|^2_{L^\infty(\Omega)}\le C_2e^{-\mu t}\quad\text{and}\quad\|\nabla(c-\overline{n}_0)\|^2_{L^q(\Omega)}\le C_2e^{-\mu t}$$

for all $t\in(0,\infty)$ and any $q>1$ with some $\mu>0$, we can use the Hölder inequality, the boundedness of n_ϵ and n, Corollary 4.1 and the Young inequality to obtain that for any fixed $q_0>p$,

$$
\begin{aligned}
&\int_\Omega \widehat{n}^{p-2}\left(\widehat{n}^2+\widehat{c}^2\right)|\nabla(c-\overline{n}_0)|^2\\
&\le \left(\|\widehat{n}\|^2_{L^p(\Omega)}+\|\widehat{c}\|^2_{L^p(\Omega)}\right)\|\widehat{n}\|^{p-2}_{L^{q_0}(\Omega)}\|\nabla(c-\overline{n}_0)\|^2_{L^{\frac{2pq_0}{(p-2)(q_0-p)}}(\Omega)}\\
&\le \left(\|\widehat{n}\|^2_{L^p(\Omega)}+\|\widehat{c}\|^2_{L^p(\Omega)}\right)\|\widehat{n}\|^{\frac{p(p-2)}{q_0}}_{L^p(\Omega)}\|n_\epsilon-n\|^{\frac{(p-2)(q_0-p)}{q_0}}_{L^\infty(\Omega)}\|\nabla(c-\overline{n}_0)\|^2_{L^{\frac{2pq_0}{(p-2)(q_0-p)}}(\Omega)}\\
&\le C_3e^{-\mu t}\left(\|\widehat{n}\|^2_{L^p(\Omega)}+\|\widehat{c}\|^2_{L^p(\Omega)}\right)\|\widehat{n}\|^{\frac{p(p-2)}{q_0}}_{L^p(\Omega)}\\
&\le C_4e^{-\mu t}\left(\|\widehat{n}\|^2_{L^p(\Omega)}+\|\widehat{c}\|^2_{L^p(\Omega)}\right)\left(\|\widehat{n}\|^{p-2}_{L^p(\Omega)}+1\right)\\
&\le C_5e^{-\mu t}\left(\|\widehat{n}\|^p_{L^p(\Omega)}+\|\widehat{c}\|^p_{L^p(\Omega)}\right)+C_5e^{-\mu t}\left(\|\widehat{n}\|^2_{L^p(\Omega)}+\|\widehat{c}\|^2_{L^p(\Omega)}\right).
\end{aligned}
$$

Similarly, we have

$$
\int_\Omega \widehat{n}^p|\nabla(c_\epsilon-\overline{n}_0)|^2\le C_6e^{-\mu t}\|\widehat{n}\|^p_{L^p(\Omega)}+C_6e^{-\mu t}\|\widehat{n}\|^2_{L^p(\Omega)}
$$

and thus deduce from the Young inequality that

$$
\begin{aligned}
&\frac{d}{dt}\|\widehat{n}^{\frac{p}{2}}\|^2_{L^2(\Omega)}+\frac{3(p-1)}{p}\|\nabla\widehat{n}^{\frac{p}{2}}\|^2_{L^2(\Omega)}\\
&\le C_7e^{-\mu t}\Big(\|\widehat{n}\|^p_{L^p(\Omega)}+\|\widehat{u}\|^p_{L^p(\Omega)}\Big)+C_7\overline{n}_0\|\widehat{n}\|^p_{L^p(\Omega)}+C_7\big(e^{-\mu t}+\overline{n}_0\big)\|\widehat{c}\|^p_{W^{1,p}(\Omega)}\\
&\qquad +C_7e^{-\mu t}\|\widehat{n}\|^2_{L^p(\Omega)}+C_7e^{-\mu t}\|\widehat{c}\|^2_{L^p(\Omega)}.
\end{aligned}
$$

Without loss of generality, we assume $C_7>1$ and take suitable small δ such that $\overline{n}_0<1$. Then since

$$
\begin{aligned}
\|\widehat{n}\|^p_{L^p(\Omega)}=\|\widehat{n}^{\frac{p}{2}}\|^2_{L^2(\Omega)}&\le C_8\left(\|\nabla\widehat{n}^{\frac{p}{2}}\|^{\frac{2(p-2)}{p}}_{L^2(\Omega)}\|\widehat{n}^{\frac{p}{2}}\|^{\frac{4}{p}}_{L^{\frac{4}{p}}(\Omega)}+\|\widehat{n}^{\frac{p}{2}}\|^2_{L^{\frac{4}{p}}(\Omega)}\right)\\
&\le\frac{p-1}{C_7p}\|\nabla\widehat{n}^{\frac{p}{2}}\|^2_{L^2(\Omega)}+C_9\|\widehat{n}^{\frac{p}{2}}\|^2_{L^{\frac{4}{p}}(\Omega)}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\widehat{n}\|^2_{L^p(\Omega)}=\|\widehat{n}^{\frac{p}{2}}\|^{\frac{4}{p}}_{L^2(\Omega)}&\le C_{10}\left(\|\nabla\widehat{n}^{\frac{p}{2}}\|^{\frac{p-2}{p}}_{L^2(\Omega)}\|\widehat{n}^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^{\frac{4}{p}}(\Omega)}+\|\widehat{n}^{\frac{p}{2}}\|_{L^{\frac{4}{p}}(\Omega)}\right)^{\frac{4}{p}}\\
&\le\frac{p-1}{2C_7p}\|\nabla\widehat{n}^{\frac{p}{2}}\|^2_{L^2(\Omega)}+C_{11}\|\widehat{n}^{\frac{p}{2}}\|^{\frac{8}{p^2-2p+4}}_{L^{\frac{4}{p}}(\Omega)}+C_{11}\|\widehat{n}^{\frac{p}{2}}\|^{\frac{4}{p}}_{L^{\frac{4}{p}}(\Omega)}
\end{aligned}
$$

we can use the fact

$$
\|\widehat{u}\|^p_{L^p(\Omega)}\le C_{12}\Big(\|\widehat{u}\|^2_{L^2(\Omega)}\|\nabla\widehat{u}\|^{p-2}_{L^2(\Omega)}+\|\widehat{u}\|^p_{L^2(\Omega)}\Big),
$$

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Lemma 4.9 and Corollary 4.2 to obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\widehat{n}^{\frac{p}{2}}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\widehat{n}^{\frac{p}{2}}\|^2_{L^2(\Omega)} \\
&\leq \frac{d}{dt}\|\widehat{n}^{\frac{p}{2}}\|^2_{L^2(\Omega)} + \frac{p-1}{2p}\|\nabla\widehat{n}^{\frac{p}{2}}\|^2_{L^2(\Omega)} + \frac{C_9}{2}\|\widehat{n}^{\frac{p}{2}}\|^2_{L^{\frac{4}{p}}(\Omega)} \\
&\leq \left(C_7C_9e^{-\mu t} + C_7C_9 + \frac{C_9}{2}\right)\|\widehat{n}^{\frac{p}{2}}\|^2_{L^{\frac{4}{p}}(\Omega)} + C_7e^{-\mu t}\|\widehat{u}\|^p_{L^p(\Omega)} + C_7\left(e^{-\mu t}+1\right)\|\widehat{c}\|^p_{W^{1,p}(\Omega)} \\
&\quad + C_7C_{11}e^{-\mu t}\|\widehat{n}^{\frac{p}{2}}\|^{\frac{8}{p^2-2p+4}}_{L^{\frac{4}{p}}(\Omega)} + C_7C_{11}e^{-\mu t}\|\widehat{n}^{\frac{p}{2}}\|^{\frac{4}{p}}_{L^{\frac{4}{p}}(\Omega)} + C_7e^{-\mu t}\|\widehat{c}\|^2_{L^p(\Omega)} \\
&\leq C_{13}(1+t)^{\frac{p}{2}}\epsilon^{\frac{p}{2}} + C_{13}(1+t)^{\frac{2p}{p^2-2p+4}}\epsilon^{\frac{2p}{p^2-2p+4}} + C_{13}(1+t)\epsilon \\
&\qquad + C_7\left(e^{-\mu t}+1\right)\|\widehat{c}\|^p_{W^{1,p}(\Omega)} + C_7e^{-\mu t}\|\widehat{c}\|^2_{L^p(\Omega)} \\
&\leq 3C_{13}(1+t)^{\frac{p}{2}}\epsilon^{\frac{2p}{p^2-2p+4}} + C_7\left(e^{-\mu t}+1\right)\|\widehat{c}\|^p_{W^{1,p}(\Omega)} + C_7e^{-\mu t}\|\widehat{c}\|^2_{L^p(\Omega)} \qquad \text{for all} \quad t\in(0,\infty)
\end{aligned}
$$

due to $p > 2$ and $\epsilon \in (0, 1)$. By setting

$$
y(t) := \|\widehat{n}(\cdot,t)\|^p_{L^p(\Omega)} = \|\widehat{n}^{\frac{p}{2}}(\cdot,t)\|^2_{L^2(\Omega)},
$$

we have

$$
y'(t) + \frac{1}{2}y(t) \leq C_{14}(1+t)^{\frac{p}{2}}\epsilon^{\frac{2p}{p^2-2p+4}} + C_{14}\|\widehat{c}\|^p_{W^{1,p}(\Omega)} + C_{14}\|\widehat{c}\|^2_{L^p(\Omega)} \qquad \text{for all} \quad t\in(0,\infty).
$$

Thus, a direct calculation implies that

$$
y(t) \leq C_{14}\epsilon^{\frac{2p}{p^2-2p+4}}\int_0^t (1+s)^{\frac{p}{2}}e^{-\frac{(t-s)}{2}}ds + C_{14}\int_0^t e^{-\frac{(t-s)}{2}}\left(\|\widehat{c}(\cdot,s)\|^p_{W^{1,p}(\Omega)} + \|\widehat{c}(\cdot,s)\|^2_{L^p(\Omega)}\right)ds \tag{4.23}
$$

for all $t \in (0,\infty)$ due to $y(0) = 0$. For the first integral on the right-hand side of (4.23), we have

$$
\int_0^t (1+s)^{\frac{p}{2}}e^{-\frac{(t-s)}{2}}ds \leq (1+t)^{\frac{p}{2}}\int_0^t e^{-\frac{(t-s)}{2}}ds \leq (1+t)^{\frac{p}{2}},
$$

while for the second one, we can make use of the interpolation inequality, Lemmas 3.8, 4.9 and the Hölder inequality to deduce that

$$
\begin{aligned}
\int_0^t e^{-\frac{(t-s)}{2}}\|\widehat{c}(\cdot,s)\|^p_{W^{1,p}(\Omega)}ds &\leq C_{15}\int_0^t e^{-\frac{(t-s)}{2}}\|\widehat{c}(\cdot,s)\|^{\frac{2(q-p)}{q-2}}_{W^{1,2}(\Omega)}\|\widehat{c}(\cdot,s)\|^{\frac{q(p-2)}{q-2}}_{W^{1,q}(\Omega)}ds \\
&\leq C_{16}\int_0^t e^{-\frac{(t-s)}{2}}\|\widehat{c}(\cdot,s)\|^{\frac{2(q-p)}{q-2}}_{W^{1,2}(\Omega)}ds \\
&\leq C_{17}\left(\int_0^t \|\widehat{c}(\cdot,s)\|^2_{W^{1,2}(\Omega)}ds\right)^{\frac{q-p}{q-2}}\left(\int_0^t e^{-\frac{q-2}{p-2}\frac{(t-s)}{2}}ds\right)^{\frac{p-2}{q-2}} \\
&\leq C_{18}(1+t)^{\frac{q-p}{q-2}}\epsilon^{\frac{q-p}{q-2}} \qquad \text{for all} \quad t\in(0,\infty)
\end{aligned}
$$

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with $q > p+2 > 4$. Similarly, for the third integral, we also have

$$\int_0^t e^{-\frac{(t-s)}{2}} \|\widehat{c}(\cdot,s)\|^2_{L^p(\Omega)} ds \le C_{19} \int_0^t e^{-\frac{(t-s)}{2}} \|\widehat{c}(\cdot,s)\|^{\frac{4}{p}}_{L^2(\Omega)} \|\widehat{c}(\cdot,s)\|^{\frac{2(p-2)}{p}}_{L^\infty(\Omega)} ds$$
$$\le C_{20} \left(\int_0^t \|\widehat{c}(\cdot,s)\|^2_{L^2(\Omega)} ds\right)^{\frac{2}{p}} \left(\int_0^t e^{-\frac{p}{p-2}\frac{(t-s)}{2}} ds\right)^{\frac{p-2}{p}}$$
$$\le C_{21}(1+t)^{\frac{2}{p}} \epsilon^{\frac{2}{p}} \qquad \text{for all} \quad t \in (0,\infty).$$

These three estimates together with (4.23) yield that

$$y(t) \le C_{22}(1+t)^{\frac{p}{2}} \epsilon^{\frac{2p}{p^2-2p+4}} + C_{22}(1+t)^{\frac{q-p}{q-2}} \epsilon^{\frac{q-p}{q-2}} + C_{22}(1+t)^{\frac{2}{p}} \epsilon^{\frac{2}{p}} \le C_{23}(1+t)^{\frac{p}{2}} \epsilon^{\frac{2}{p}}$$

for all $t \in (0,\infty)$. This completes the proof of Lemma 4.11. □

Proof of Theorem 1.2. A direct application of Corollary 4.1 and Lemmas 4.8–4.11 implies the desired conclusion. □

5. Numerical experiments

In this section, we carry out the numerical simulations to look into the convergence behaviour depending on ϵ and t. Set $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$. We solve the PP-fluid system (1.9) and the PE-fluid system (1.10) with the chemotactic sensitivity

$$S(x,n,c) = \frac{1}{(1+n)^\alpha} \begin{bmatrix} a_1 & b_1 \\ b_2 & a_2 \end{bmatrix}.$$

We will compute and plot the following norms to observe the experimental convergence rates:

$$\text{nL2} = \|\widehat{n}(t)\|_{L^2}/\|n(t)\|_{L^2}, \qquad \text{nL4} = \|\widehat{n}(t)\|_{L^4}/\|n(t)\|_{L^4}, \qquad \text{nH1} = \|\widehat{n}(t)\|_{H^1}/\|n(t)\|_{H^1},$$
$$\text{cH1} = \|\widehat{c}(t)\|_{H^1}/\|c(t)\|_{H^1}, \qquad \text{uLInf} = \|\widehat{u}(t)\|_{L^\infty}/\|u(t)\|_{L^\infty}, \qquad \text{uH1} = \|\widehat{u}(t)\|_{H^1}/\|u(t)\|_{H^1}.$$

In all numerical examples, we take the initial velocity $u_0 = (0,0)$ and assign the initial concentration c_0 as the solution of

$$-\Delta c_0 + c_0 = n_0 \quad \text{in } \Omega, \qquad \nabla c_0 \cdot \nu = 0 \quad \text{on } \partial\Omega.$$

We select three initial cell densities n_0 (presented below), which result in three different types of solution.

Example 1. We set $\alpha = \frac{1}{4}$, $a_1 = a_2 = b_1 = b_2 = \frac{1}{2}$, $\nabla\phi = (1,-1)$. The initial density n_0 is given by:

$$n_0 = 30\left(e^{-5\left|x-\left(\frac{1}{2},-\frac{1}{2}\right)\right|^2} + e^{-5\left|x-\left(-\frac{1}{2},-\frac{1}{2}\right)\right|^2}\right) + 20\left(e^{-5\left|x-\left(\frac{1}{2},\frac{1}{2}\right)\right|^2} + e^{-5\left|x-\left(-\frac{1}{2},\frac{1}{2}\right)\right|^2}\right).$$

The evolution of the PE-fluid system and PP-fluid system with $\epsilon = 0.5$ is shown in Figures 1 and 2, respectively, where the colour bars illustrate the magnitude of $(n, c, |u|)$ and $(n_\epsilon, c_\epsilon, |u_\epsilon|)$, and the arrows in the figures of u and u_ϵ represent the direction of velocity. Both two solutions

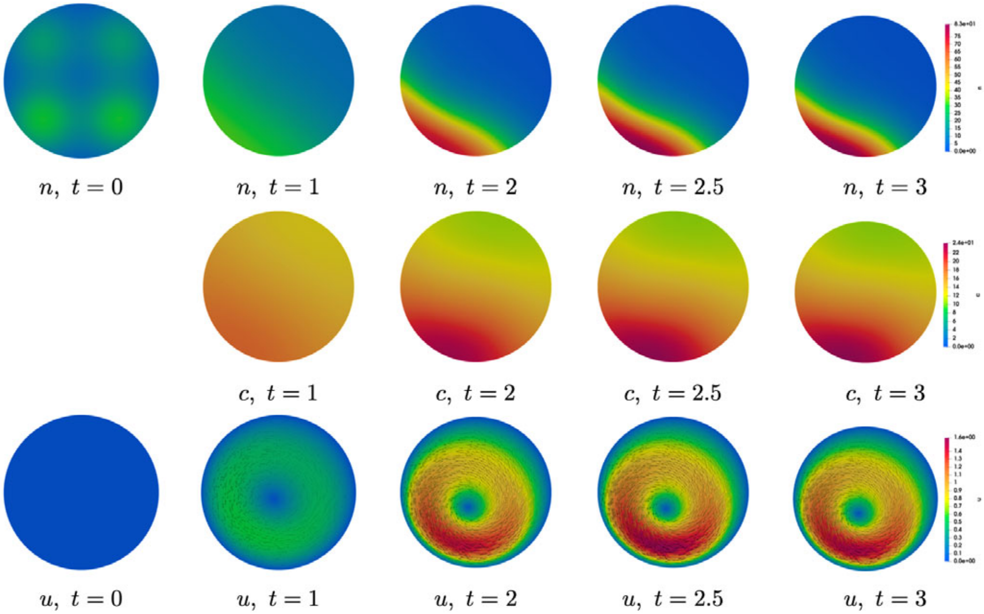

FIGURE 1. Example 1. (n, c, u) of the PE-fluid system.

tend to the nontrivial equilibrium with the cells and chemical concentrating at the lower left quarter, whereas the PE-fluid system approaches the equilibrium quicker, and the motion of the PP-fluid system seems to be a delay of the PE-fluid system. In the following, we investigate the convergence of $(\widehat{n}, \widehat{c}, \widehat{u})$ depending on ϵ and t, respectively.

Now let us fix $t = 2$ and compute the norms of $(\widehat{n}(t), \widehat{c}(t), \widehat{u}(t))$ on different ϵ. The results are plotted in log-scale in Figure 3 (Ex1-ϵ). All the errors decrease in the same rate to the solid straight line $y = \epsilon$, which means that the experiment enjoys the $O(\epsilon)$-convergence. It is better than the theoretical results (Theorems 1.1 and 1.2), but not a surprise because any sufficiently smooth solution (in particular, the norms of $\partial_t c_\epsilon$ are bounded independent of the reciprocal of ϵ) will result the optimal convergence.

Next, we fix $\epsilon = 2^{-5}$ and compute the errors for various t (see Figure 4 (Ex1-t)). Be aware of the PE-fluid system and PP-fluid system tend to the same nontrivial equilibrium, the deviation $(\widehat{n}, \widehat{c}, \widehat{u})$ will extinguish. Figure 4 (Ex1-t) exhibits the decreasing of the error over t. However, it is saturated near 10^{-6} due to the numerical approximation.

Example 2. We replace the initial value n_0 of the previous example by a symmetry function:

$$n_0 = 30\left(e^{-5\left|x-\left(\frac{1}{2},-\frac{1}{2}\right)\right|^2} + e^{-5\left|x-\left(-\frac{1}{2},\frac{1}{2}\right)\right|^2}\right) + 20\left(e^{-5\left|x-\left(\frac{1}{2},\frac{1}{2}\right)\right|^2} + e^{-5\left|x-\left(-\frac{1}{2},-\frac{1}{2}\right)\right|^2}\right).$$

Both (n, c) and (n_ϵ, c_ϵ) rapidly tend to the constant equilibriums $(\overline{n}_0, \overline{n}_0)$ (see Figures 5 and 6). The difference between the PE-fluid system and PP-fluid system is tiny. In this example, the velocity u and u_ϵ vanish quickly. In fact, the amplitudes of u and u_ϵ are very small (see the colour bar of u and u_ϵ), and the pressures P and P_ϵ tend to the function $\overline{n}_0\phi = \overline{n}_0(x - y)$ almost instantly (note that $\nabla\phi = (1, -1)$ and $\nabla P = n\nabla\phi$ when $u = 0$).

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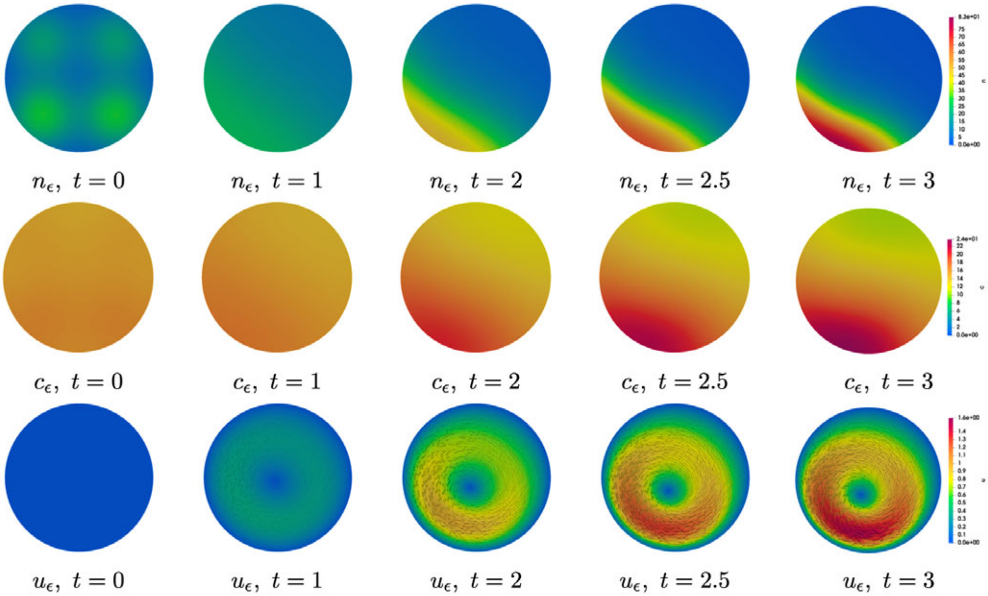

FIGURE 2. Example 1. $(n_\epsilon, c_\epsilon, u_\epsilon)$ of the PP-fluid system with $\epsilon = 0.5$.

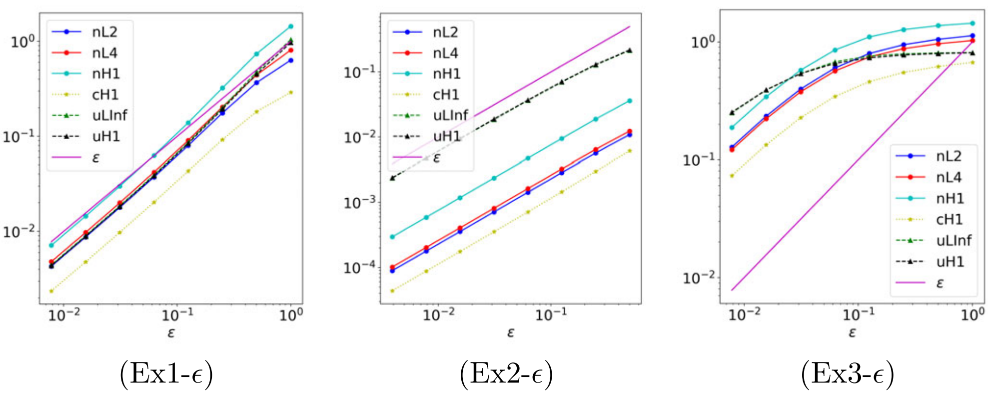

FIGURE 3. We fix $t = 2$ for Example 1, $t = 0.3$ for Example 2 and $t = 4$ for Example 3. The three figures, (Ex1-ϵ), (Ex2-ϵ) and (Ex3-ϵ), show the norms of $(\widehat{n}(t), \widehat{c}(t), \widehat{u}(t))$ depending on ϵ for the Example 1, 2 and 3, respectively. The errors plotted in log-scale decrease as with ϵ (the solid straight line), which indicates the convergence rate $O(\epsilon)$ for all three numerical examples.

Now we fix $t = 0.3$ and plot the error for various ϵ (see Figure 3 (Ex2-ϵ)), which reveals the $O(\epsilon)$-convergence of the fast signal diffusion limit. Next for fixed $\epsilon = 2^{-5}$, we compute and plot the error for different t in Figure 4 (Ex2-t). Since both the PE-fluid system and PP-fluid system tend to the trivial equilibrium $(\overline{n}_0, \overline{n}_0, 0, \overline{n}_0\phi)$, the norms of $(\widehat{n}, \widehat{c}, \widehat{u})$ will diminish to 0. The experimental errors of n and c decrease over t but are bounded below because the numerical error exists. We mention that the experimental $\|\widehat{u}(t)\|_{L^\infty}$ and $\|\widehat{u}(t)\|_{H^1}$ also decrease to 0 in the simulation. But the computation results of the relative errors uLInf $= \|\widehat{u}(t)\|_{L^\infty} / \|u(t)\|_{L^\infty}$ and

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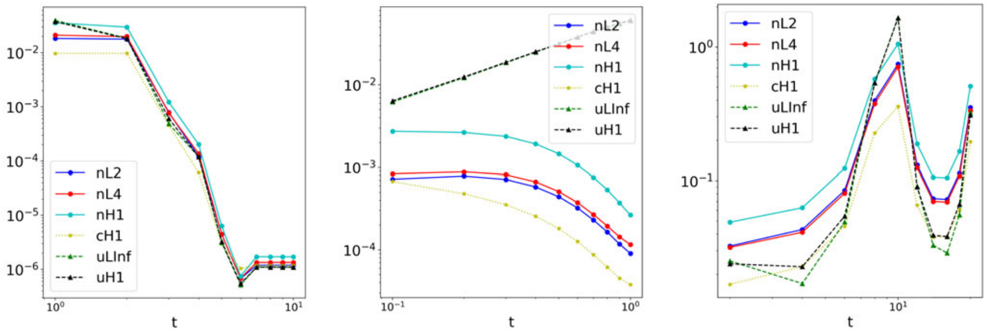

FIGURE 4. We fix $\epsilon = 2^{-5}$. The three figures, (Ex1-t), (Ex2-t) and (Ex3-t), display the norms of $(\widehat{n}, \widehat{c}, \widehat{u})$ on various t for the Example 1, 2 and 3, respectively. We plot the errors in log-scale.

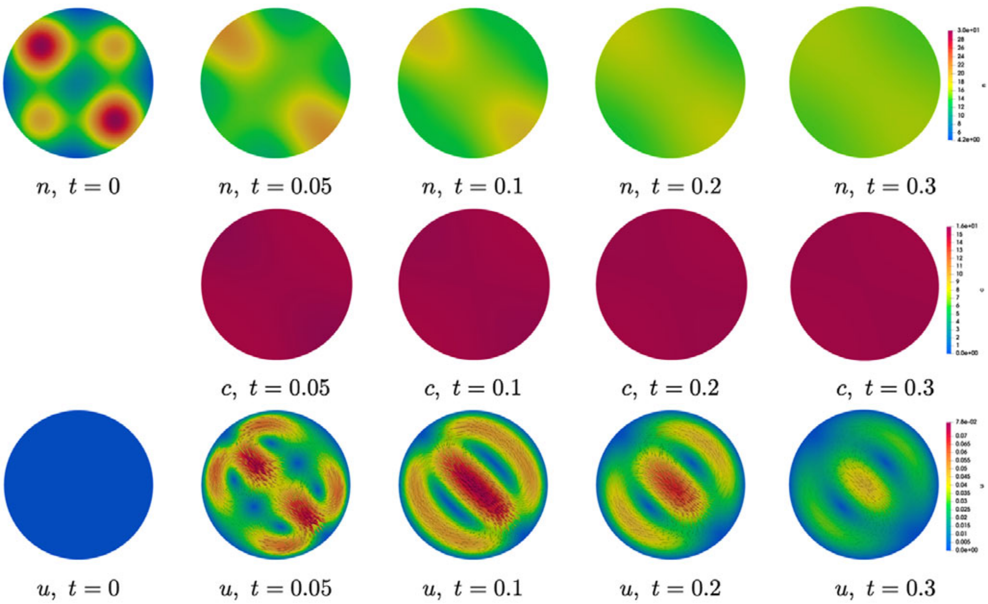

FIGURE 5. Example 2. (n, c, u) of the PE-fluid system.

$\text{uH1} = \|\widehat{u}(t)\|_{H^1}/\|u(t)\|_{H^1}$ are polluted because the denominators also tend to zero. Therefore, we shall ignore the curves of uLInf and uH1 in (Ex2-t), which cannot demonstrate the real convergence behaviour of $\|\widehat{u}\|$.

Example 3. Change the parameters $\alpha = \frac{1}{2}$, $a_1 = a_2 = 2$, $-b_1 = b_2 = 1$ of the chemotactic sensitivity function and take the initial density

$$n_0 = 60\left(e^{-10|x|^2} + 20e^{-5\left|x-\left(-\frac{1}{2},-\frac{1}{2}\right)\right|^2}\right).$$

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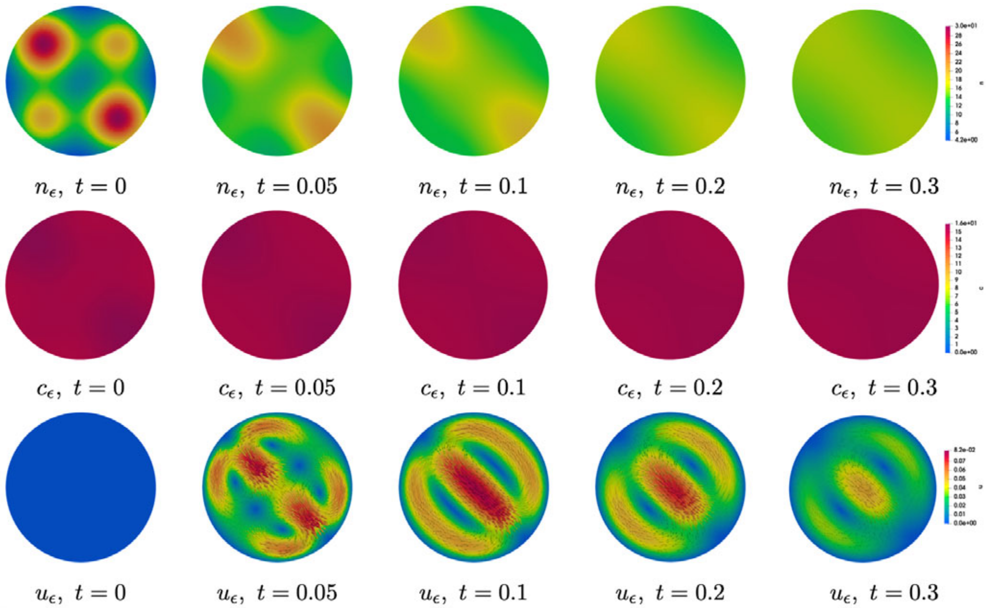

FIGURE 6. Example 2. $(n_\epsilon, c_\epsilon, u_\epsilon)$ of the PP-fluid system with $\epsilon = 0.25$.

We carry out the simulation with the anticlockwise rotating aggregations (see Figures 7 and 8). The cells and chemicals first disperse and move to the lower left quarter, and then gather and rotate anticlockwise along the boundary. The circular motion of the aggregation is nonuniform. Roughly speaking, the rotation accelerates at the lower left quarter and decelerates near the upper right quarter. The PP-fluid system with smaller ϵ has a shorter rotation period and the solution is closer to the PE-fluid system.

Figure 3. (Ex3-ϵ) indicates the $O(\epsilon)$-convergence of $(\widehat{n}(t), \widehat{c}(t), \widehat{u}(t))$ for fixed $t = 4$. Figure 4 (Ex3-t) shows the errors on t for fixed $\epsilon = 2^{-5}$. When the rotation of the PE-fluid system is tardy near $t = 4$ and $t = 14$, the evolution of the PP-fluid system catches up and the difference becomes small. On the contrary, the rotation of the PE-fluid system speeds up around $t = 10$ and $t = 20$, while motion of the PP-fluid system is delayed, which causes the big increment of error.

Remark 5.1. *Theorem 1.1 concludes the* $O\left(e^{Ct}\epsilon^{\frac{1}{2}}\right)$*-convergence for the general global bounded solution, and Theorem 1.2 improves the estimate to* $O\left(t^{\frac{1}{2}}\epsilon^{\frac{1}{2}}\right)$ *for the solutions with the trivial equilibrium. Above three numerical examples further explore this topic by investigating the experimental convergence behaviour on different types of solutions. If* $\partial_t c_\epsilon$ *is smooth enough and bounded independent of* $\epsilon^{-\ell}$ $(\ell > 0)$*, it seems quite possible to obtain the* $O(\epsilon)$*-convergence, and the numerical results confirm that. When the PE-fluid system and PP-fluid system tend to the same equilibrium, nontrivial or trivial, the deviation between them shall vanish as t increases, which is revealed in the Example 1 and 2. The case of the rotating solution is complicated and interesting. The error between the PE-fluid system and PP-fluid system fluctuates drastically in consequence of the nonuniform rotating speed together with the different rotation periods, which brings challenges to the elaborate convergence analysis.*

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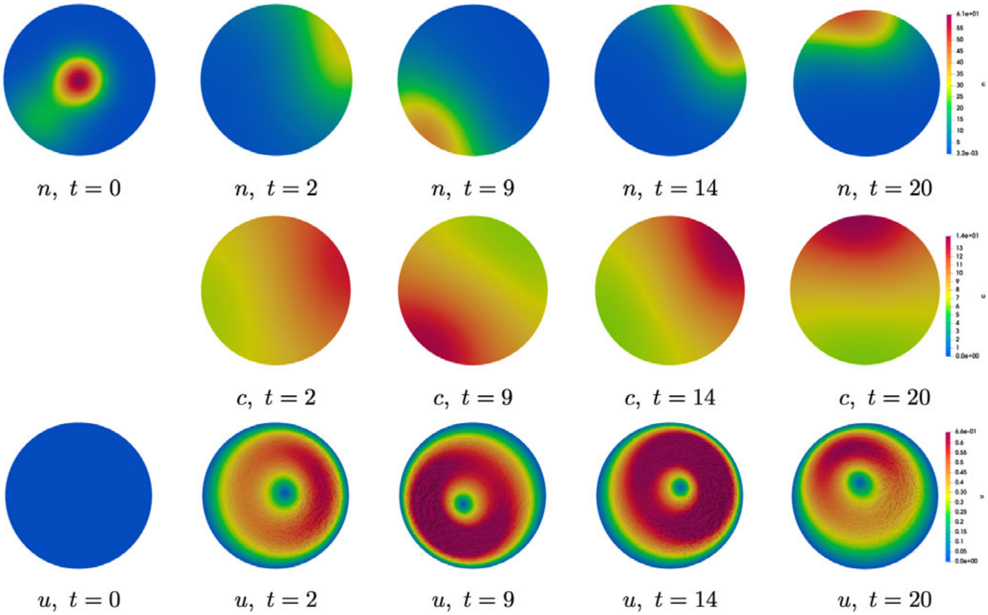

FIGURE 7. Example 3. (n, c, u) of the PE-fluid system.

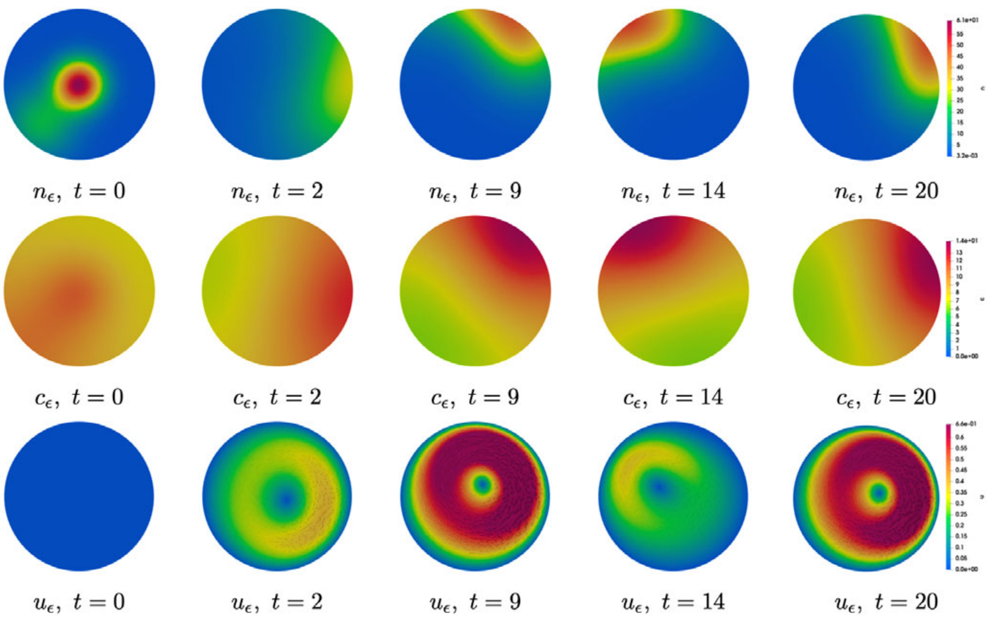

FIGURE 8. Example 3. $(n_\epsilon, c_\epsilon, u_\epsilon)$ of the PP-fluid system with $\epsilon = 0.5$.

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