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# ZETA FUNCTIONS OF TWISTED MODULAR CURVES

**CRISTIAN VIRDOL**

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**Abstract**

In this paper we compute and continue meromorphically to the whole complex plane the zeta function for twisted modular curves. The twist of the modular curve is done by a mod $p$ representation of the absolute Galois group.

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## 1. Introduction

In Chapter 7 of his book [9], Shimura computed the zeta function for modular curves and modular abelian varieties by relating the Frobenius morphism with Hecke operators using some congruence relations. We will use some of his ideas to compute the zeta function of the curves that we will define below. When the mod $p$ representation is associated to a rational elliptic curve, such a twisted modular curve was defined and used in a paper by Wiles [12, Remark 2]. Let $X(p)/\mathbb{Q}$ be the modular curve of the principal congruence subgroup $\Gamma(p)$ of $SL_2(\mathbb{Z})$ for a prime $p \geq 7$ (we do not consider $5 \geq p$, since for these values, the modular curve has genus 0), which is a geometrically disconnected curve whose connected components are $p-1$ copies of the half upper plane quotient out by $\Gamma(p)$. Let $\overline{X(p)}$ be the compactification of $X(p)$. The curve $\overline{X(p)}$ has an action of $GL_2(\mathbb{Z}/p\mathbb{Z})$ as specified later (see Section 2.1). For a number field $F$ we denote by $G_F$ the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$. We consider a continuous Galois representation $\rho : G_{\mathbb{Q}} \to GL_2(\mathbb{Z}/p\mathbb{Z})$, and let $\overline{X'(p)}/\mathbb{Q}$ be the curve obtained from $\overline{X(p)}/\mathbb{Q}$ via twisting by $\rho$ composed with the action of $GL_2(\mathbb{Z}/p\mathbb{Z})$ on $\overline{X(p)}$ (see Section 2.2 for the definition of $\overline{X'(p)}$).

Let $\hat{\Gamma}(p)$ be the adelic principal congruence subgroup of level $p$ in $GL_2(\mathbb{A}_f)$, where $\mathbb{A}_f$ is the finite part of the adele ring $\mathbb{A}_{\mathbb{Q}}$ of $\mathbb{Q}$. Let $\pi = \pi_f \otimes \pi_\infty$ be a cuspidal automorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$, where $\pi_f$ and $\pi_\infty$ are representations of $GL_2(\mathbb{A}_f)$ and $GL_2(\mathbb{R})$, respectively. If $K$ is an open compact subgroup of $GL_2(\mathbb{A}_f)$, let $\pi_f^K$ denote the space of $K$-fixed vectors of $\pi_f$. One can associate to a representation $\pi$ an $L$-function $L(s,\pi)$ which has an analytic continuation to the whole complex plane as an entire function and verifies a functional equation: $s \leftrightarrow 1-s$ (see [2, Theorems 6.15 and 6.16]).

We fix an isomorphism $j : \overline{\mathbb{Q}}_l \to \mathbb{C}$ and from now on we identify these two fields. Let $\rho_{\pi,l} : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_l) \cong GL_2(\mathbb{C})$, $l$ prime, $l \neq p$ be the two dimensional continuous Galois representation associated to the cuspidal automorphic representation $\pi$. Define

$$L(s,\rho_{\pi,l}) := \prod_q L_q(s),$$

where

$$L_q(s) := \det\left(1 - j\left(\rho_{\pi,l}(\mathrm{Frob}_q)\right)\big|_{V^{I_q}} q^{-s}\right)^{-1},$$

and $\mathrm{Frob}_q$ is a Frobenius element at $q$, $I_q$ is the inertia group at $q$ and $V$ is the space corresponding to $\rho_{\pi,l}$. Then $L(s,\rho_{\pi,l})$ has an analytic continuation to the whole complex plane as an entire function and verifies a functional equation: $s \leftrightarrow 2-s$.

As Shimura proved, we have $L(s-1/2,\pi) = L(s,\rho_{\pi,l})$. From the work of Shimura and others (see [9, Theorems 7.11 and 7.13]), we know that the $H^1$ part of the Hasse-Weil zeta functions of $\overline{X(p)}$ is given by

$$L(s,\overline{X(p)}) = \prod_\pi L(s-1/2,\pi)^{\dim \pi_f^{\hat{\Gamma}(p)}},$$

where the cuspidal automorphic representations $\pi$ that appear in the product are of weight 2, satisfy $\pi_f^{\hat{\Gamma}(p)} \neq 0$ and are cohomological, which means that

$$H^1(gl_2(\mathbb{R}), SO_2(\mathbb{R}); \pi_\infty) \neq 0.$$

Here, $H^1(gl_2(\mathbb{R}), SO_2(\mathbb{R}); \pi_\infty)$ is the Lie algebra cohomology group with respect to $gl_2(\mathbb{R})$ relative to the maximal compact subgroup $SO_2(\mathbb{R})$.

The group $GL_2(\mathbb{Z}/p\mathbb{Z})$ acts on the modular curve $\overline{X(p)}$. The composition of this action with $\rho$ gives us an action of $G_{\mathbb{Q}}$ on $\overline{X(p)}$. Taking complex points of $\overline{X(p)}$ we get that $G_{\mathbb{Q}}$ acts on $\overline{X(p)}(\mathbb{C})$ through this geometric action on $\overline{X(p)}$. Thus $G_{\mathbb{Q}}$ acts on $H^1(\overline{X(p)},\mathbb{C})$. Using this commutativity of this action and of the Hecke operators outside $p$ we obtain the representation $\tilde{\varphi}_\pi \circ \rho$ of $G_{\mathbb{Q}}$ on $\pi_f^{\hat{\Gamma}(p)}$ (see the beginning of Section 2.6 for the definition of $\tilde{\varphi}_\pi$).

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Let

$$L(s, \rho_{\pi,l} \otimes (\tilde{\varphi}_\pi \circ \rho)) := \prod_q L'_q(s),$$

where we denote

$$L'_q(s) := \det\left(1 - j\left(\rho_{\pi,l}(\mathrm{Frob}_q)\big|_{V^{I_q}} \otimes (\tilde{\varphi}_\pi \circ \rho)\ (\mathrm{Frob}_q)\big|_{U^{I_q}}\right) q^{-s}\right)^{-1}$$

and $U$ is the space corresponding to $\tilde{\varphi}_\pi \circ \rho$.

We have two curves $\overline{X(p)}/\mathbb{Q}$ and the twisted one $\overline{X'(p)}/\mathbb{Q}$. Their jacobians $J$ and $J'$ are identical over $\overline{\mathbb{Q}}$, but the Galois actions on $J$ and $J'$ are different. The difference is described by the representation $\tilde{\varphi}_\pi \circ \rho$. Then we go through Shimura's computation of the zeta function of $\overline{X(p)}$ modifying the Galois action by $\tilde{\varphi}_\pi \circ \rho$ and we obtain the first part of the following theorem (which is a consequence of Proposition 2.2):

THEOREM 1.1. *We have*

$$L\big(s, \overline{X'(p)}\big) = \prod_\pi L(s, \rho_{\pi,l} \otimes (\tilde{\varphi}_\pi \circ \rho)),$$

*where the cuspidal automorphic representations $\pi$'s that appear in the product are of weight 2, verify $\pi_f^{\hat{\Gamma}(p)} \neq 0$ and are cohomological. If the representation $\rho$ factors through the Galois group of a solvable Galois extension of a totally real field (that is, the field $K := (\overline{\mathbb{Q}})^{\ker(\rho)}$ is a solvable extension of a totally real field), then the $L$-function $L\big(s, \overline{X'(p)}\big)$ has a meromorphic continuation to the whole complex plane and verifies a functional equation.*

In this theorem $L\big(s, \overline{X'(p)}\big)$ represents the $H^1$ part of the zeta function of $\overline{X'(p)}$. Meromorphic continuation is done combining the technique of Artin-Brauer with a recent result of Taylor [11] and the $GL_2$-base change for cyclic extensions proven by Langlands [6]. We shall compute the $L$-function in the following section and prove the meromorphic continuation in Section 3.

## 2. Computation of the zeta function

**2.1. Known facts** Let us recall some known facts (see [3] or [5]) which will be used in the proof of Theorem 1.1. Let $N$ be a positive integer with $N > 2$, $S$ a scheme, and $E/S$ an elliptic curve over $S$. If $N : E \to E$ is the multiplication by $N$, then the kernel of this morphism $E[N]/S = \ker[N]$ is a locally free group scheme of rank $N^2$ over $S$. A level $N$-structure is by definition a group scheme isomorphism $\phi : (\mathbb{Z}/N\mathbb{Z})^2/S \to$

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$E[N]/S$. Let SCH be the category of schemes and SETS be the category of sets. We consider the following functor $\epsilon_N : \mathrm{SCH} / \operatorname{Spec}(\mathbb{Z}[1/N]) \to \mathrm{SETS}$:

$$\epsilon_N(S) = [(E, \phi)/S \mid E/S \text{ an elliptic curve}, \phi \text{ a level } N \text{ structure}],$$

where '[$\cdot$]' means the set of isomorphism classes of the objects in the brackets. Two structures $(E, \phi)$ and $(E', \phi')$ are isomorphic by $\varphi : E \to E'$ if $\varphi$ is an isomorphism and $\varphi \circ \phi = \phi'$. It is known that the functor $\epsilon_N$ is representable over $\operatorname{Spec}(\mathbb{Z}[1/N])$ by an affine curve $X(N)$. There is a natural action of $GL_2(\mathbb{Z}/N\mathbb{Z})$ on $\epsilon_N$ which is given by $(E, \phi) \to (E, \phi \circ g)$, if $g \in GL_2(\mathbb{Z}/N\mathbb{Z})$. The action of $-1 \in GL_2(\mathbb{Z}/N\mathbb{Z})$ is trivial because $-1 : E \to E$ induces an isomorphism $(E, \phi) \cong (E, \phi \circ (-1))$. Since $\epsilon_N$ is representable by $X(N)$ over $\operatorname{Spec}(\mathbb{Z}[1/N])$, the group $GL_2(\mathbb{Z}/N\mathbb{Z})$ acts on $X(N)/\operatorname{Spec}(\mathbb{Z}[1/N])$.

It is known also that (see [5, Lemma 10.3.2]):

PROPOSITION 2.1. *The group $GL_2(\mathbb{Z}/N\mathbb{Z})$ acts on the compactified modular curve $\overline{X(N)}$.*

**2.2. Construction of the twisted curve** We fix a continuous representation

$$\rho : G_{\mathbb{Q}} \to GL_2(\mathbb{Z}/p\mathbb{Z})$$

where $p$ is a prime number. Let $K$ be the finite Galois extension of $\mathbb{Q}$ defined by $K := (\overline{\mathbb{Q}})^{\ker(\rho)}$.

Suppose that a group $G$ acts on an affine scheme $X = \operatorname{Spec}(R)$. Then $G$ determines an action on $R$. If we consider $R^G = \{r \in R \mid gr = r, \forall g \in G\}$, then $R^G$ is a ring. We have that $\operatorname{Spec}(R^G) = \operatorname{Spec}(R)/G$ as a geometric quotient if $R/R^G$ is étale (see [3, Proposition 1.8.4]). If $X$ is not affine and we can cover $X$ by affine schemes that are stable under $G$, we similarly obtain a geometric quotient $X/G$.

Let $X' = \overline{X(p)} \times_{\operatorname{Spec}(\mathbb{Z}[1/pd])} \operatorname{Spec}(O_K[1/pd])$, where $O_K$ is the ring of integers of $K$, $d$ is the discriminant of $K/\mathbb{Q}$, and $O_K[1/pd]$ is the sub-ring of $K$ in which $pd$ is inverted. The group $GL_2(\mathbb{Z}/p\mathbb{Z})$ acts on $\overline{X(p)}$. Since $\rho : \operatorname{Gal}(K/\mathbb{Q}) \hookrightarrow GL_2(\mathbb{Z}/p\mathbb{Z})$, the group $\operatorname{Gal}(K/\mathbb{Q})$ acts on $\overline{X(p)}$. The Galois group $\operatorname{Gal}(K/\mathbb{Q})$ has a natural action on $\operatorname{Spec}(O_K[1/pd])$ and we can descend via the quotient process $X'$ to $\overline{X'(p)}/\operatorname{Spec}(\mathbb{Z}[1/pd])$ using the diagonal action

$$\operatorname{Gal}(K/\mathbb{Q}) \ni \sigma \to \rho(\sigma) \otimes \sigma$$

on $X'$. Thus, we obtain a smooth projective curve $\overline{X'(p)}/\operatorname{Spec}(\mathbb{Z}[1/pd])$. This is the twisted curve that we mentioned in the title. If we do descend as above the jacobian of $\overline{X(p)}/\operatorname{Spec}(\mathbb{Z}[1/pd])$, we obtain the jacobian of $\overline{X'(p)}/\operatorname{Spec}(\mathbb{Z}[1/pd])$.

**2.3. Zeta Function for curves** We recall briefly the definition of the $H^1$ part of the Hasse-Weil zeta function of a smooth projective curve $X$ over $S = \mathrm{Spec}(\mathbb{Z}[1/N])$ where $N$ is a positive integer. We hereafter call the $H^1$ part, the zeta function of $X$ for simplicity. First let us consider $\mathbb{F}_p$ a finite field of characteristic $p$ and $J/\mathbb{F}_p$ an abelian variety. Let $J[l^n]$ the $l^n$-torsion points for a prime number $l \neq p$. The Galois group $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ acts on $J[l^n]$ for all natural numbers $n$ and taking the limit

$$T_l(J) = \varprojlim J[l^n],$$

we get the $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$-module $T_l(J)$. We write $\rho_l : \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \to GL(T_l(J))$ for the resulting representation. Let $L_p(T) = \det(1 - \rho_l(\phi_p)T)$, where $\phi_p$ is the Frobenius element

$$\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \ni (x \to x^p).$$

The polynomial $L_p(T) \in \mathbb{Z}[T]$ does not depend on $l$.

Now, let $X$ be a smooth proper curve over $S = \mathrm{Spec}(\mathbb{Z}[1/N])$, and $J$ be the jacobian of $X$. Then $J$ is an abelian scheme over $S = \mathrm{Spec}(\mathbb{Z}[1/N])$. We denote

$$J(l) = J \times_{\mathrm{Spec}(\mathbb{Z}[1/N])} \mathrm{Spec}(\mathbb{F}_l),$$

for $l$ prime, $l \nmid N$. We define the zeta function of $X$ over $S = \mathrm{Spec}(\mathbb{Z}[1/N])$ as the product

$$L(s, X/S) = \prod_{l \nmid N} L_l(l^{-s})^{-1}$$

Here the definition of the zeta function is given up to the factors at $l \mid N$.

**2.4. Twisted Galois action on $J$** To simplify the notations we regard our curves $\overline{X(p)}$ and $\overline{X'(p)}$ as curves over $\mathrm{Spec}(\mathbb{Q})$. Let $J$ and $J'$ be the jacobians of $\overline{X(p)}$ and $\overline{X'(p)}$ respectively.

We obtained $\overline{X'(p)}$ from $\overline{X(p)}$ first tensoring by $\mathrm{Spec}(O_K[1/pd])$ and then making the diagonal quotient. The difference of the action of $G_{\mathbb{Q}}$ on the Tate modules $T_l(J)$ and $T_l(J')$ can be described in the following way: As $\mathbb{Z}_l$-modules, we have $T_l(J) = T_l(J')$, but the Galois action is different. We write the Galois action of $\sigma \in G_{\mathbb{Q}}$ on $T_l(J)$ as $x \to x^\sigma$. We want to describe the action of $G_{\mathbb{Q}}$ on $T_l(J')$ in terms of the action of $G_{\mathbb{Q}}$ on $T_l(J)$ and

$$\rho : G_{\mathbb{Q}} \to GL_2(\mathbb{Z}/p\mathbb{Z}).$$

The Galois representation $\rho$ composed with the action of $GL_2(\mathbb{Z}/p\mathbb{Z})$ on $\overline{X(p)}$ induces a representation

$$\rho'' : G_{\mathbb{Q}} \to \mathrm{Aut}(T_l(J)).$$

PROPOSITION 2.2. *The action of $\sigma \in G_{\mathbb{Q}}$ on $T_l(J')$ is given by $x \mapsto \rho''(\sigma)x^\sigma$.*

PROOF. Let $\rho' : G_{\mathbb{Q}} \to \mathrm{Aut}(\overline{X(p)})$ be the composition of $\rho : G_{\mathbb{Q}} \to GL_2(\mathbb{Z}/p\mathbb{Z})$ and of the action of $GL_2(\mathbb{Z}/p\mathbb{Z})$ on $\overline{X(p)}$. Let $W = \overline{X(p)} \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\overline{\mathbb{Q}})$. We have

$$W(\overline{\mathbb{Q}}) = \mathrm{Hom}_{\mathrm{Spec}(\mathbb{Q})}(\mathrm{Spec}(\overline{\mathbb{Q}}), \overline{X(p)}) \times \mathrm{Hom}_{\mathbb{Q}}(\overline{\mathbb{Q}}, \overline{\mathbb{Q}}).$$

An element $\tau \in G_{\mathbb{Q}}$ acts on $W(\overline{\mathbb{Q}})$ by

$$\tau(x, g) = (\rho'(\tau)(x), g\tau^{-1}),$$

where $(x, g) \in W(\overline{\mathbb{Q}})$. This is the diagonal action on $W(\overline{\mathbb{Q}})$ that we use to do descent. In this circumstance, we can realize the descent as a geometric quotient of $W$ by the action of $G_{\mathbb{Q}}$.

Since $\overline{X'(p)}$ is obtained from $\overline{X(p)} \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(K)$ by a twist of $\mathrm{Gal}(K/\mathbb{Q})$-action, we have

$$\overline{X(p)} \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(K) = \overline{X'(p)} \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(K).$$

Thus, we get $\overline{X(p)}(\overline{\mathbb{Q}}) = \overline{X'(p)}(\overline{\mathbb{Q}})$. Let $u = [(x, 1)] \in \overline{X'(p)}(\overline{\mathbb{Q}}) = \overline{X(p)}(\overline{\mathbb{Q}})$ be a class of the quotient of $W(\overline{\mathbb{Q}})$, determined by the above action. The group $G_{\mathbb{Q}}$ acts through its arithmetic action on $\overline{X(p)}(\overline{\mathbb{Q}})$ sending $u \to u^{\sigma} = [(x^{\sigma}, 1)]$. Then we describe the action of $G_{\mathbb{Q}}$ on $\overline{X'(p)}(\overline{\mathbb{Q}})$ in terms of the action of $G_{\mathbb{Q}}$ on $\overline{X(p)}(\overline{\mathbb{Q}})$:

$$\begin{aligned} u \to u^{\sigma'} &= [(x, 1)]^{\sigma'} = [(x^{\sigma}, \sigma)] = [\sigma^{-1}(\rho'(\sigma)x^{\sigma}, 1)] \\ &= [(\rho'(\sigma)x^{\sigma}, 1)] = \rho'(\sigma)[(x^{\sigma}, 1)] = \rho'(\sigma)u^{\sigma}, \end{aligned}$$

where we attach a $'$ to $\sigma$ to indicate when we refer to the action of $G_{\mathbb{Q}}$ on $\overline{X'(p)}(\overline{\mathbb{Q}})$.

Thus $\sigma \in G_{\mathbb{Q}}$ acts on $\overline{X'(p)}(\overline{\mathbb{Q}})$ by sending $u \to \rho'(\sigma)u^{\sigma}$. We explained above the action of $G_{\mathbb{Q}}$ on $\overline{X'(p)}(\overline{\mathbb{Q}})$ in terms of the action of $G_{\mathbb{Q}}$ on $\overline{X(p)}(\overline{\mathbb{Q}})$. We obtain the action in the proposition replacing $\overline{X(p)}$ and $\overline{X'(p)}$ by their jacobians and by their Tate modules. □

**2.5. Complex points on the modular curve** We have

$$X(p)(\mathbb{C}) = GL_2^+(\mathbb{Q})\backslash GL_2^+(\mathbb{A}_{\mathbb{Q}})/\hat{\Gamma}(p)SO_2(\mathbb{R})\mathbb{R}^{\times},$$

where

$$GL_2^+(\mathbb{Q}) = \{g \in GL_2(\mathbb{Q}) \mid \det g > 0\},$$

the ring $\mathbb{A}_{\mathbb{Q}}$ is the adele ring of $\mathbb{Q}$ and $GL_2^+(\mathbb{A}_{\mathbb{Q}}) = GL_2(\mathbb{A}_f)GL_2^+(\mathbb{R})$, $\mathbb{A}_f$ is the finite part of the adele ring $\mathbb{A}_{\mathbb{Q}}$, and

$$\hat{\Gamma}(p) = \{x \in GL_2(\hat{\mathbb{Z}}) \mid x \equiv 1(p)\}$$

with $\hat{\mathbb{Z}} = \Pi_p\mathbb{Z}_p$.

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The group $GL_2(\mathbb{Z}/p\mathbb{Z})$ acts on $X(p)(\mathbb{C})$. This action can be described in terms of the following action: $GL_2(\mathbb{Z}_p) \hookrightarrow GL_2(\mathbb{A}_{\mathbb{Q}})$ by $\alpha \mapsto (1, \ldots, \alpha, 1, \ldots, 1)$ ($\alpha$ is the $p$ component). Using the isomorphism $GL_2(\mathbb{Z}/p\mathbb{Z}) \cong GL_2(\mathbb{Z}_p)/\hat{\Gamma}(p)_p$, where

$$\hat{\Gamma}(p)_p = \{x \in GL_2(\mathbb{Z}_p) | x \equiv 1(p)\},$$

we get the action of $g \in GL_2(\mathbb{Z}_p)$ on $X(p)(\mathbb{C})$ which is given by the left multiplication at the $p$ component.

**2.6. The zeta function of the twisted curve** Let $\pi = \pi_f \otimes \pi_\infty$ be a cuspidal automorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$, where $\pi_f$ and $\pi_\infty$ are representations of $GL_2(\mathbb{A}_f)$ and $GL_2(\mathbb{R})$, respectively. Let

$$\rho_{\pi,l} : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_l) \cong GL_2(\mathbb{C}),$$

$l$ prime, $l \neq p$ be the two dimensional continuous Galois representation associated to $\pi$. If $K$ is an open compact subgroup of $GL_2(\mathbb{A}_f)$, let $\pi_f^K$ denote the space of $K$-fixed vectors of $\pi_f$.

We write $\pi_f^{\hat{\Gamma}(p)} = \pi^{\prod_{l\neq p} GL_2(\mathbb{Z}_l)} \otimes \pi_p^{\hat{\Gamma}(p)_p}$. By the work of Shimura and others we know that

$$L(\overline{X(p)}, s) = \prod_{\pi} L(s - 1/2, \pi)^{\dim \pi_f^{\hat{\Gamma}(p)}},$$

where the $\pi$'s that appear in the product are of weight 2, verify $\pi_f^{\hat{\Gamma}(p)} \neq 0$ and are cohomological, that is, $H^1(gl_2(\mathbb{R}), SO_2(\mathbb{R}); \pi_\infty) \neq 0$.

We consider the decomposition of the cohomology with compact support of $\overline{X(p)}$:

$$H_c^1(\overline{X(p)}, \mathbb{C}) = \oplus_\pi H^1(gl_2(\mathbb{R}), SO_2(\mathbb{R}); \pi_\infty) \otimes \pi_f^{\hat{\Gamma}(p)},$$

where the $\pi$'s that appear in the product are of weight 2, verify $\pi_f^{\hat{\Gamma}(p)} \neq 0$ and are cohomological. The space $H^1(gl_2(\mathbb{R}), SO_2(\mathbb{R}); \pi_\infty)$ is a 2-dimensional complex vector space. On each of the above summands, $GL_2(\mathbb{Z}/p\mathbb{Z})$ acts through a representation of the form $1 \otimes \varphi_\pi^{-1}$, where $\varphi_\pi$ is a representation of $GL_2(\mathbb{Z}/p\mathbb{Z})$ on $\pi_f^{\hat{\Gamma}(p)}$. The space $H_{et}^1(\overline{X(p)}, \overline{\mathbb{Q}}_l)$ has a decomposition of the same form as $H_c^1(\overline{X(p)}, \mathbb{C})$:

$$H_{et}^1(\overline{X(p)}, \overline{\mathbb{Q}}_l) = \oplus_\pi U_{\overline{\mathbb{Q}}_l}(\pi) \otimes_{\overline{\mathbb{Q}}_l} \tilde{\pi}_f^{\hat{\Gamma}(p)},$$

where the $\pi$'s that appear in the product are of weight 2, verify $\pi_f^{\hat{\Gamma}(p)} \neq 0$ and are cohomological, $U_{\overline{\mathbb{Q}}_l}(\pi)$ is the $\overline{\mathbb{Q}}_l$-space of dimension 2 and $\tilde{\pi}_f^{\hat{\Gamma}(p)}$ is a $\overline{\mathbb{Q}}_l$-space. The group $G_{\mathbb{Q}}$ acts on each summand of $H_{et}^1(\overline{X(p)}, \overline{\mathbb{Q}}_l)$ by a representation of the form

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${}^t\rho_{\pi,l}^{-1} \otimes 1$. We denote by $\tilde{\varphi}_\pi$ the representation on $\tilde{\pi}_f^{\hat{\Gamma}(p)}$ that corresponds to $\varphi_\pi$. Since $H^1_{et}(\overline{X(p)}, \overline{\mathbb{Q}}_l) \cong T_l(J)^\vee \otimes_{\mathbb{Z}_l} \overline{\mathbb{Q}}_l$ (here $T_l(J)^\vee$ is the dual space of $T_l(J)$) we obtain a decomposition of the same form as above for $V = T_l(J) \otimes_{\mathbb{Z}_l} \overline{\mathbb{Q}}_l$ and write $V[\pi]$ (see the above decomposition) for the $\pi$ component. By the result of Shimura and the irreducibility of $\rho_{\pi,l}$ (the irreducibility of $\rho_{\pi,l}$ is proved in Section 3.2) and multiplicity one of $\pi$ combined, $V[\pi]$ is isomorphic to $\rho_{\pi,l} \otimes \tilde{\varphi}_\pi$ as $(G_\mathbb{Q}, GL_2(\mathbb{Z}/p\mathbb{Z}))$-module. For the twist $J'$ of $J$ we put $V' = T_l(J') \otimes_{\mathbb{Z}_l} \overline{\mathbb{Q}}_l$. Then by Proposition 2.2, the action of $G_\mathbb{Q}$ on $V'[\pi]$ is given by $\rho_{\pi,l} \otimes (\tilde{\varphi}_\pi \circ \rho)$. Thus, we get

$$L(s, \overline{X'(p)}) = \prod_\pi L(s, \rho_{\pi,l} \otimes (\tilde{\varphi}_\pi \circ \rho)).$$

Hence, we proved the following result, which is the first part of the main theorem from the introduction:

PROPOSITION 2.3. *The zeta function of the curve $\overline{X'(p)}$ that is obtained from the compactified modular curve $\overline{X(p)}$ via twisting by a continuous Galois representation $\rho : G_\mathbb{Q} \to GL_2(\mathbb{Z}/p\mathbb{Z})$ composed with the natural action of $GL_2(\mathbb{Z}/p\mathbb{Z})$ on $\overline{X(p)}$ is equal to*

$$L\big(s, \overline{X'(p)}\big) = \prod_\pi L(s, \rho_{\pi,l} \otimes (\tilde{\varphi}_\pi \circ \rho)),$$

*where the $\pi$'s that appear in the product are of weight 2, with $\pi_f^{\hat{\Gamma}(p)} \neq 0$ and are cohomological.*

REMARK 1. Here we have used the fact that $L(s, \rho_{\pi,l}) = L(s - 1/2, \pi)$ by the solution of the local Langlands conjecture for $GL_2$. We computed the zeta function of $\overline{X'(p)}$ only up to Euler factors at the prime numbers $l \mid pd$, where $d$ is the discriminant of $K/\mathbb{Q}$.

REMARK 2. We can replace $p$ and $\overline{X(p)}$ in the proof of the theorem by an arbitrary positive integer $N$ and $\overline{X(N)}$ and obtain essentially the same result.

Actually we studied the twisted curves slightly different from those used in [12] in order to treat the general $\rho$. The Galois representation $\rho$ that Wiles used in [12] comes from an elliptic curve over $\mathbb{Q}$. Thus its action on $\mathbb{Q}(\zeta_p)$ is given by $\det \rho$ composed with the cyclotomic character $(\mathbb{Z}/p\mathbb{Z})^\times \cong \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. Thus the action coincides with the action of $GL_2(\mathbb{Z}/p\mathbb{Z}) \subset \mathrm{Aut}(\overline{X(p)})$ on $\mathbb{Q}(\zeta_p)$; so, we can actually make quotient of $\overline{X(p)} \otimes_{\mathrm{Spec}(\mathbb{Q}(\zeta_p))} \mathrm{Spec}(K)$ by the diagonal action. The new curve thus obtained, slightly different from the one we studied, is the curve Wiles used whose zeta function can be computed in the same manner as we described.

## 3. Meromorphic continuation

Now we try to continue meromorphically the zeta function $L(s, \overline{X'(p)})$ to the whole complex plane. Since

$$L\big(s, \overline{X'(p)}\big) = \prod_{\pi} L(s, \rho_{\pi,l} \otimes (\tilde{\varphi}_\pi \circ \rho)),$$

it is sufficient to continue meromorphically the function $L(s, \omega \otimes \rho_l)$, where $\rho_l$ is the Galois representation

$$\rho_l : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_l) \cong GL_2(\mathbb{C}),$$

$l$ prime, $l \neq p$ associated to a cuspidal automorphic representation $\pi$ of weight 2, with $\pi_f^{\hat{\Gamma}(p)} \neq 0$ and $\omega : G_{\mathbb{Q}} \to GL_N(\mathbb{C})$ is an Artin representation.

Let $Q_f := \mathbb{Q}(a(q) \mid q \text{ prime}, q \nmid lp)$, where $a(q) := \mathrm{Tr}(\rho_l(\mathrm{Frob}_q))$ if $q \nmid lp$, and $\mathrm{Frob}_q$ is the Frobenius element at $q$ ($\rho_l$ is unramified outside $lp$). If we change the prime $l$, then we obtain also the value of $a(l)$. It is known that $Q_f$ is a finite extension of $\mathbb{Q}$. The field $Q_f$ is the minimal field of rationality of $\pi$. Let $O_f$ be the integer ring of $Q_f$ and $O_{\mathfrak{l}}$ the $\mathfrak{l}$ completion of $O_f$ for a prime factor $\mathfrak{l}$ of $l$ in $Q_f$. Then, $\rho_l : G_{\mathbb{Q}} \to GL_2(O_{\mathfrak{l}})$ is continuous, unramified outside $lp$ and satisfies $\mathrm{Tr}(\rho_l(\mathrm{Frob}_q)) = a(q)$ and $\det(\rho_l(\mathrm{Frob}_q)) = \epsilon(q)q$ for $q$ prime, $q \nmid lp$, where $\epsilon$ is a Dirichlet character. Strictly speaking, we should have written $\rho_{\mathfrak{l}}$ instead of $\rho_l$, but we keep the symbol $\rho_l$ to simplify our notation. We say that $\pi$ is of CM type if the associated representation $\rho_l$ is an induced from a Galois character of $G_M$ for a quadratic imaginary extension $M/\mathbb{Q}$.

Define $K$ to be the fixed field of $\mathrm{Ker}(\omega)$.

**3.1. CM case** First we consider the case when $\pi$ is of CM type. By the work of Langlands and Jacquet (see [2, Theorem 7.4]) for any number field $E$, one can find an automorphic representation $\varphi$ of $GL_2(\mathbb{A}_E)$ and a place $\lambda$ of the minimal field of rationality of $\varphi$ above $l$ such that $\rho_{\varphi,\lambda} \sim \rho_l|_{G_E}$. We take the number field $E$ to be a Galois extension of $\mathbb{Q}$ that contains $K$.

By Brauer's theorem (see [8, Theorems 16 and 19]), we can find $F_i \subset E$ such that $\mathrm{Gal}(E/F_i)$ is solvable and the characters $\chi_i : \mathrm{Gal}(E/F_i) \to \mathbb{C}^\times$ and the integers $m_i$ such that the representation

$$\omega : \mathrm{Gal}(E/\mathbb{Q}) \to \mathrm{Gal}(K/\mathbb{Q}) \to GL_N(\mathbb{C}),$$

can be written as

$$\omega = \sum_{i=1}^{i=k} m_i \,\mathrm{Ind}_{G_{F_i}}^{G_{\mathbb{Q}}} \chi_i,$$

as a virtual sum. We denote also by $\chi_i$ the corresponding character of adele class group $\mathbb{A}_{F_i}^\times/F_i^\times$ by class field theory. We know (see [6, Lemma 11.6]) that there is a cuspidal automorphic representation $\varphi_i$ of $GL_2(\mathbb{A}_{F_i})$ and a prime $\lambda_1$ above $l$ such that $\rho_l|_{G_{F_i}} \sim \rho_{\varphi_i,\lambda_1}$. Then we have

$$\begin{aligned}
L(s,\rho_l\otimes\omega) &= \prod_{i=1}^{i=k} L\left(s,\rho_l\otimes \mathrm{Ind}_{G_{F_i}}^{G_{\mathbb{Q}}}\chi_i\right)^{m_i} = \prod_{i=1}^{i=k} L\left(s,\mathrm{Ind}_{G_{F_i}}^{G_{\mathbb{Q}}}(\rho_l|_{G_{F_i}}\otimes\chi_i)\right)^{m_i}\\
&= \prod_{i=1}^{i=k} L(s-1/2,\varphi_i\otimes\chi_i\circ\det)^{m_i}
\end{aligned}$$

which is a product of $L$-functions that have a meromorphic continuation to the whole complex plane and verify a functional equation. Thus $L(s,\rho_l\otimes\omega)$ can be meromorphically continued to the whole complex plane when $\pi$ is of CM type.

**3.2. Solvable extension of a totally real field and non CM case** We consider the case when $K/\mathbb{Q}$ is a solvable extension of Galois totally real field $F$ and $\pi$ is not of CM type. We prove in this case that $L(s,\rho_l\otimes\omega)$ has a meromorphic continuation and verifies a functional equation. Let $\epsilon_l$ be the $l$-adic cyclotomic character: $\epsilon_l : G_F \to \mathbb{Z}_l^\times$ for $l$ a prime number and $F$ a number field. We want to use the following theorem of R. Taylor:

THEOREM 3.1 (Taylor, [11]). *Suppose that $l$ is an odd prime and that $k/\mathbb{F}_l$ is a finite extension. Let $F$ be a totally real field and $\rho' : G_F \to GL_2(k)$ a continuous representation. Suppose that the following conditions hold:*

(1) *The representation $\rho'$ is irreducible.*

(2) *For every place $v$ of $F$ above $l$, we have*

$$\rho'|_{G_v} \simeq \begin{pmatrix} \epsilon_l\chi_{v,1} & * \\ 0 & \chi_{v,2}\end{pmatrix},$$

*where $G_v$ is the decomposition group above $v$ and $\chi_{v,1}$ and $\chi_{v,2}$ finitely ramified characters.*

(3) *For every complex conjugation $c$, we have* $\det\rho'(c) = -1$.

*Then there is a finite Galois totally real extension $E/F$ in which every prime of $F$ above $l$ splits completely, a cuspidal automorphic representation $\varphi$ of $GL_2(\mathbb{A}_E)$ and a place $\lambda'$ of the minimal field of rationality of $\varphi$ above $l$ such that $\bar\rho_{\varphi,\lambda'} \simeq \rho'|_{G_E}$, where $\rho_{\varphi,\lambda'} : G_E \to GL_2(M_{\lambda'})$ is the continuous irreducible representation associated to $\varphi$, the field $M$ is the minimal field of rationality of $\varphi$ and $\bar\rho_{\varphi,\lambda'}$ is the reduction of $\rho_{\varphi,\lambda'}$ modulo $\lambda'$.*

*Moreover, if $\rho'(I_{v'})$ does not consist of scalar matrices for every place $v'$ of $E$ above $l$ ($I_{v'}$ is the inertia group at $v'$), then the representation $\varphi$ can be chosen such that*

$$\rho_{\varphi,\lambda'}|_{G_{v'}} \simeq \begin{pmatrix} \mu_{v',1} & * \\ 0 & \mu_{v',2} \end{pmatrix},$$

*where $G_{v'}$ is the decomposition group above $v'$ and the characters $\mu_{v',1}$ and $\mu_{v',2}$ are the lifts of $\chi_{v,1}$ and $\chi_{v,2}$ respectively, if $v'$ devides $v$.*

This statement is a combination of Theorem 1.6 and Corollary 1.7 of [11] (in [11, Theorem 1.6 and Corollary 1.7] the representation $\rho'|_{G_v}$ verifies $\det \rho'|_{G_v} = \epsilon_l$, but this condition was imposed only to simplify some notations). In our case where the field $F$ is a Galois extension of $\mathbb{Q}$, one can prove that the field $E$ that appears in the above theorem can be taken to be Galois over $\mathbb{Q}$ by the following argument. By a $M$-HBAV over a field $E$ we mean a triple $(A, i, j)$, where

(1) $A/E$ is an abelian variety of dimension $[M : \mathbb{Q}]$;
(2) $i : O_M \hookrightarrow \mathrm{End}(A/E)$ (algebra homomorphism which takes 1 to identity);
(3) $j$ is an $O_M$-polarization (see [11, page 133] for details).

In his paper ([11, page 136]), Taylor finds a prime $p$, a totally real field $M$, a Galois totally real extension $E/F$ in which every place above $l$ and $p$ splits completely, a quadratic extension $L/F$ in which every place above $l$ and $p$ splits and a $M$-HBAV $(A, i, j)/E$ such that the representation of $G_E$ on $A[\lambda]$ is equivalent to $\rho'|_{G_E}$, and the representation of $G_E$ on $A[\mathfrak{p}]$ is equivalent to $\mathrm{Ind}_{G_L}^{G_F} \psi|_{G_E}$ for some character $\psi$ of $G_L$. Here $\lambda$ and $\mathfrak{p}$ are primes of $M$ over $l$ and $p$. Taking the Galois closure $E^{\mathrm{gal}}$ of $E$, the primes above $p$ and $l$ in $F$ also split completely in $E^{\mathrm{gal}}$ and the above proprieties are verified for $M$-HBAV $(A, i, j)/E^{\mathrm{gal}}$. Thus we obtain the result that we wanted.

We shall now verify the conditions of Theorem 3.1 for some prime number $l$ and $\rho' := \bar{\rho}_l|_{G_F}$. We remark that in order to find a $M$-HBAV $(A, i, j)$ as above, in [11] it was assumed that the image of the representation $\rho'$ is not solvable, but using Proposition 6 below, we can assume this fact.

For $l$ rational prime we say that $\pi$ is $l$-ordinary if $a(l)$ is a unit in $O_{\mathfrak{l}}$. We have the following proposition (see [1, Proposition 2.2]):

PROPOSITION 3.2 (Serre). *Any cuspidal automorphic representation $\pi$ of weight* 2 *as above is $l$-ordinary for a set of primes of density* 1.

Using the same notations as above, we know by the work of Deligne, Mazur and Wiles the following theorem (see [4, Theorem 3.26]):

THEOREM 3.3 (Deligne, Mazur-Wiles). *If $a(l)$ is a unit in $O_{\mathfrak{l}}$ for a prime factor $\mathfrak{l}$ of $l$, then $\rho_l|_{G_l} \simeq \left(\begin{smallmatrix} \epsilon_l\delta_2 & * \\ 0 & \delta_1 \end{smallmatrix}\right)$, where $G_l$ is the decomposition group at $l$, the character $\delta_1$ is unramified and $\delta_2$ is finitely ramified.*

We remark that we can use Proposition 3.2 to find a prime $l$ and a prime ideal $\mathfrak{l}$, such that $a(l)$ is a unit in $O_{\mathfrak{l}}$. Thus, the conditions of Theorem 3.3 are verified and we can choose $l$ such that $\bar{\rho}_l|_{G_F}$ verifies condition (2) of Theorem 3.1. Also, it is known that $\det \rho_l(c) = -1$, so we have $\det \rho_l|_{G_F}(c) = -1$ for all complex conjugations $c$, thus we verified condition (3) of Theorem 3.1.

We now verify condition (1) of Theorem 3.1, that is, the irreducibility of $\bar{\rho}_l|_{G_F}$.

Let $\mathbb{F}_{l^r}$ be the residue field of $O_{\mathfrak{l}}$ mod the maximal ideal and $\bar{\rho}_l$ be the reduction of $\rho_l : G_{\mathbf{Q}} \to GL_2(O_{\mathfrak{l}})$. By a nice result of Ribet (see[7, Section 4]) we have:

PROPOSITION 3.4 (Ribet). *For all but finite $l$, the representation $\overline{\rho}_l$ is full if $\pi$ is not of CM type, that is, $SL_2(\mathbb{F}_l) \subset \bar{\rho}_l(G_{\mathbf{Q}})$.*

Actually Ribet proved a slightly stronger result concerning $SL_2(\mathbb{F}_{l^s})$ for an explicit $0 < s \le r$. Thus we can choose an odd prime $l$ such that $SL_2(\mathbb{F}_l) \subset \bar{\rho}_l(G_{\mathbf{Q}})$. We prove the following proposition:

PROPOSITION 3.5. *For all but finite $l$, the representation $\overline{\rho}_l|_{G_E}$ is full and hence irreducible for any totally real extension $E/F$.*

PROOF. Using Proposition 3.4 we may assume that $l$ is odd and that $SL_2(\mathbb{F}_l) \subset \bar{\rho}_l(G_{\mathbf{Q}})$. For any $x \in G_{\mathbf{Q}}$, $xcx^{-1}$ fixes $E$, because $E$ is totally real. Since $\mathrm{im}(\bar{\rho}_l)$ contains $SL_2(\mathbb{F}_l)$, we have that $\mathrm{im}(\bar{\rho}_l|_{G_E})$ contains all the $SL_2(\mathbb{F}_l)$ conjugates of $\bar{\rho}_l(c)$. We can choose a basis for $\rho_l$ such that $\rho_l(c)$ is the diagonal matrix with diagonal entries 1 and $-1$. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{F}_l)$. Then we have that

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \begin{pmatrix} ad+bc & -2ab \\ 2cd & -bc-da \end{pmatrix} \in \mathrm{im}(\bar{\rho}_l|_{G_E}).\end{aligned}$$

For $a = d = 1$ and $c = 0$ we get that $\left(\begin{smallmatrix} 1 & 2b \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{im}(\bar{\rho}_l|_{G_E})$. Thus, we have

$$\begin{pmatrix} 1 & -2b \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -2b \\ 0 & 1 \end{pmatrix} \in \mathrm{im}(\bar{\rho}_l|_{G_E}).$$

Since 2 is invertible in $\mathbb{F}_l$ and $b$ is an arbitrary element of $\mathbb{F}_l$, we get that $\mathrm{im}(\bar{\rho}_l|_{G_E})$ contains all the elements of the form $\left(\begin{smallmatrix} 1 & e \\ 0 & -1 \end{smallmatrix}\right)$ with $e \in \mathbb{F}_l$. For $a = d = 1$ and $b = 0$ we get $\left(\begin{smallmatrix} 1 & 0 \\ 2c & -1 \end{smallmatrix}\right) \in \mathrm{im}(\bar{\rho}_l|_{G_E})$. Thus,

$$\begin{pmatrix} 1 & 0 \\ 2c & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2c & 1 \end{pmatrix} \in \mathrm{im}(\bar{\rho}_l|_{G_E})$$

and we obtain that $\mathrm{im}(\bar\rho_l|_{G_E})$ contains all the elements of the form $\left(\begin{smallmatrix}1&0\\f&1\end{smallmatrix}\right)$ for $f\in\mathbb{F}_l$. But the elements $\left(\begin{smallmatrix}1&e\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\f&1\end{smallmatrix}\right)$ with $e,f\in\mathbb{F}_l$ generate $SL_2(\mathbb{F}_l)$, so $\bar\rho_l|_{G_E}$ is full. Thus, $\bar\rho_l|_{G_E}$ is irreducible. □

So we proved that we can find $l$ such that $\bar\rho_l|_{G_F}$ is irreducible and verifies conditions (2) and (3) of Theorem 3.1. We fix an $l$ that verifies these proprieties. Thus, we can conclude using Theorem 3.1, that there is a Galois totally real extension $E$ of $\mathbb{Q}$, which contains $F$, a cuspidal automorphic representation $\varphi'$ of $GL_2(\mathbb{A}_E)$ and a place $\lambda'$ of the field of coefficients of $\varphi'$ above $l$ such that $\bar\rho_{\varphi',\lambda'}\sim\bar\rho_l|_{G_E}$.

Now we use the following theorem (this is [10, Theorem 5.1]):

THEOREM 3.6. *Let $F$ be a totally real number field and let $\rho:\mathrm{Gal}(\overline{F}/F)\to GL_2(\overline{\mathbb{Q}}_l)$ be a representation satisfying*:

(1) *$\rho$ is continuous and irreducible*;

(2) *$\rho$ is unramified at all but a finite number of finite places*;

(3) $\det\rho(c)=-1$ *for all complex conjugations $c$*;

(4) $\det\rho=\psi\epsilon_l$, *where $\psi$ is a character of finite order*;

(5) $\rho|_{D_i}\simeq\begin{pmatrix}\psi_1^i&*\\0&\psi_2^i\end{pmatrix}$, *with $\psi_2|_{I_i}$ having finite order, where $D_i$, for $i=1,\dots,t$ are the decomposition groups at the places $v_1,\dots,v_t$ of $F$ dividing $l$ and $I_i\subset D_i$ are the inertia groups*;

(6) *$\bar\rho$ is irreducible and* $\bar\rho|_{D_i}\simeq\begin{pmatrix}\chi_1^i&*\\0&\chi_2^i\end{pmatrix}$, $i=1,\dots,t$, *with* $\chi_1^i\neq\chi_2^i$ *and* $\chi_2^i=\psi_2^i \mod \lambda$;

(7) *there exists an automorphic representation $\pi_0$ of $GL_2(\mathbb{A}_F)$ and a prime $\lambda_0$ of the field of coefficients of $\pi_0$ above $l$ such that* $\bar\rho_{\pi_0,\lambda_0}\simeq\bar\rho$ *and* $\rho_{\pi_0,\lambda_0}|_{D_i}\simeq\begin{pmatrix}\phi_1^i&*\\0&\phi_2^i\end{pmatrix}$, $i=1,\dots,t$, *and* $\chi_2^i=\phi_2^i \mod \lambda$.

*Then we have $\rho\simeq\rho_{\pi,\lambda_1}$ for some automorphic representation $\pi$ and some prime $\lambda_1$ of the field of coefficients of $\pi$ above $l$.*

We show now, that the representation $\rho_l|_{G_E}$ verifies all the conditions of the Theorem 4: the representation $\rho_l|_{G_E}$ is irreducible, since we have chosen $l$ so that $\bar\rho_l|_{G_E}$ is irreducible; conditions (1)–(4) are verified (see the beginning of Section 2); condition (5) is proved by Theorem 3.3 out of our choice of $l$; condition (6) is satisfied also (for a big $l$), since we proved that $\bar\rho_l|_{G_E}$ is irreducible by our choice of $l$ and (using the notations of Theorem 3.3) we have $\epsilon_l\delta_2 \mod \lambda\neq\delta_1 \mod \lambda$ for $l$ sufficiently large, since $\delta_2$ is a finite character independent of $l$ and $\epsilon_l(I_i)$ increases linearly with $l$, while $\delta_1$ is unramified; condition (7) is satisfied by Theorem 3.1 by our choice of $l$. Thus we can choose $l$ such that Theorem 3.6 is verified.

Hence we can apply Theorem 3.6 to find an automorphic representation $\varphi$ of $GL_2(\mathbb{A}_E)$ and a place $\lambda$ of the field of coefficients of $\varphi$ above $l$ such that $\rho_{\varphi,\lambda}\sim\rho_l|_{G_E}$. The field $K$ is a Galois solvable extension of $F$, so the field $KE$ is a Galois solvable extension of $E$. By Langlands base change for Galois cyclic extensions (see [6, Proposition 11.5]), we get a automorphic representation $\varphi''$ of $GL_2(\mathbb{A}_{KE})$ and a place $\lambda''$ of the field of the coefficients of $\varphi''$ above $\lambda$ such that $\rho_{\varphi'',\lambda''} \sim \rho_l|_{G_{KE}}$.

Thus we proved the following theorem, which is a combination of the above discussion and the beginning of Section 3.1:

THEOREM 3.7. *If $\pi$ is a cuspidal automorphic representation of $GL_2(\mathbb{A}_{\mathbf{Q}})$, of weight 2 and $K$ is a solvable extension of a totally real field, then there is a solvable extension of a totally real field $K'$ that contains $K$ and an automorphic representation $\varphi''$ of $GL_2(\mathbb{A}_{K'})$ and a prime $\lambda''$ of the field of coefficients of $\varphi''$ above $l$ such that $\rho_{\pi,l}|_{G_{K'}} \sim \rho_{\varphi'',\lambda''}$.*

In order to prove the meromorphic continuation of $L(s, \rho_l \otimes \omega)$ we can use the same method as in Section 3.1. To find $\varphi_i$ as in Section 3.1 out of $\varphi''$ (in Section 3.1 the representation was $\varphi$), we use the result of Langlands (see [6, Lemma 11.6]) that $\varphi''$ descends to $\varphi_i$, because $KE/F_i$ is a solvable Galois extension. We deduce that $L(s, \rho_l \otimes \omega)$ can be meromorphically continued to the whole complex plane and verifies a functional equation when $K$ is solvable extension of a totally real field and $\pi$ is not of CM type.

Combining this section where we treated non CM type case and Section 3.1 (read the last sentence of Section 3.1) where we treated the CM type case, we can conclude in particular that when the field $K$ is a solvable extension of a Galois totally real field, $L\big(s, \overline{X'(p)}\big)$ has a meromorphic continuation to the complex plane and verifies a functional equation. Thus, we proved the second part of the main theorem from the introduction.

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University of California, Los Angeles
Department of Mathematics
Los Angeles, CA
USA
e-mail: cvirdol@math.ucla.edu

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