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ON AN INEQUALITY RELATING TO SUM SETS

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Abstract

We show how a short and elementary proof can be provided for a recently-published inequality ([6], [4]) which has found a number of applications.

1. Introduction

Probability-measure inequalities for sum sets have found a number of applications, see, for example, Brown and Williamson [3] for a coin-tossing application and Newhouse [7] and Palis and Takens [8] in connection with dynamical systems. They are often also intimately associated with combinatorial counting problems, as is the case in the present context.

In 1974, G. Brown and W. Moran [1] showed that a key probability inequality for uncountable sum sets could be deduced if a related counting inequality held for certain discrete sum sets, and that this in turn would follow from the truth of the inequality

$$x^\alpha y^\alpha + \max\left[x^\alpha(1-y)^\alpha, y^\alpha(1-x)^\alpha\right] + (1-x)^\alpha(1-y)^\alpha \geq 1 \tag{1}$$

for $0 \leq x, y \leq 1$ and $\alpha = \log_4 3$. Brown and Moran were unable to establish (1) at the time. Quite a rich literature, an historical perspective on some of which is recounted by Brown in [2], has developed around both (1) and the original problem.

Relation (1) possesses an m-variable generalization

$$\prod_{i=1}^{m} x_i^\alpha + \sum_{j=1}^{m-1} \max_\pi \prod_{i=1}^{j} x_{\pi(i)}^\alpha \prod_{i=j+1}^{m} (1 - x_{\pi(i)})^\alpha + \prod_{i=1}^{m} (1 - x_i)^\alpha \geq 1, \tag{2}$$

where π denotes a permutation of $\{1, \ldots, m\}$ and

$$\alpha = \alpha(m) = \frac{\log(m+1)}{m \log 2}.$$

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If the x_i are ordered by

$$1 \geq x_1 \geq \ldots \geq x_m \geq 0,$$

this may be expressed more simply as

$$f(x_1, \ldots, x_m) \geq 1, \tag{3}$$

where

$$f(x_1, \ldots, x_m) \equiv x_1^\alpha \ldots x_m^\alpha + x_1 \ldots x_{m-1}^\alpha (1 - x_m)^\alpha + \ldots + (1 - x_1)^\alpha \ldots (1 - x_m)^\alpha. \tag{4}$$

It was established by Landau, Logan and Shepp [6] and independently by Brown, Keane, Moran and Pearce [4].

This apparently new and superficially inoccuous inequality turned out to be quite tricky to prove. As noted in [6]: "since the inequality seems to be a matter of real variables, it is perhaps surprising that our proof is based on conformal mapping and Hadamard's three-circle theorem." The proof of [4] is more elementary.

In fact, in [6] the following extension of (3) is proved.

Suppose $\alpha > 0$ and $1 \geq x_1 \geq \ldots \geq x_m \geq 0$ and let f be defined by (4). Then

$$f(x_1, \ldots, x_m) \geq \min\left[1, (m+1)2^{-m\alpha}\right]. \tag{5}$$

Here we shall note that we can use the simple method of [4] to obtain the general inequality (5). We show also how the argument of [4] may be shortened considerably.

2. Results

The following lemma was proved in [4].

LEMMA 1. *Denote by $w(m) \geq 0$ the infimum of f, so that*

$$f(x_1, \ldots, x_m) \geq w(m).$$

If the values $x_1, \ldots, x_m$ are such that $f(x_1, \ldots, x_m) = w(m)$, then $x_1 = \ldots = x_m = x$, say.

Now we have the following lemma, which extends Lemma 3 of [4].

LEMMA 2. (a) *For $\alpha = \alpha(m)$,*

$$\frac{\sinh(m+1)\alpha t}{(\sinh \alpha t)\cosh^{m\alpha} t} \geq m + 1 \quad (t \geq 0); \tag{6}$$

(b) *for all* β *satisfying* $0 \le \beta \le \alpha(m)$

$$1 + b^{\beta} + b^{2\beta} + \ldots + b^{m\beta} \ge (1+b)^{m\beta} \quad (b \ge 0); \tag{7}$$

(c) *for all* $\beta \ge \alpha(m)$

$$1 + b^{\beta} + b^{2\beta} + \ldots + b^{m\beta} \ge (m+1)2^{-m\beta}(1+b)^{m\beta} \quad (b \ge 0). \tag{8}$$

PROOF. Parts (a) and (b) are established in [4], while for $\beta \ge \alpha(m)$ we have

$$\begin{aligned} 1 + b^{\beta} + b^{2\beta} + \ldots + b^{m\beta} &= 1 + \left(b^{\beta/\alpha(m)}\right)^{\alpha(m)} + \ldots + \left(b^{\beta/\alpha(m)}\right)^{m\alpha(m)} \\ &\ge \left[1 + b^{\beta/\alpha(m)}\right]^{m\alpha(m)} \qquad \text{by (7)} \\ &\ge 2^{m(\alpha(m)-\beta)}(1+b)^{m\beta} \qquad \text{by Jensen's inequality} \\ &= (m+1)2^{-m\beta}(1+b)^{m\beta}. \end{aligned}$$

From (7) and (8) we can formulate the following result.
If $\alpha > 0$, then

$$1 + b^{\alpha} + b^{2\alpha} + \ldots + b^{m\alpha} \ge \min\left[1, (m+1)2^{-m\alpha}\right](1+b)^{m\alpha}. \tag{9}$$

With the substitutions $x = 1/(1+b)$, $1 - x = b/(1+b)$, (5) now follows from Lemma 1 and (9).

REMARK. As observed in [4], (6) and (7) are equivalent. However (6) arises as a special case of a theorem of Pittenger [9] (see Bullen, Mitrinović and Vasić [5, Theorem 5, page 349] for a more accessible account). For $r > 0$, Pittenger's theorem gives in particular that (modulo an obvious misprint in [5])

$$(\cosh r_1 t)^{1/r_1} \le \left[\frac{\sinh(r+1)t}{(r+1)\sinh t}\right]^{1/r} \le (\cosh r_2 t)^{1/r_2}, \tag{10}$$

where

$$r_1 = \min\left[\frac{r+2}{3}, \frac{r\log 2}{\log(r+1)}\right],$$
$$r_2 = \max\left[\frac{r+2}{3}, \frac{r\log 2}{\log(r+1)}\right].$$

Replace t by αt. Since

$$r_1 = r\log 2/\log(r+1)$$

for $r \ge 1$, we have $r/r_1 = r\alpha(r)$ and $r_1\alpha(r) = 1$, and (6) follows at once from the left-hand relation of (10). This enables the end result of Lemmas 2 and 3 of [4] to be deduced directly, thereby shortening considerably the argument of [4] to provide a conveniently short proof of (2) and (5).

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