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DIRECT SUMS OF TORSION-FREE COVERS

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In [**4**, Theorem 4.1, p. 45], Enochs characterizes the integral domains with the property that the direct product of any family of torsion-free covers is a torsion-free cover. In a setting which includes integral domains as a special case, we consider the corresponding question for direct sums. We use the notion of torsion introduced by Goldie [**5**]. Among commutative rings, we show that the property "any direct sum of torsion-free covers is a torsion-free cover" characterizes the semi-simple Artinian rings.

In what follows, R will denote an associative ring with identity, and modules will be unital left R-modules. The modules M for which $\mathrm{Hom}_R(M, —)$ commute with direct sums have been called Σ-*modules* by Rentschler [**7**]. A systematic study of such modules is given in his thesis [**6**]. It will be useful to note that M is a Σ-module if and only if $\mathrm{Hom}_R(M, —)$ commutes with direct sums of injective modules. We will also make use of the following result of Rentschler [**7**, Remark 7, p. 931]: Over a left Noetherian ring a Σ-module is finitely generated.

Let A and B be R-modules with $A \subseteq B$. We say that A is large (or essential) in B if (0) is the only submodule of B which has trivial intersection with A, and in this case we write $A \subseteq 'B$. The singular submodule is defined as

$$Z(B) = \{x | x \in B \text{ and } (0 : x) \subseteq 'R\}.$$

Throughout, $(\mathscr{G}, \mathscr{F})$ will denote the Goldie torsion theory; i.e., $\mathscr{G}$ is the collection of all modules B with $Z(B) \subseteq 'B$, and $\mathscr{F}$ consists of those modules A with $Z(A) = (0)$. We say that A is $(\mathscr{G} -)$ *torsion-free* if and only if $A \in \mathscr{F}$.

It follows from [**9**, Theorem 2.7, p. 459] and [**3**, Theorem 3] that every R-module has a unique (up to isomorphism) $\mathscr{G}$-torsion-free cover if and only if $Z(R) = (0)$ and R has finite (left) Goldie dimension (i.e., R contains no nontrivial infinite direct sum of left ideals). We now impose these two restrictions on the ring R. Then R has a semi-simple Artinian maximal left quotient ring Q [**8**, Theorem 1.6, p. 115]. For a module A, $T(A)$ will denote its torsion-free covering module. Exactly as in the integral domain case [**4**, Corollary 1, p. 42], we can prove that A is injective if and only if $T(A)$ is injective.

Note that any torsion-free injective R-module A becomes a Q-module in a natural way: if $x \in A$, define $t_x : R \rightarrow A$ via $t_x(r) = rx$. Since $t_x \in \mathrm{Hom}_R(R, A)$, it has an extension $t_x' \in \mathrm{Hom}_R(Q, A)$. Hence, we define $qx = t_x'(q)$ for $q \in Q$.

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Using the torsion-freeness of A, we easily see that this scalar multiplication gives A a left Q-module structure extending the action of R.

LEMMA 1. $\mathrm{Hom}_R(Q, —)$ *commutes with the direct sum of any family* $\{A_i\}$ *of torsion-free injective R-modules.*

Proof. If each A_i is a torsion-free injective module, so is their direct sum, $\bigoplus A_i$, (cf. [**8**, Theorem 2.5, p. 119]). Hence $\bigoplus A_i$ and each A_i are left Q-modules. But for any left Q-module F we have, $\mathrm{Hom}_R(Q, F) = \mathrm{Hom}_Q(Q, F)$. The desired conclusion follows since $\mathrm{Hom}_Q(Q, —)$ commutes with direct sums of Q-modules.

THEOREM 1. *The R-module* $_RQ$ *is a* Σ*-module if and only if the canonical R-homomorphism*

$$\mathrm{Hom}_R(Q, \bigoplus T(E_i) \to \mathrm{Hom}_R(Q, \bigoplus E_i)$$

is a surjection, for each family $\{E_i\}$ *of injective modules.*

Proof. Let $\{E_i\}$ be any family of injective modules. Since each $T(E_i) \to E_i$ is a torsion-free cover, the sequence

$$(0) \to \mathrm{Hom}_R(Q, T(E_i)) \to \mathrm{Hom}_R(Q, E_i) \to (0)$$

is exact for each i. Thus we have the following exact sequences:

$$(1) \qquad (0) \to \bigoplus \mathrm{Hom}_R(Q, T(E_i)) \xrightarrow{\alpha} \bigoplus \mathrm{Hom}_R(Q, E_i) \to (0),$$

$$(2) \qquad (0) \to \mathrm{Hom}_R(Q, \bigoplus T(E_i)) \xrightarrow{\beta} \mathrm{Hom}_R(Q, \bigoplus E_i),$$

$$(3) \qquad (0) \to \bigoplus \mathrm{Hom}_R(Q, T(E_i)) \xrightarrow{\sigma} \mathrm{Hom}_R(Q, \bigoplus T(E_i)),$$

$$(4) \qquad (0) \to \bigoplus \mathrm{Hom}_R(Q, E_i) \xrightarrow{\delta} \mathrm{Hom}_R(Q, \bigoplus E_i).$$

It is straightforward to check that $\delta\alpha = \beta\sigma$. Since E_i is injective, so is $T(E_i)$. Hence σ is an isomorphism by Lemma 1. A simple diagram chase now shows that β is a surjection if and only if δ is an isomorphism. Since δ being an isomorphism is equivalent to Q being a Σ-module, the desired result is established.

LEMMA 2. *For any family* $\{\phi_i : T(E_i) \to E_i\}$ *of torsion-free covers, if* $\bigoplus\phi_i : \bigoplus T(E_i) \to \bigoplus E_i$ *is also a torsion-free cover, then R is left Noetherian.*

Proof. Let $\{E_i\}$ be a family of injective modules. As remarked earlier, each $T(E_i)$ is injective. Therefore, $\bigoplus T(E_i)$ is injective. Thus, if $\bigoplus \phi_i : \bigoplus T(E_i) \to \bigoplus E_i$ is a torsion-free cover, $\bigoplus E_i$ is injective. The proof is completed by Bass' observation [**2**] that left Noetherian rings are characterized by the property that their injective modules are preserved by direct sums.

THEOREM 2. *The following statements are equivalent:*

(a) *For each family* $\{\phi_i : T(E_i) \to E_i\}$ *of torsion-free covers,*

$$\bigoplus \phi_i : \bigoplus T(E_i) \to \bigoplus E_i$$

is a torsion-free cover.

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(b) *R is a left Noetherian ring, and the left R-module Q is a Σ-module.*

Proof. (a) $\Rightarrow$ (b). By Lemma 2, (a) implies that R is left Noetherian. It also follows from (a) that, for any family of modules $\{E_i\}$, the canonical homomorphism $\mathrm{Hom}_R(Q, \bigoplus T(E_i)) \to \mathrm{Hom}_R(Q, \bigoplus E_i)$ is a surjection. Therefore, Theorem 1 implies that Q is a Σ-module.

(b) $\Rightarrow$ (*a*). In [**1**, Proposition 1], it is shown that a torsion-free cover $(T(E), g')$ for a left R-module E can be obtained as follows: $T(E) = \{f \in \mathrm{Hom}_R(Q, I(E)) | f(1) \in E\}$, where $I(E)$ denotes an injective hull of the module E; and $g' : T(E) \to E$ is defined by $g'(f) = f(1)$. For each module E_i in a family of modules $\{E_i\}$, let $g_i : \mathrm{Hom}_R(Q, I(E_i)) \to I(E_i)$ denote the evaluation map; that is, $g_i(f) = f(1)$ for $f \in \mathrm{Hom}_R(Q, I(E_i))$. The restriction g_i' of g_i to the submodule $T(E_i) = \{f \in \mathrm{Hom}_R(Q, I(E_i)) | f(1) \in E_i\}$ is a torsion-free cover of E_i. We shall show that $\bigoplus g_i : \bigoplus T(E_i) \to \bigoplus E_i$ is a torsion-free cover.

Consider the commutative diagram

$$\begin{array}{ccc} \bigoplus \mathrm{Hom}_R(Q, I(E_i)) & \xrightarrow{h} & \mathrm{Hom}_R(Q, \bigoplus I(E_i)) \\ \Big\downarrow{\scriptstyle \bigoplus g_i} & & \Big\downarrow{\scriptstyle g} \\ \bigoplus I(E_i) & \xrightarrow{\text{identity}} & \bigoplus I(E_i), \end{array}$$

where g is the obvious evaluation map and h is the canonical injection. This diagram induces a second commutative diagram

$$\begin{array}{ccc} \bigoplus T(E_i) & \xrightarrow{h'} & T(\bigoplus E_i) \\ \Big\downarrow{\scriptstyle (\bigoplus g_i)'} & & \Big\downarrow{\scriptstyle g'} \\ \bigoplus E_i & \xrightarrow{\text{identity}} & \bigoplus E_i, \end{array}$$

where the $'$ denotes the obvious restrictions. Using this convention we have, $(\bigoplus g_i)' = \bigoplus g_i$. Since R is left Noetherian, $\bigoplus I(E_i)$ is an injective hull of $\bigoplus E_i$. It follows that g' is a torsion-free cover for $\bigoplus E_i$.

We shall complete the proof by showing that h' is an isomorphism. Since h is an injection, so is the restriction h'. To see that h' is a surjection, let $f' \in T(\bigoplus E_i) = \{f \in \mathrm{Hom}_R(Q, \bigoplus I(E_i) | f(1) \in \bigoplus E_i\}$. The assumption that Q is a Σ-module yields an element (f_i) in $\bigoplus \mathrm{Hom}_R(Q, I(E_i))$ such that $h((f_i)) = f'$. Noting that $(f_i(1)) = f'(1) \in \bigoplus E_i$, we see that $(f_i) \in \bigoplus T(E_i)$. Therefore, $h'((f_i)) = h((f_i)) = f'$; so, h' is an isomorphism, and $\bigoplus g_i : \bigoplus T(E_i) \to \bigoplus E_i$ is a torsion-free cover. The proof of Theorem 2 is complete.

COROLLARY. *For a commutative, finite dimensional ring R, with $Z(R) = (0)$ and maximal quotient ring Q, statement* (a) *of Theorem* 2 *is equivalent to $R = Q$.*

Proof. Assuming statement (a) of Theorem 2, the R-module Q is a Σ-module. By the result of Rentschler quoted above, it follows that Q is a finitely generated R-module, say $Q = \sum Rq_i$, where $q_i \in Q$ for each $i = 1, 2, \ldots, n$. Then the ideal $I = \bigcap_{i=1}^{n}(R : q_i)$ is essential in R. Clearly $IQ \subseteq R$. But, by [**8**, Theorem 1.6, p. 155], $IQ = Q$; so $Q = R$.

Over a semi-simple ring, every module is Goldie torsion-free. Thus, the converse is clear.

Remark. It follows that property (a) of Theorem 2 holds over an integral domain if and only if the domain is a field. It should also be pointed out that if $\{\phi_i : T(E_i) \to E_i\}$ is a finite family of torsion-free covers, then $\bigoplus \phi_i : \bigoplus T(E_i) \to \bigoplus E_i$ is a torsion-free cover, regardless of whether (b) of Theorem (2) is valid. The ring of 2×2 upper triangular matrices over a division ring provides a non-commutative example of ring for which the corollary does not hold.

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