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H^∞ Functional Calculus and Mikhlin-Type Multiplier Conditions

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Abstract. Let T be a sectorial operator. It is known that the existence of a bounded (suitably scaled) H^∞ calculus for T, on every sector containing the positive half-line, is equivalent to the existence of a bounded functional calculus on the Besov algebra $\Lambda^\alpha_{\infty,1}(\mathbb{R}^+)$. Such an algebra includes functions defined by Mikhlin-type conditions and so the Besov calculus can be seen as a result on multipliers for T. In this paper, we use fractional derivation to analyse in detail the relationship between $\Lambda^\alpha_{\infty,1}$ and Banach algebras of Mikhlin-type. As a result, we obtain a new version of the quoted equivalence.

1 Introduction

On the basis of the work done by A. McIntosh for Hilbert spaces [12], an H^∞ functional calculus is given for sectorial operators on general Banach spaces [4]. When the operators under discussion are of type 0, the existence of the (suitably scaled) H^∞ calculus is shown to be equivalent to the existence of a functional calculus defined on a certain Besov space $\Lambda^\alpha_{\infty,1}(\mathbb{R}^+)$ [4, Theorem 4.10].

Every n-differentiable function F on $\mathbb{R}^+ := (0,\infty)$ obeying Mikhlin-type conditions like

$$\sup_{t>0} t^k |F^{(k)}(t)| < \infty \quad (k = 0, 1, \dots, n)$$

belongs to $\Lambda^\alpha_{\infty,1}$ if $n > \alpha$; see [4, p. 73], [5, p. 416]. This reinforces the view of the Besov functional calculus as a theorem about multipliers. We study more closely such a link by using fractional derivation, in Section 2 and Section 3 of this paper. The equivalence between the H^∞ calculus and the Besov calculus is proven in [4, Theorem 4.10] through the Paley–Wiener theorem. We show in Section 4 that to go from (bounded) analytic functions to functions in $\Lambda^\alpha_{\infty,1}$, the way is in fact paved with a formula of Cauchy type for fractional derivatives. In Section 5, we apply the results of previous sections to give a characterization of the (scaled) H^∞ calculus in terms of Mikhlin algebras.

On the other hand, the sectorial H^∞ calculus provides us, in general, with operators which are not necessarily bounded [4, 16]. It has been shown [8, 9] that these operators can always be regarded as certain generalized multipliers, or *regular quasi-multipliers* in the sense defined by J. Esterle [7]. It may be worth pointing out that as

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a consequence of the results in Sections 3 and 4, an unbounded calculus is available where operating functions of Mikhlin type yield regular quasimultipliers.

2 Mikhlin Algebras Defined by Fractional Derivation

Let h be a locally integrable function on $\mathbb{R}^+ := (0, \infty)$. For δ such that $0 < \delta < 1$ and $\omega > 0$, we put

$$I_\omega^\delta h(t) := \frac{1}{\Gamma(\delta)} \int_t^\omega (s-t)^{\delta-1} h(s)\, ds,$$

if $0 < t < \omega$, and $I_\omega^\delta h(t) := 0$, if $t \geq \omega$. Then, assuming that the following limit exists, we write

$$h^{(\delta)}(t) := \lim_{\omega\to\infty} \Big(-\frac{d}{dt}\Big) (I_\omega^{1-\delta} h)(t).$$

If α is a positive number with $\alpha = n + \delta$ where $n := [\alpha]$ is the integer part of α, we define

$$h^{(\alpha)}(t) := \Big(\frac{d}{dt}\Big)^n h^{(\alpha-n)}(t), \quad t > 0.$$

Whenever we write $h^{(\alpha)}$, we understand that the limit exists and that $I_\omega^{1-\delta} h$ for $\omega > 0$ and $h^{(\delta)}, \dots, h^{(\alpha-1)}$ are locally absolutely continuous functions on $\mathbb{R}^+$.

The above definition of $h^{(\alpha)}$ is a kind of Riemann–Liouville fractional derivative introduced by Cossar [3] and reconsidered by Trebels [15]. Here, we call $h^{(\alpha)}$ the *Cossar–Riemann–Liouville* derivative of h. In some cases, the definition of $h^{(\alpha)}$ can be done more directly. For example, when h is assumed to be, additionally, of compact support in $\mathbb{R}^+$, then we may use the Fourier transform so that

$$\widehat{h^{(\alpha)}}(\xi) = (-i\xi)^\alpha \hat{h}(\xi), \quad \xi \in \mathbb{R},$$

in the distributional sense.

Let $\mathrm{WBV}_{\infty,\alpha}$ denote the space of functions of *weak bounded variation* formed by the functions in $L^\infty \cap C(\mathbb{R}^+)$ for which there exist $h^{(\alpha)}$ and $\|h\|_{\infty,\alpha} := \|h\|_\infty + \|t^\alpha h^{(\alpha)}(t)\|_\infty < \infty$. The space $\mathrm{WBV}_{\infty,\alpha}$ is a Banach space with respect to the norm $\|\cdot\|_{\infty,\alpha}$. Moreover, it coincides with corresponding (concerning order α and sup-norm) localized Riesz potential spaces and localized Riemann–Liouville spaces. In particular, the norm $\|h\|_{\infty,\alpha}$ is equivalent to the norm

$$\sup_{t>0} \|(\phi h_t)^{(\alpha)}\|_\infty$$

for *any*, fixed, non-negative $\phi \in C_c^{(\infty)}(\mathbb{R}^+)$, and where $h_t(s) := h(ts)$, for a.e. $s, t > 0$, see [2, Theorem 2]. If $h \in \mathrm{WBV}_{\infty,\alpha}$ is of compact support, then

$$h(s) = \frac{(-1)^n}{\Gamma(\alpha)} \int_s^\infty (t-s)^{\alpha-1} h^{(\alpha)}(t)\, dt, \quad \text{a.e. } s > 0,$$

see [2, p. 252]. Note that in particular if $h(s) = 0$ for $s \geq r$, then $h^{(\alpha)}(s) = 0$ for $s \geq r$.

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Although for generic elements of $\mathrm{WBV}_{\infty,\alpha}$ the above formula need not hold, there is also a reproducing formula for derivatives. This is

$$g^{(\nu)}(t) = \frac{(-1)^{[\alpha]-[\nu]}}{\Gamma(\alpha-\nu)} \int_t^\infty (s-t)^{\alpha-\nu-1} g^{(\alpha)}(s)\, ds$$

for a.e. $t > 0$ if $g \in \mathrm{WBV}_{\infty,\alpha}$ and $0 < \nu < \alpha$, see [11, p. 250]. This formula readily implies that $\mathrm{WBV}_{\infty,\beta} \subset \mathrm{WBV}_{\infty,\alpha}$ with $\|t^\alpha g^\alpha(t)\|_\infty \le \|t^\beta g^\beta(t)\|_\infty$, if $g \in \mathrm{WBV}_{\infty,\beta}$ and $0 < \alpha \le \beta$.

For convenience, we are interested here in elements f of $\mathrm{WBV}_{\infty,\alpha}$ with f and $f^{(\alpha)}$ continuous.

Definition 2.1 For $\alpha > 0$, let $\mathcal{M}_\infty^{(\alpha)}$ denote the closure in $\mathrm{WBV}_{\infty,\alpha}$ of the linear subspace $\mathrm{WBV}_{\infty,\alpha} \cap C^{(\infty)}(\mathbb{R}^+)$.

Clearly, $\mathcal{M}_\infty^{(\beta)} \subset \mathcal{M}_\infty^{(\alpha)}$ for $0 < \alpha \le \beta$. It is possible to endow $\mathcal{M}_\infty^{(\alpha)}$ with another norm which is equivalent to $\|\cdot\|_{\infty,\alpha}$ and involves the fractional power operator $(-s\frac{d}{ds})^\alpha$. Let us first recall some well-known facts about such an operator when $\alpha = n \in \mathbb{N}$.

If $F \in C^{(n)}(\mathbb{R})$ and $x \in \mathbb{R}$, we have $(x\frac{d}{dx})^n F(x) = \sum_{j=1}^n c_j x^j F^{(j)}(x)$, for specific coefficients c_j, $j = 1, \dots, n$. If $F(x) := f(e^x)$, where f is a $C^{(n)}$ function on $\mathbb{R}^+$, then

$$F^{(n)}(x) = \sum_{j=1}^n c_j e^{jx} f^{(j)}(e^x) \equiv \sum_{j=1}^n c_j s^j f^{(j)}(s)$$

for every $s = e^x > 0$. That is, the operators $\frac{d^n}{dx^n}$ on $\mathbb{R}$ and $(s\frac{d}{ds})^n$ on $\mathbb{R}^+$ are in correspondence under exponential (or, conversely, logarithmic) change of variable. Indeed, the set of functions $F \in C^{(n)}(\mathbb{R})$ such that $\sup_{j=0,1,\dots,n} \|F^{(j)}\|_\infty < \infty$ is bijective with the set of functions $f \in C^{(n)}(\mathbb{R}^+)$ for which $\sup_{j=0,1,\dots,n} \|f^{(j)}(s)s^j\|_\infty < \infty$. On the other hand, using induction, we obtain that $\sup_{j=0,1,\dots,n} \|f^{(j)}(s)s^j\|_\infty < \infty$ if and only if $\sup_{j=0,1,\dots,n} \|(s\frac{d}{ds})^j f\|_\infty < \infty$. In order to find an analog of this equivalence for fractional derivation, we replace the usual derivation on $\mathbb{R}^+$ with the Marchaud derivation, and use the Hadamard fractional version of $(-s\frac{d}{ds})^n$.

Let $0 < \delta < 1$. If $f \in \mathrm{WBV}_{\infty,\delta}$, then

$$f^{(\delta)}(s) = \frac{1}{\Gamma(-\delta)} \int_s^\infty \frac{f(t) - f(s)}{(t-s)^{1+\delta}}\, dt$$

for every $s > 0$ [11, p. 256]. Recall that the above integral is known as the *Marchaud derivative* of f of order δ [14, p. 110]. For higher order derivation, let $\alpha = n + \delta > 0$ with $n = [\alpha]$ and let f be a $C^{(n+1)}$ function in $\mathcal{M}_\infty^{(n)}$. From the above we get for $s > 0$,

$$f^{(\alpha)}(s) = \frac{1}{\Gamma(-\delta)} \frac{d^n}{ds^n} \int_s^\infty \frac{f(t) - f(s)}{(t-s)^{1+\delta}}\, dt = \frac{1}{\Gamma(-\delta)} \frac{d^n}{ds^n} \Big(s^{-\delta} \int_1^\infty \frac{f(st) - f(s)}{(t-1)^{1+\delta}}\, dt \Big).$$

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In a similar way, if $f \in \mathcal{M}_\infty^{(n)} \cap C^{(n+1)}(\mathbb{R}^+)$, first note that the Hadamard operator of order δ is defined by

$$\Big(-s\frac{d}{ds}\Big)^{\delta} f(s) := \frac{1}{\Gamma(-\delta)} \int_1^\infty [f(st) - f(s)]\, \frac{dt}{t(\log t)^{1+\delta}},$$

see [14, (18.53), (18.56′)]. Thus the action of the Hadamard operator of order α on f can be expressed as

$$\Big(-s\frac{d}{ds}\Big)^{\alpha} f(s) = \frac{1}{\Gamma(-\delta)} \int_1^\infty \Big(-s\frac{d}{ds}\Big)^{n} [f(st) - f(s)]\, \frac{dt}{t(\log t)^{1+\delta}}$$

for every $s > 0$, $\alpha = n + \delta$, $0 < \delta < 1$.

Before passing to the result about equivalent norms, note that for $0 < \delta < 1$, the function $\kappa(t) := t^{-1}(\log t)^{-(1+\delta)} - (t-1)^{-(1+\delta)}$ is integrable on $(1, \infty)$. In fact, we only need to check integrability near $t = 1$, and this is straightforward.

$$\begin{aligned}\int_1^2 |\kappa(t)|\, dt &\le \int_1^2 (1+\delta)\Big(\int_{\log t}^{t-1} u^\delta\, du\Big) \frac{dt}{(t-1)^{1+\delta}(\log t)^{1+\delta}} + \int_1^2 (t-1)^{-\delta}\, dt \\ &\le (1+\delta)\int_1^2 \frac{t-1-\log t}{(t-1)(\log t)^{1+\delta}}\, dt + (1-\delta)^{-1} \equiv C_\delta < \infty.\end{aligned}$$

Put $(\frac{d}{ds})^\alpha f := f^{(\alpha)}$.

Proposition 2.2 *Let $\alpha = n + \delta$, $n = [\alpha]$. Let f be a bounded $C^{(n+1)}$ function on $\mathbb{R}^+$. The following are equivalent.*

(i) $\sup_{s>0} |s^\alpha (\frac{d}{ds})^\alpha f(s)| < \infty$.

(ii) $\sup_{s>0} |(-s\frac{d}{ds})^\beta f(s)| < \infty$, *for every* $0 < \beta \le \alpha$.

Proof Put $\mu_k := \sup_{s>0} |s^k f^{(k)}(s)|$ where $k = 0, 1, \dots, n$. Assuming either (i) or (ii) implies that $\mu_k < \infty$ for all $k = 0, 1, \dots, n$ (if we assume (i), then f is in $\mathcal{M}_\infty^{(\alpha)}$ and so is in $\mathcal{M}_\infty^{(k)}$; if we assume (ii), then we can take $\beta = k$ and proceed by induction).

By Leibniz' rule we get

$$\begin{aligned} s^\alpha \Big(\frac{d}{ds}\Big)^\alpha f(s) &= \frac{s^\alpha}{\Gamma(-\delta)} \sum_{k=0}^n \binom{n}{k} \Big(\frac{d^k}{ds^k} \int_1^\infty \frac{f(st) - f(s)}{(t-1)^{1+\delta}}\, dt\Big) \frac{d^{n-k}}{ds^{n-k}} s^{-\delta} \\ &= \frac{1}{\Gamma(-\delta)} \sum_{k=0}^n a_{k,\delta} \int_1^\infty \frac{f^{(k)}(st)(st)^k - f^{(k)}(s)s^k}{(t-1)^{1+\delta}}\, dt,\end{aligned}$$

where $a_{n,\delta} = 1$. On the other hand,

$$\begin{aligned}\Big(-s\frac{d}{ds}\Big)^\alpha f(s) &= \frac{(-1)^n}{\Gamma(-\delta)} \int_1^\infty \Big(s\frac{d}{ds}\Big)^n [f(st) - f(s)]\, \frac{dt}{t(\log t)^{1+\delta}} \\ &= \frac{(-1)^n}{\Gamma(-\delta)} \sum_{k=1}^n c_k \int_1^\infty \frac{f^{(k)}(st)(st)^k - f^{(k)}(s)s^k}{t(\log t)^{1+\delta}}\, dt,\end{aligned}$$

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where $c_n = 1$.

Let us now consider the difference $(-s\frac{d}{ds})^\alpha f(s) - (-1)^n s^\alpha (\frac{d}{ds})^\alpha f(s)$. In this expression the terms that correspond to $k = 0, 1, \dots, n-1$ are bounded uniformly in s. So are

$$\Big| \int_1^\infty \frac{f^{(k)}(st)(st)^k - f^{(k)}(s)s^k}{(t-1)^{1+\delta}}\, dt \Big| \le \int_1^2 \frac{\int_s^{ts}(\mu_{k+1} + k\mu_k)(du/u)}{(t-1)^{1+\delta}}\, dt + \int_2^\infty \frac{2\mu_k\, dt}{(t-1)^{1+\delta}} < \infty,$$

for $k = 0, 1, \dots, n-1$. Terms of the form

$$\int_1^\infty [f^{(k)}(st)(st)^k - f^{(k)}(s)s^k]\, t^{-1}(\log t)^{-(1+\delta)}\, dt,$$

with $k = 1, \dots, n-1$, are estimated analogously.

Hence the only term which is really significant for comparing both derivatives is

$$\frac{(-1)^n}{\Gamma(-\delta)} \int_1^\infty [f^{(n)}(st)(st)^n - f^{(n)}(s)s^n] \Big\{ \frac{1}{t(\log t)^{1+\delta}} - \frac{1}{(t-1)^{1+\delta}} \Big\}\, dt.$$

This integral is bounded by $2\mu_n \Gamma(-\delta)^{-1} \int_1^\infty \left| t^{-1}(\log t)^{-(1+\delta)} - (t-1)^{-(1+\delta)} \right| dt$, and this is finite as shown prior to the proposition.

Finally, noting that in the direction (i) ⇒ (ii) β can play the role of α, we end the proof. ∎

Corollary 2.3 *The expression* $\sup_{0\le\beta\le\alpha} \sup_{s>0} |(-s\frac{d}{ds})^\beta f(s)|$ *defines a norm in* $\mathcal{M}_\infty^{(\alpha)}$ *which is equivalent to* $\|\cdot\|_{\infty,\alpha}$.

Cossar–Riemann–Liouville derivatives become simpler in certain spaces of absolutely continuous functions of higher order. For $\alpha = n + \delta > 0$, $0 < \delta < 1$, $f \in C_c^{(\infty)}([0,\infty))$ and $s \ge 0$, set

$$W^{-\alpha} f(s) := \frac{1}{\Gamma(\alpha)} \int_s^\infty (t-s)^{\alpha-1} f(t)\, dt,$$

$$W^{\alpha} f(s) := \frac{(-1)^{n+1}}{\Gamma(1-\delta)} \frac{d^{n+1}}{ds^{n+1}} \int_s^\infty (t-s)^{-\delta} f(t)\, dt.$$

Then, with $W^0 f \equiv f$, $(W^\alpha)_{\alpha\in\mathbb{R}}$ is a group (acting on f). In [10], the space of the functions $AC_{2,1}^{(\alpha)}$ has been defined as the completion of $C_c^{(\infty)}([0,\infty))$ in the norm

$$\|f\|_{(\alpha);2,1} := \int_0^\infty \Big(\int_t^{2t} |W^\alpha f(s)\, s^\alpha|^2 \frac{ds}{s} \Big)^{1/2} \frac{dt}{t}.$$

Then for every f in $AC_{2,1}^{(\alpha)}$, the symbol $W^\alpha f$ can be given a precise sense, and $W^\alpha f$ is called the Weyl derivative of f. Note that if h is in $C_c^{(\infty)}([0,\infty))$, then $h^{(\alpha)} =$

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$(-1)^{[\alpha]}W^\alpha h$. We extend this definition to every f in $AC^{(\alpha)}_{2,1}$, and we will use $f^{(\alpha)}$ rather than $W^\alpha f$ in the sequel.

The space $AC^{(\alpha)}_{2,1}$ is a Banach algebra for pointwise multiplication provided that $\alpha > 1/2$. This is proved in [10, Proposition 3.8] as an application of the following Leibniz formula for fractional derivatives [10, Proposition 2.5]:

For $f, g \in C_c^{(\infty)}([0,\infty))$ and $\alpha > 0$,

$$(fg)^{(\alpha)}(s) = f^{(\alpha)}(s)g(s) + f(s)g^{(\alpha)}(s) + (-1)^{[\alpha]+1}\int_s^\infty\int_s^\infty (\varphi^{\alpha-1}_{t,u})'(s) f^{(\alpha)}(t)\, g^{(\alpha)}(u)\, dt du, \tag{2.1}$$

where $\varphi^{\alpha-1}_{r,u}$ is the function defined in [10, p. 313].

We shall need to consider a certain ideal of $AC^{(\alpha)}_{2,1}$.

Definition 2.4 For $\alpha > 0$, let $\mathcal{M}^{(\alpha)}_{2,1}$ denote the completion of $C_c^{(\infty)}(\mathbb{R}^+)$ in the norm

$$\|f\|_{\mathcal{M},\alpha} := \max\Big\{ \int_0^\infty \Big(\int_t^{2t} |f^{(k)}(s)s^k|^2 \,\frac{ds}{s}\Big)^{1/2} \frac{dt}{t} : k = 0, \alpha \Big\}.$$

It is readily seen that $\mathcal{M}^{(\alpha)}_{2,1}$ is a Banach algebra for pointwise multiplication, and an ideal of $AC^{(\alpha)}_{2,1}$ such that $\|fh\|_{\mathcal{M},\alpha} \le C_\alpha \|f\|_{(\alpha);2,1}\, \|h\|_{\mathcal{M},\alpha}$ for every $f \in AC^{(\alpha)}_{2,1}$ and $h \in \mathcal{M}^{(\alpha)}_{2,1}$, if $\alpha > 1/2$ (for this we need to observe that $\|f\|_\infty \le C\|f\|_{(\alpha);2,1}$ if $f \in AC^{(\alpha)}_{2,1}$ and $\alpha > 1/2$ [10, Lemma 3.6]).

We finish this section with two more results about the multiplicative structure of $\mathcal{M}^{(\alpha)}_\infty$ and $\mathcal{M}^{(\alpha)}_{2,1}$.

Theorem 2.5 *For every $\alpha > 0$, $\mathcal{M}^{(\alpha)}_\infty$ is a Banach algebra with respect to pointwise multiplication.*

Proof Take $\phi \in C_c^{(\infty)}(\mathbb{R}_+)$, $\phi \ge 0$, with $\sigma := \max(\operatorname{supp}\phi)$. Let f, g be $C^{(\infty)}$ functions in $\mathcal{M}^{(\alpha)}_\infty$ and let $s, t > 0$. From the Leibniz formula (2.1) we have

$$|(\phi^2 f_t g_t)^{(\alpha)}(s)| \le |(\phi f_t)^{(\alpha)}(s)(\phi g_t)(s)| + |(\phi g_t)^{(\alpha)}(s)(\phi f_t)(s)| + \Big|\int_s^\infty\int_s^\infty (\varphi^{\alpha-1}_{r,u})'(s)(\phi f_t)^{(\alpha)}(r)(\phi g_t)^{(\alpha)}(u)\, drdu\Big|.$$

If $0 < \alpha \le 1/2$, then $(\varphi^{\alpha-1}_{r,u})'(s) \ge 0$ for $s < \min\{r, u\}$ [10, Lemma 2.2], whence the double integral in the previous equality is bounded by

$$\|(\phi f_t)^{(\alpha)}\|_\infty\, \|(\phi g_t)^{(\alpha)}\|_\infty \int_s^\sigma\int_s^\sigma (\varphi^{\alpha-1}_{r,u})'(s)\, drdu.$$

In turn, the above double integral is equal to $c_\sigma(\sigma - s)^\alpha$ for a certain constant c_σ [10, Lemma 2.4], and so it is bounded by $c_\sigma\sigma^\alpha$.

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Now assume that $\alpha > 1/2$. Then $|(\varphi_{r,u}^{\alpha-1})'(s)| \leq c_\alpha (u-s)^{\alpha-2}$ if $s < r < u$ [10, Lemma 2.2]. Take ε such that $0 < \varepsilon < \min\{1, \alpha\}$. Then the double integral at the beginning of the proof is bounded by the sum (up to constant coefficients) of

$$\int_s^\infty \int_s^u (u-s)^{\alpha-2} |(\phi f_t)^{(\alpha)}(r)|\, |(\phi g_t)^{(\alpha)}(u)|\, dr du$$

plus a similar term where u and r exchange places. Since $(u-s)^{\varepsilon-1} \leq (r-s)^{\varepsilon-1}$ for $r \leq u$, the last integral is bounded by

$$\Big(\int_s^\sigma (r-s)^{\varepsilon-1} |(\phi f_t)^{(\alpha)}(r)|\, dr\Big) \Big(\int_s^\sigma (u-s)^{\alpha-\varepsilon-1} |(\phi g_t)^{(\alpha)}(u)|\, du\Big)$$
$$\leq C_\varepsilon (\sigma - s)^\alpha \|(\phi f_t)^{(\alpha)}\|_\infty \, \|(\phi g_t)^{(\alpha)}\|_\infty \leq C_\varepsilon \sigma^\alpha \|(\phi f_t)^{(\alpha)}\|_\infty \, \|(\phi g_t)^{(\alpha)}\|_\infty .$$

The second term in the aforementioned sum is treated similarly.

Hence, for any $\alpha > 0$,

$$\begin{aligned} \|fg\|_{\infty,\alpha} &\approx \sup_{t>0} \|(\phi^2 f_t g_t)^{(\alpha)}\|_\infty \\ &\leq \big(\sup_{t>0} \|(\phi f_t)^{(\alpha)}\|_\infty\big) \big(\sup_{t>0} \|\phi g_t\|_\infty\big) + \big(\sup_{t>0} \|\phi f_t\|_\infty\big) \big(\sup_{t>0} \|(\phi g_t)^{(\alpha)}\|_\infty\big) \\ &\quad + C_\sigma \big(\sup_{t>0} \|(\phi f_t)^{(\alpha)}\|_\infty\big) \big(\sup_{t>0} \|(\phi g_t)^{(\alpha)}\|_\infty\big) \approx C \|f\|_{\infty,\alpha} \|g\|_{\infty,\alpha} \end{aligned}$$

as we wanted to show. ∎

The relationship between Mikhlin algebras and algebras of absolutely continuous functions of higher order is given by the following result.

Theorem 2.6 *For every $\alpha > 1/2$, $\mathcal{M}_{2,1}^{(\alpha)}$ is a Banach $\mathcal{M}_\infty^{(\alpha)}$-module, that is,*

$$\|fg\|_{\mathcal{M},\alpha} \leq C_\alpha \|f\|_{\infty,\alpha} \|g\|_{\mathcal{M},\alpha}$$

for every $f \in \mathcal{M}_\infty^{(\alpha)}$, $g \in \mathcal{M}_{2,1}^{(\alpha)}$.

Proof Take ϕ in $C_c^{(\infty)}([0,\infty))$ with $\phi(s) = 1$ if $0 \leq s \leq 1$, and $\phi(s) = 0$ if $s \geq 2$. Put $\phi_k(s) = \phi(s/k)$ for $s \geq 0$, $k \in \mathbb{N}$. Then $\operatorname{supp} \phi_k \subset [0, 2k]$, $\phi_k(s) = 1$ if $0 \leq s \leq k$ and $\sup_{s\geq 0} |s^m \phi_k^{(m)}(s)| \leq 2^m \|\phi^{(m)}\|_\infty$ for $k, m \in \mathbb{N}$.

Let $f \in \mathcal{M}_\infty^{(\alpha)} \cap C^{(\infty)}(\mathbb{R}^+)$ and let $g \in C_c^{(\infty)}(\mathbb{R}^+)$. Fix k such that $\operatorname{supp} g \subset [0, k]$ and put $\varphi = \phi_k$, so that $fg = (f\varphi)g$. Later on we will apply Leibniz formula (2.1) to $f\varphi$ and g, but before doing so, note that

$$\begin{aligned} \int_s^\infty (t-x)^{\gamma-1} |(f\varphi)^{(\alpha)}(t)|\, dt &\leq \|f\varphi\|_{\infty,\alpha} x^{\gamma-\alpha} \int_1^\infty (s-1)^{\gamma-1} s^{-\alpha}\, ds \\ &= C_{\alpha,\gamma} \|f\varphi\|_{\infty,\alpha} x^{\gamma-\alpha}, \end{aligned} \tag{2.2}$$

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for all $x > 0$ and whenever $0 < \gamma < \alpha$. Also, if $\tilde{g}(x) := \int_x^\infty (u-x)^{\alpha-1}|g^{(\alpha)}(u)|\,\frac{du}{\Gamma(\alpha)}$ for $x \geq 0$, then $\tilde{g} \in AC^{(\alpha)}_{2,1}$ and $\|\tilde{g}\|_{(\alpha);2,1} = \|g\|_{(\alpha);2,1}$ [10, p. 325].

Now, in formula (2.1) for $f\varphi$ and g the double integral is bounded by

$$C_1 \int_x^\infty \int_x^u (u-x)^{\alpha-2}|(f\varphi)^{(\alpha)}(t)|\,dt|g^{(\alpha)}(u)|\,du \\ + C_2 \int_x^\infty \int_x^t (t-x)^{\alpha-2}|g^{(\alpha)}(u)|\,du\,|(f\varphi)^{(\alpha)}(t)|dt \equiv \text{(I)} + \text{(II)}.$$

see [10, p. 313, 314]. To estimate (I), we choose ε such that $1/2 < \varepsilon < \min(1, \alpha)$. Then, as in [10, p. 325],

$$\begin{aligned} \text{(I)} &\leq C_1 \int_x^\infty \int_x^\infty (t-x)^{\varepsilon-1}|(f\varphi)^{(\alpha)}(t)|dt\,(u-x)^{\alpha-\varepsilon-1}|g^{(\alpha)}(u)|\,du \\ &\leq C_1'\|f\varphi\|_{\infty,\alpha}\, x^{\varepsilon-\alpha}\,\tilde{g}^{(\varepsilon)}(x), \quad x > 0, \end{aligned}$$

where the second inequality is obtained from (2.2) with $\gamma = \varepsilon$.

Analogously, for δ such that $0 < \delta < \min\{1, \alpha - (1/2)\}$, we have

$$\begin{aligned} \text{(II)} &\leq C_2 \int_x^\infty \int_x^\infty (u-x)^{\delta-1}|g^{(\alpha)}(u)|\,du(t-x)^{\alpha-\delta-1}|(f\varphi)^{(\alpha)}(t)|\,dt \\ &\leq C_2'\|f\varphi\|_{\infty,\alpha}\,\tilde{g}^{(\alpha-\delta)}(x)\,x^{-\delta}, \quad x > 0. \end{aligned}$$

Hence, for every $x > 0$,

$$|(fg)^{(\alpha)}(x)|\,x^\alpha \leq |(f\varphi)^{(\alpha)}(x)|\,x^\alpha\,|g(x)| + |(f\varphi)(x)|\,|g^{(\alpha)}(x)|\,x^\alpha \\ + \big(C\,x^\varepsilon\,\tilde{g}^{(\varepsilon)}(x) + C'\,x^{\alpha-\delta}\,\tilde{g}^{(\alpha-\delta)}(x)\big)\,\|f\varphi\|_{\infty,\alpha}$$

and therefore

$$\begin{aligned} \|fg\|_{(\alpha);2,1} &\leq \|f\varphi\|_{\infty,\alpha}\,\|g\|_{(0);2,1} + \|f\varphi\|_\infty\,\|g\|_{(\alpha);2,1} \\ &\quad + C\|f\varphi\|_{\infty,\alpha}\big(\|\tilde{g}\|_{(\varepsilon);2,1} + \|\tilde{g}\|_{(\alpha-\delta);2,1}\big) \\ &\leq C\|f\varphi\|_{\infty,\alpha}\|g\|_{\mathcal{M}^\alpha_{2,1}}, \end{aligned}$$

in particular because $\varepsilon, \alpha - \delta > 1/2$ [10, Proposition 3.7(i)] . Moreover, $\|f\varphi\|_{\infty,\alpha} \leq C\|f\|_{\infty,\alpha}\,\|\varphi\|_{\infty,\alpha}$ and therefore $\|\varphi\|_{\infty,\alpha} \leq C'\|\varphi\|_{\infty,n+1} \leq C'2^{n+1}\|\varphi^{(n+1)}\|_\infty \equiv C_n$ where $n = [\alpha]$. Thus we have that $\|fg\|_{(\alpha);2,1} \leq C\|f\|_{\infty,\alpha}\|g\|_{\mathcal{M}^\alpha_{2,1}}$. Finally,

$$\int_0^\infty \Big(\int_y^{2y} |(fg)(x)|\,\frac{dx}{x}\Big)^{1/2}\frac{dy}{y} \leq \|f\|_\infty\,\|g\|_{(0);2,1} \leq \|f\|_{\infty,\alpha}\,\|g\|_{\mathcal{M}^\alpha_{2,1}}.$$

In conclusion we have obtained that $\|fg\|_{\mathcal{M}^\alpha_{2,1}} \leq C\|f\|_{\infty,\alpha}\,\|g\|_{\mathcal{M}^\alpha_{2,1}}$. ■

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3 Mikhlin Algebras and Besov Spaces

For $\alpha > 0$ let $\Lambda^{\alpha}_{\infty,1}(\mathbb{R}^+)$ denote the Besov space formed by all bounded continuous functions f on $\mathbb{R}^+$ such that $\|f\|_{\Lambda,\alpha} < \infty$, where

$$\|f\|_{\Lambda,\alpha} = \sum_{k=-\infty}^{\infty} 2^{|k|\alpha} \|F * \check{\phi}_k\|_\infty .$$

Here $F(x) := f(e^x)$, $x \in \mathbb{R}$, and $\{\phi_k\}_k$ is a suitable family of functions in $C_c(\mathbb{R})$, see [4, p. 73], [5, p. 415].

It is clear that $\Lambda^{\beta}_{\infty,1}(\mathbb{R}^+)$ is contained in $\Lambda^{\alpha}_{\infty,1}(\mathbb{R}^+)$ whenever $\beta \geq \alpha$, and that the inclusion $\Lambda^{\beta}_{\infty,1}(\mathbb{R}^+) \hookrightarrow \Lambda^{\alpha}_{\infty,1}(\mathbb{R}^+)$ is a *contraction*. Moreover, the space $\Lambda^{\alpha}_{\infty,1}(\mathbb{R}^+)$ is a Banach algebra for pointwise multiplication [1, p. 163], and this algebra can be described alternatively as the set of functions f on $\mathbb{R}^+$ of $C^{(n)}$ class such that

$$\|f\|_\infty + \int_0^\infty \frac{\|F^{(n)}(x+y) - F^{(n)}(x)\|_\infty}{y^{1+\delta}}\, dy < \infty,$$

where $n = [\alpha]$, $\delta = \alpha - n$ and $F = f \circ \exp$ [13, pp. 9, 11]. The above sum defines a norm in $\Lambda^{\alpha}_{\infty,1}(\mathbb{R}^+)$ which is equivalent to the norm $\|f\|_{\Lambda,\alpha}$. After exponential change of variable in the integral, we will use that norm in the form

$$\|f\|_\infty + \int_1^\infty \frac{\|\sum_{j=1}^n c_j \{f^{(j)}(st)(st)^j - f^{(j)}(s)s^j\|_\infty}{(\log t)^{1+\delta}}\, \frac{dt}{t},$$

where c_j are the Stirling numbers defined by $(x\frac{d}{dx})^n = \sum_{j=1}^n c_j x^j \frac{d^j}{dx^j}$.

As part of the motivation for [4, Theorem 4.10], it has been pointed out there that $\mathcal{M}^{(k)}_\infty \hookrightarrow \Lambda^{\alpha}_{\infty,1}(\mathbb{R}^+)$, provided that k is a natural number with $k > \alpha$. We will now refine this inclusion.

Theorem 3.1 *Let $\alpha > 0$.*

(i) $\mathcal{M}^{(\beta)}_\infty \hookrightarrow \Lambda^{\alpha}_{\infty,1}(\mathbb{R}^+)$ *for every* $\beta > \alpha$.

(ii) $\Lambda^{\alpha}_{\infty,1}(\mathbb{R}^+) \hookrightarrow \mathcal{M}^{(\alpha)}_\infty$.

Proof (i) Let $\alpha = n+\delta$, $n = [\alpha]$, $0 < \delta < 1$. Take $\beta > \alpha$ and f in $\mathcal{M}^{(\beta)}_\infty \cap C^{(\infty)}(\mathbb{R}^+)$. For $k = 1, \dots, n$ and $s > 0$, put

$$I_k = \int_1^\infty \frac{\|f^{(k)}(st)(st)^k - f^{(k)}(s)s^k\|_\infty}{(\log t)^{1+\delta}}\, \frac{dt}{t}.$$

If $1 \leq k \leq n-1$,

$$\begin{aligned} I_k &\leq \int_1^2 \Big(\sup_{s>0} \int_s^{st} |f^{(k+1)}(u)u^k + kf^{(k)}(u)u^{k-1}|\, du \Big) \frac{dt}{t(\log t)^{1+\delta}} + \int_2^\infty \frac{2\|f\|_{\infty,k}}{(\log t)^{1+\delta}}\, \frac{dt}{t} \\ &\leq \int_1^2 \frac{\|f\|_{\infty,k+1} + k\|f\|_{\infty,k}}{(\log t)^{1+\delta}} \Big(\sup_{s>0} \int_s^{st} \frac{du}{u} \Big) \frac{dt}{t} + C_\delta \|f\|_{\infty,k} \\ &= C'_\delta \|f\|_{\infty,k+1} + C''_\delta \|f\|_{\infty,k} \leq C_\delta \|f\|_{\infty,\beta}. \end{aligned}$$

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If $k = n$ and $t > 2$, we have as before $\|f^{(n)}(st)(st)^n - f^{(n)}(s)s^n\|_\infty \le C\|f\|_{\infty,\beta}$. For $k = n$ and $1 < t \le 2$ we use the representation

$$f^{(n)}(ts) - f^{(n)}(s) = \frac{\pm 1}{\Gamma(\beta - n)} \int_0^\infty \{(u - ts)_+^{\beta-n-1} - (u - s)_+^{\beta-n-1}\} f^{(\beta)}(u)\, du,$$

if $s > 0$, which holds even for $n = 0$, see [11, pp. 250, 252]. Then

$$\begin{aligned} &|f^{(n)}(st)(st)^n - f^{(n)}(s)s^n| \\ &\quad = s^n |f^{(n)}(st)(t^n - 1) + f^{(n)}(st) - f^{(n)}(s)| \\ &\quad \le \|f\|_{\infty,n} t^{-n}(t^n - 1) \\ &\qquad + \frac{s^n}{\Gamma(\beta - n)} \Big| \int_0^\infty \{(u - ts)_+^{\beta-n-1} - (u - s)_+^{\beta-n-1}\} f^{(\beta)}(u)\, du \Big|. \end{aligned}$$

The module of the integral is in turn bounded by $\|f\|_{\infty,\beta}$ times the sum of

$$\int_s^{ts} (u - s)^{\beta-n-1} u^{-\beta}\, du \le (\beta - n)^{-1} s^{-n} (t - 1)^{\beta-n}$$

and

$$\begin{aligned} s^{-n} \int_t^\infty [(r - t)^{\beta-n-1} - (r - 1)^{\beta-n-1}] r^{-\beta}\, dr &= s^{-n} t^{-\beta} \frac{(t - 1)^{\beta-n}}{\beta - n} \\ &\quad + s^{-n} \frac{\beta}{\beta - n} \int_t^\infty \Big(\int_1^t (\beta - n)(r - u)^{\beta-n-1}\, du \Big) r^{-(\beta+1)}\, dr. \end{aligned}$$

Without loss of generality we can assume that $\beta \le n + 1$, and therefore we obtain $\int_1^t (r - u)^{\beta-n-1} du \le \int_1^t (t - u)^{\beta-n-1} du = (\beta - n)^{-1}(t - 1)^{\beta-n}$. Thus, in summary, we have

$$\|f^{(n)}(st)(st)^n - f^{(n)}(s)s^n\|_\infty \le C_{\beta,n} \|f\|_{\infty,\beta} (t^n - 1 + (t - 1)^{\beta-n}),$$

whenever $1 < t \le 2$.

Hence,

$$\begin{aligned} I_n &\le C\Big(\int_1^2 \frac{t^n - 1 + (t - 1)^{\beta-n}}{(\log t)^{1+\delta}} \frac{dt}{t} \Big) \|f\|_{\infty,\beta} + \int_2^\infty \frac{2\|f\|_{\infty,\beta}}{(\log t)^{1+\delta}} \frac{dt}{t} \\ &\le C\|f\|_{\infty,\beta}, \end{aligned}$$

since $\beta - n > \alpha - n = \delta$. (Note that if $n = 0$, the first term is missing.)

In conclusion, we have proved that $\|F\|_{\Lambda,\alpha} \le C\|F\|_{\infty,\beta}$, for $\beta > \alpha$.

(ii) The elements of $\Lambda^\alpha_{\infty,1}(\mathbb{R}^+)$ can be approximated in its norm by analytic functions on $\mathbb{R}^+$ [4, p. 74]. So it is enough to check the required estimates for $C^{(\infty)}$

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functions in $\Lambda^{\alpha}_{\infty,1}(\mathbb{R}^+)$. For such a function f, we use the former Proposition 2.2. Thus,

$$\begin{aligned}\Big|\Big(-s\frac{d}{ds}\Big)^{\alpha} f(s)\Big| &= \frac{1}{\Gamma(-\delta)}\Big|\int_1^{\infty}\Big(s\frac{d}{ds}\Big)^{n}[f(st)-f(s)]\,\frac{dt}{t(\log t)^{1+\delta}}\Big| \\ &\le \frac{1}{\Gamma(-\delta)}\int_1^{\infty}\Big|\sum_{j=1}^{n} c_j[f^{(j)}(st)(st)^j - f^{(j)}(s)s^j]\Big|\,\frac{dt}{t(\log t)^{1+\delta}} \\ &\le \frac{1}{\Gamma(-\delta)}\int_1^{\infty}\Big\|\sum_{j=1}^{n} c_j[f^{(j)}(st)(st)^j - f^{(j)}(s)s^j]\Big\|_{\infty}\,\frac{dt}{t(\log t)^{1+\delta}},\end{aligned}$$

and therefore $\sup_{s>0}|(-s\frac{d}{ds})^{\alpha} f(s)| \le C\|f\|_{\Lambda,\alpha}$.

Analogouosly, if $0 \le \beta \le \alpha$, $\sup_{s>0}|(-s\frac{d}{ds})^{\beta} f(s)| \le C\|f\|_{\Lambda,\beta} \le C\|f\|_{\Lambda,\alpha}$ since $\Lambda^{\alpha}_{\infty,1}(\mathbb{R}^+) \hookrightarrow \Lambda^{\beta}_{\infty,1}(\mathbb{R}^+)$ is a contraction. In conclusion, $\Lambda^{\alpha}_{\infty,1}(\mathbb{R}^+) \hookrightarrow \mathcal{M}^{(\alpha)}_{\infty}$, as wanted. ■

Remark 3.2. It is noticed in [4, p. 73] that for every integer $m > \alpha$, the inclusion $\mathcal{M}^{(m)}_{\infty} \hookrightarrow \Lambda^{\alpha}_{\infty,1}(\mathbb{R}^+)$ can be established using the norm

$$\|f\|_{\Lambda,\alpha} = \sum_{k=-\infty}^{\infty} 2^{|k|\alpha}\|F * \check{\phi}_k\|_{\infty}$$

in $\Lambda^{\alpha}_{\infty,1}(\mathbb{R}^+)$. The way to do this is to apply the estimate $\|\mathcal{I}^m\check{\phi}_k\|_1 \le C_m 2^{-|k|m}$ in the convolution $F * \check{\phi}_k = F^{(m)} * \mathcal{I}^m\check{\phi}_k$. Here $\mathcal{I}$ is the integration operator $\mathcal{I}h(x) := \int_{-\infty}^{x} h(y)\,dy$ on $\mathbb{R}$. This argument also works for fractional $\beta > \alpha$, but it turns out to be more involved. In this case it is also convenient to replace the usual derivation with the Hadamard derivation $(-s(d/ds))^{\beta}$, as well as to replace $\mathcal{I}$ with the corresponding adjoint operator of $(-s(d/ds))^{\beta}$ on $\mathbb{R}^+$.

4 Algebras of Analytic Functions on Sectors

The algebras which we consider here are those linked to the H^{∞} calculus such as they are introduced in [4], see also [16]. We present these algebras under a slightly different viewpoint which is more suitable for our aims. In this section we show that such algebras are closely related to the Mikhlin algebras of Section 2, via a Cauchy formula for fractional derivatives.

For τ such that $0 < \tau < \pi$, set $S_\tau = \{\lambda \in \mathbb{C}\setminus\{0\} : |\arg(\lambda)| < \tau\}$, where $\arg(\lambda)$ is the argument of λ which takes values in $[-\pi,\pi)$. Let $H^{\infty}(S_\tau)$ be the usual Banach algebra of bounded analytic functions on S_τ with norm $\|\cdot\|_{\infty}$ (reference to the angle τ is omitted in this norm; it will not cause any trouble). Let $\mathcal{A}_b(S_\tau)$ denote the Banach subalgebra of $H^{\infty}(S_\tau)$ formed by all functions of $H^{\infty}(S_\tau)$ which are continuous on $\overline{S_\tau}\setminus\{0\}$. Set $\psi(\lambda) := \lambda(1+\lambda)^{-2}$, if $\lambda \in S_\tau$. For $\delta > 0$, we define $\mathcal{A}^{\delta}_0(S_\tau)$ as the subalgebra of all functions f of $\mathcal{A}_b(S_\tau)$ for which $f(\lambda)\psi^{-\delta}(\lambda) \to 0$ as $|\lambda| \to \infty$ or $|\lambda| \to 0$. Endowed with the norm $\|f\|_{\delta,\infty} := \|f\psi^{-\delta}\|_{\infty}$, $\mathcal{A}^{\delta}_0(S_\tau)$ is a Banach algebra

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and a Banach module of $\mathcal{A}_b(S_\tau)$. Moreover, $\mathcal{A}_b(S_\tau)$ is the multiplier algebra of $\mathcal{A}_0^\delta(S_\tau)$ for every $\delta > 0$, $\mathcal{A}_b(S_\tau) = \mathrm{Mul}(\mathcal{A}_0^\delta(S_\tau))$ [9].

Extensions of Cauchy formulae on suitable paths are tools usually considered to define complex fractional derivatives [14, p. 422]. The following lemma is a sort of Cauchy formula for Weyl and Cossar–Weyl derivatives of functions in $\mathcal{A}_b(S_\tau)$. In the statement and proof, the mapping $z \mapsto z^{\alpha+1} = |z|^{\alpha+1}e^{(\alpha+1)\arg(z)}$, $\alpha > 0$, corresponds to the continuous branch of the argument on $\mathbb{C} \setminus (-\infty, 0]$ defined by $\arg(z^{\alpha+1}) = 0$ when $z > 0$.

Lemma 4.1 *Let $\alpha > 0$. For every $0 < \tau < \pi/2$ and $h \in \mathcal{A}_b(S_\tau)$ there exists $h^{(\alpha)}$ and we have*

$$h^{(\alpha)}(x) = (-1)^{[\alpha]+1}\frac{\Gamma(\alpha+1)}{2\pi i}\int_{\gamma(\tau,x)} \frac{h(\lambda)}{(x-\lambda)^{\alpha+1}}\,d\lambda + (-1)^{[\alpha]+1}\frac{\sin\alpha\pi}{\pi}\Gamma(\alpha+1)\int_{(1+\sin\tau)x}^{+\infty}\frac{h(u)}{(u-x)^{\alpha+1}}\,du$$

for each $x > 0$, where $\gamma(\tau, x)$ is the circle $|\lambda - x| = (\sin\tau)x$ positively oriented.

Proof If $h \in \mathcal{A}_b(S_\tau)$, then $h \in \mathcal{M}_\infty^{(m+1)}$ for all integer m. This follows from the Cauchy formula $h^{(m+1)}(x) = (2\pi i)^{-1}(m+1)!\int_{\gamma(\tau,x)} h(\lambda)(\lambda - x)^{-(m+2)}d\lambda$, $x > 0$. We will use this fact for $n = [\alpha]$. So in particular we have

$$h^{(\alpha)}(x) = \frac{-1}{\Gamma(n+1-\alpha)}\int_0^\infty y^{n-\alpha}h^{(n+1)}(x+y)\,dy$$

for every $x > 0$. We want to represent $h^{(n+1)}(x+y)$ as an integral on a path independent of y. Fix $x > 0$. For $R > 0$, set $\gamma(R,\tau) := \{\lambda : |\lambda| = R, |\arg(\lambda)| \le \tau\}$, $\rho^\pm(\tau,x) := \{\lambda : (\cos\tau)x \le |\lambda|, \arg(\lambda) = \pm\tau\}$ and denote by $\gamma^l(\tau,x)$ the sub-arc of $\gamma(\tau,x)$ which joins $(\cos\tau)xe^{i\tau}$ and $(\cos\tau)xe^{-i\tau}$ to the left of x. Take $y > 0$.

For $R > 2(x+y)$,

$$\Big|\int_{\gamma(R,\tau)}\frac{h(\lambda)}{[\lambda-(x+y)]^{n+2}}d\lambda\Big| \le C\frac{R}{[R-(x+y)]^{n+2}}\|h\|_\infty \to_{R\to\infty} 0,$$

and therefore the Cauchy formula implies that

$$h^{(n+1)}(x+y) = \frac{(-1)^n(n+1)!}{2\pi i}\int_{\Lambda(\tau,x)}\frac{h(\lambda)}{(x+y-\lambda)^{n+2}}d\lambda,$$

where $\Lambda(\tau,x) = \rho^+(\tau,x) \cup \gamma^l(\tau,x) \cup \rho^-(\tau,x)$ is positively oriented.

Put $z = x+y$. Then,

$$\begin{aligned}\int_{\rho^+(\tau,x)}\frac{|h(\lambda)|}{|z-\lambda|^{n+2}}|d\lambda| &\le \|h\|_\infty\int_{(\cos\tau)x}^\infty\frac{dr}{|z-re^{i\tau}|^{n+2}}\\ &= Cz^{-(n+1)}\int_0^{\frac{z}{(\cos\tau)x}}\frac{s^n}{|s-e^{i\tau}|^{n+2}}\,ds\\ &\le C_{n,\tau}z^{-(n+1)}\int_0^\infty\frac{ds}{|s-e^{i\tau}|^2} \equiv Cz^{-(n+1)}.\end{aligned}$$

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A similar estimate is obtained on $\rho^+(\tau, x)$. Further,

$$\int_{\gamma^l(\tau,x)} \frac{|h(\lambda)|}{|z-\lambda|^{n+2}}|d\lambda| \le C(x)[z-(\cos\tau)x]^{-(n+2)}.$$

Then Fubini's theorem can be applied to get

$$h^{(\alpha)}(x) = \frac{(-1)^{n+1}\,(n+1)!}{2\pi i\Gamma(n+1-\alpha)} \int_{\Lambda(\tau,x)} \int_0^\infty \frac{y^{n-\alpha}}{(x+y-\lambda)^{n+2}}\, dy h(\lambda) d\lambda.$$

The integral in the variable y defines an analytic mapping in $\lambda \in \mathbb{C} \setminus [x, +\infty)$ and then its value is readily obtained, using the identity principle, as $c_n(x-\lambda)^{-(\nu+1)}$, with $c_n = \int_0^\infty r^{n-\nu}(1+r)^{-(n+2)}dr = B(n-\alpha+1, \alpha+1)$. Thus we have that

$$h^{(\alpha)}(x) = (-1)^{n+1}\frac{\Gamma(\alpha+1)}{2\pi i}\int_{\Lambda(\tau,x)} \frac{h(\lambda)}{(x-\lambda)^{\alpha+1}}\, d\lambda$$

for every $x > 0$.

Take $\varepsilon > 0$. By $z_\varepsilon^\pm$ we denote the intersection point of $\gamma(\tau, x)$ and the line $\Im\lambda = \pm\varepsilon$ such that $\Re z_\varepsilon^\pm > x$. Put $\sigma(\varepsilon)^\pm := \{\lambda : \Im\lambda = \pm\varepsilon, \Re\lambda \ge \Re z_\varepsilon^\pm\}$. Let $\gamma_\pm^r(\tau, x)$ be the sub-arc of $\gamma(\tau, x)$ joining $(\cos\tau)xe^{\pm i\tau}$ and $z_\varepsilon^\pm$ in the shortest way. Application of Cauchy's theorem to suitable domains implies now that

$$h^{(\alpha)}(x) = (-1)^{n+1}\frac{\Gamma(\alpha+1)}{2\pi i}\int_{K(\tau,x)} \frac{h(\lambda)}{(x-\lambda)^{\alpha+1}}\, d\lambda, \quad x > 0,$$

where $K(\tau, x)$ is the path $K(\tau, x) = \sigma(\varepsilon)^+ \cup \gamma_+^r(\tau, x) \cup \gamma^l(\tau, x) \cup \gamma_-^r(\tau, x) \cup \sigma(\varepsilon)^-$, positively oriented. It is readily seen that

$$\lim_{\varepsilon\to 0^+} \int_{\sigma(\varepsilon)^\pm} h(\lambda)(x-\lambda)^{-(\alpha+1)}\, d\lambda = \mp e^{\pm(\alpha+1)\pi i} \int_{(1+\sin\tau)x}^{+\infty} h(u)(u-x)^{-(\alpha+1)}\, du,$$

and from this we obtain that

$$h^{(\alpha)}(x) = (-1)^{n+1}\frac{\Gamma(\alpha+1)}{2\pi i}\int_{\gamma(\tau,x)} \frac{h(\lambda)}{(x-\lambda)^{\alpha+1}} d\lambda \\ + (-1)^{n+1}\frac{\sin\alpha\pi}{\pi}\Gamma(\alpha+1)\int_{(1+\sin\tau)x}^{+\infty} \frac{h(u)}{(u-x)^{\alpha+1}}\, du. \quad \blacksquare$$

The lemma tells us in particular that $H^\infty(S_\tau)$ is contained in $\mathcal{M}_\infty^{(\nu)}$. More precisely, we have the following.

Proposition 4.2 *Let $\alpha, \delta > 0$, and let τ be such that $0 < \tau < \pi$.*

(i) *$\mathcal{A}_b(S_\tau) \hookrightarrow \mathcal{M}_\infty^{(\alpha)}$, with $\|h\|_{\infty,\alpha} \le C\tau^{-\alpha}\|h\|_\infty$ for every $h \in \mathcal{A}_b(S_\tau)$.*

(ii) *$\mathcal{A}_0^\delta(S_\tau) \hookrightarrow \mathcal{M}_{2,1}^{(\alpha)}$, with $\|h\|_{\mathcal{M},\alpha} \le C_\delta\tau^{-\alpha}\|h\|_{\delta,\infty}$ for every $h \in \mathcal{A}_0^\delta(S_\tau)$. Moreover, $\mathcal{A}_0^\delta(S_\tau)$ generates a dense ideal of $\mathcal{M}_{2,1}^{(\alpha)}$.*

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Proof (i) This is immediately obtained from the formula in Lemma 4.1.

(ii) Take τ such that $0 < \tau < \pi/6$ and put $\kappa := 1 + \sin\tau$. We need to estimate the functional $L_\alpha(\,\cdot\,) \equiv \int_0^\infty \big(\int_y^{2y} |\cdot|^2 x^{2\alpha-1} dx\big)^{1/2} \frac{dy}{y}$ on each integral in the Cauchy formula of $h^{(\alpha)}$. First, note that

$$\Big|\int_{\gamma(\tau,x)} \frac{h(\lambda)}{(x-\lambda)^{\alpha+1}}\,d\lambda\Big| \le 2\pi(x\sin\tau)^{-\alpha}\|h\|_{\delta,\infty}\big(\max_{\lambda\in\gamma(\tau,x)}|\psi^\delta(\lambda)|\big),$$

where $|\psi^\delta(\lambda)| \le C_\delta \min(|\lambda|^{-\delta}, |\lambda|^\delta)$ and $(x/2) \le |\lambda| \le (3x/2)$ (since $0 < \tau < \pi/3$) for each $\lambda \in \gamma(\tau,x)$. Thus

$$\begin{aligned} L_\alpha\Big(\int_{\gamma(\tau,x)} \frac{h(\lambda)}{(x-\lambda)^{\alpha+1}}\,d\lambda\Big) &\le C_\delta \frac{2\pi\|h\|_{\delta,\infty}}{(\sin\tau)^\alpha}\Big[\int_0^{1/3}\Big(\frac{3}{2}\Big)^\delta \Big(\int_y^{2y} x^{2\delta-1}\,dx\Big)^{1/2}\frac{dy}{y} \\ &\quad + \int_{1/3}^2 6^\delta \Big(\int_y^{2y}\frac{dx}{x}\Big)^{1/2}\frac{dy}{y} \\ &\quad + \int_2^\infty 2^\delta \Big(\int_y^{2y} x^{-(2\delta+1)}\,dx\Big)^{1/2}\frac{dy}{y}\Big] \\ &= C_\delta(\sin\tau)^{-\alpha}\|h\|_{\delta,\infty}. \end{aligned}$$

Now, for the second integral entering the Cauchy formula of $h^{(\alpha)}$, we have

$$\begin{aligned} &L_\alpha\Big(\int_{\kappa.x}^\infty \frac{h(u)}{(x-u)^{\alpha+1}}du\Big) \\ &\le \|h\|_{\delta,\infty} L_\alpha\Big(\int_{\kappa.x}^\infty \frac{u^\delta}{(1+u)^{2\delta}}\frac{du}{(u-x)^{\alpha+1}}\Big) \\ &= \|h\|_{\delta,\infty} L_\delta\Big(\int_\kappa^\infty \frac{r^\delta}{(1+xr)^{2\delta}}\frac{dr}{(r-1)^{\alpha+1}}\Big) \\ &\le \|h\|_{\delta,\infty}\int_\kappa^\infty\int_0^\infty\Big(\int_{ry}^{2ry}\frac{z^{2\delta}}{(1+z)^{4\delta}}\frac{dz}{z}\Big)^{1/2}\frac{dy}{y}\frac{dr}{(r-1)^{\alpha+1}} \\ &= \|h\|_{\delta,\infty}\int_\kappa^\infty\int_0^\infty\Big(\int_s^{2s}\frac{z^{2\delta}}{(1+z)^{4\delta}}\frac{dz}{z}\Big)^{1/2}\frac{ds}{s}\frac{dr}{(r-1)^{\alpha+1}} \\ &\le 2^\delta(\log 2)\|h\|_{\delta,\infty}\int_\kappa^\infty\int_0^\infty\frac{s^\delta}{(1+s)^{2\delta}}\frac{ds}{s}\frac{dr}{(r-1)^{\alpha+1}} = \frac{C_\delta}{\alpha}(\sin\tau)^{-\alpha}\|h\|_{\delta,\infty}, \end{aligned}$$

where, for the third inequality, we have used the vector Minkowsky inequality as well as Fubini's rule. Moreover, since the above arguments also work for $\alpha = 0$, we have $\int_0^\infty\big(\int_y^{2y}|h(x)|^2\frac{dx}{x}\big)^{\frac12}\frac{dy}{y} \le C_\delta\|h\|_{\delta,\infty}$.

Finally, $\mathcal{M}_{2,1}^{(\alpha)}$ is a Banach algebra, and the density of $\mathcal{A}_0^\delta(S_\tau).\mathcal{M}_{2,1}^{(\alpha)}$ in $\mathcal{M}_{2,1}^{(\alpha)}$ follows from the density of $C_c^{(\infty)}(\mathbb{R}^+)$ in $\mathcal{M}_{2,1}^{(\alpha)}$. ■

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Remark 4.3. (i) In Proposition 4.2(i), the algebra $\mathcal{A}_b(S_\tau)$ can be replaced by the algebra $H^\infty(S_\tau)$ (with the same estimate). This is a consequence of the fact that $H^\infty(S_\tau) \hookrightarrow \mathcal{A}_b(S_{\tau/2})$ for every $\tau > 0$.

(ii) As a consequence of Proposition 4.2(i) and [4, Theorem 4.10], we obtain the bounded homomorphism $\Lambda^\alpha_{\infty,1}(\mathbb{R}^+) \hookrightarrow \mathcal{M}^{(\alpha)}_\infty$. This inclusion has been shown directly in the above section, see Theorem 3.1(ii).

(iii) An estimate of the same type as that of Proposition 4.2(i) is given in [6, p. 481] by interpolation. This is

$$\sup_{t>0} \sup_{\lambda \in S_\tau} \Big| \Big(I - \frac{d^2}{d\lambda^2} \Big)^{\alpha/2} (\eta h)(\lambda) \Big| \le C_\varepsilon \tau^{-(\alpha+\varepsilon)} \|h\|_\infty$$

where $\eta \in C_c^{(\infty)}(\mathbb{R})$ is fixed and $\varepsilon > 0$. Note that ε is not needed in our proposition.

5 Mikhlin Theorems for Sectorial Operators

Let X be a Banach space and let T be a closed one-to-one operator with dense domain and dense range in X. Suppose that the spectrum $\sigma(T)$ of T lies in the closed sector $\overline{S_\omega}$, where $\omega \in (0,\infty)$, and that $\|(z-T)^{-1}\| \le C_\tau |z|^{-1}$ whenever $\tau \in (\omega,\pi)$ and $z \in \mathbb{C} \setminus S_\tau$. Then T is said to be a *sectorial operator of type* ω. An operator which is of type ω for all $\omega > 0$ is called sectorial operator *of type* 0.

Set $\mathcal{DR}(S_\tau) := \bigcup_{\delta>0} \mathcal{A}_0^\delta(S_\tau)$ and $\mathcal{F}(S_\tau) := \bigcup_{\delta>0} \psi^{-\delta} H^\infty(S_\tau)$ in the notation of Section 4. Note that $\mathcal{DR}(S_\tau) \subset H^\infty(S_\tau) \subset \mathcal{F}(S_\tau)$. For a sectorial operator T (of type ω) it is possible to construct, on the basis of the Cauchy operator-valued formula, a functional calculus (the Dunford–Riesz calculus) $f \mapsto f(T), \mathcal{DR}(S_\tau) \longrightarrow \mathcal{L}(X)$, for all $\tau > \omega$, which extends to $\mathcal{F}(S_\tau)$. In general, $f(T)$ is unbounded, even though $f \in H^\infty(S_\tau)$. We say that T admits a bounded H^∞ calculus (on S_τ) if $f(T) \in \mathcal{L}(X)$ with $\|f(T)\| \le C\|f\|_\infty$ for all $f \in H^\infty(S_\tau)$.

When T is of type 0, then the H^∞ calculus for T is connected with a functional calculus for T having the Besov algebra $\Lambda^\alpha_{\infty,1}(\mathbb{R}^+)$ as domain.

Theorem 5.1 (**[4, Theorem 4.10]**) *Let T be a sectorial operator of type* 0*. Then the following are equivalent.*

(i) *There exist constants $\alpha, C > 0$ such that for every $\tau > 0$ the operator T has a functional calculus $H^\infty(S_\tau) \to \mathcal{L}(X)$ with $\|f(T)\| \le C\tau^{-\alpha}\|f\|_\infty$ for all $f \in H^\infty(S_\tau)$.*

(ii) *T admits a bounded $\Lambda^\alpha_{\infty,1}(\mathbb{R}^+)$ functional calculus, that is, a bounded algebra homomorphism $\Lambda^\alpha_{\infty,1}(\mathbb{R}^+) \longrightarrow \mathcal{L}(X)$ such that $(z-u)^{-1} \mapsto (z-T)^{-1}$ if $z \in \mathbb{C} \setminus \overline{\mathbb{R}^+}$.*

According to results obtained in previous sections we can give a variant of the above theorem, which tells us that the Besov calculus and the Mikhlin calculus are equivalent.

Theorem 5.2 *Let T be a sectorial operator of type* 0*. Let $\alpha > 0$. Then the following are equivalent.*

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(i) *T admits a bounded H^∞ calculus on S_τ, for all $\tau > 0$, such that for every $\nu > \alpha$ there exists $C_\nu > 0$ with*

$$\|f(T)\| \leq C_\nu \tau^{-\nu} \|f\|_\infty, \quad \tau > 0, f \in H^\infty(S_\tau).$$

(ii) *T admits a bounded $\Lambda^{\nu}_{\infty,1}(\mathbb{R}^+)$ calculus for every $\nu > \alpha$.*

(iii) *T admits a bounded $\mathcal{M}^{(\nu)}_\infty$ calculus for every $\nu > \alpha$.*

Proof (i) ⇒ (ii). This is the implication (i) ⇒ (ii) of Theorem 5.1.

(ii) ⇒ (iii). This is a consequence of Theorem 3.1(i).

(iii) ⇒ (i). This is a consequence of Proposition 4.2(i). See Remark 4.3(i). ∎

X. T. Duong [5] used Theorem 5.1 to establish a multiplier theorem for certain sub-Laplacians L on Lie groups, in terms of the Besov calculus. His method of proof consists in showing that the structure of L^p spaces on the group G, for $1 < p < \infty$, is good enough to obtain the appropriate scaled H^∞ calculus. In this way, we obtain the following improvement to [5, Theorem 2]. As usual, if h is a bounded Borel function on the spectrum $\sigma(L)$, then $h(L)$ denotes the corresponding bounded operator on $L^2(G)$ given by the spectral theorem for L.

Corollary 5.3 *Let L be a sub-Laplacian operator on a homogeneous nilpotent Lie group G such that the heat kernel e^{-zL}, $(\Re z > 0)$ generated by $-L$ satisfies property*

$$(HG_\alpha) \qquad \|e^{-zL}\|_1 \leq C_\alpha \left(\frac{|z|}{\Re z}\right)^\alpha, \quad (\Re z > 0),$$

where α is a fixed, non-negative, real number. Then $f(L)$ extends to a bounded operator on $L^p(G)$ for all $p \in (1,\infty)$ whenever $f \in \mathcal{M}^{(\nu)}_\infty$ with $\nu > \alpha + 1$.

Proof Let p be a real number such that $1 < p < \infty$. If L is as in the statement, it is proved in [5] that L admits a calculus $\Psi\colon H^\infty(S_\tau) \hookrightarrow \mathcal{L}(L^p(G))$, $\tau > 0$, as in Theorem 5.2(i), where $h(L) = \Psi(h)$ for every $h \in H^\infty(S_\tau)$. Then the corollary follows from the equivalence between parts (i) and (iii) of Theorem 5.2 above. ∎

Remark 5.4. (i) Condition HG_α is a natural assumption in our setting. The mapping $s \mapsto e^{-zs}$, where $s, \Re z > 0$, defines a holomorphic semigroup in $\mathcal{M}^{(\nu)}_\infty$, $(\nu > 0)$, such that

$$\sup_{s>0} |(e^{-zs})^{(\nu)}(s)s^\nu| = |z|^\nu \Big(\sup_{s>0} |s^\nu e^{-zs}|\Big) = (\nu/e)^\nu (|z|/\Re z)^\nu.$$

Hence, assuming that T admits the calculus $\mathcal{M}^{(\nu)}_\infty \to \mathcal{L}(X)$, the application of this calculus to the function e^{-zs} shows that $-T$ is the infinitesimal generator of a holomorphic semigroup $(a^z)_{\Re z>0}$ in $\mathcal{L}(X)$ satisfying condition (HG_ν) for all $\nu > \alpha$. On the other hand, there are many semigroups a^z satisfying property (HG_α) on L^1-spaces X for which, as is well known, it is not possible to get Mikhlin multiplier theorems.

(ii) It is known that the sectorial H^∞ calculus provides us in general with operators which are not necessarily bounded, see [4,16]. It has been shown [8,9] that these operators can always be regarded as regular quasimultipliers, in the sense defined by

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J. Esterle [7]. In this way, the resulting operators of the H^∞ calculus enjoy interesting algebraic and spectral properties [7, 9].

There is a link between the above two remarks. Namely, the infinitesimal generator of an analytic semigroup satisfying property (HG_α) admits a Mikhlin-type calculus, where the resulting operators are regular quasimultipliers. This calculus may be obtained as a consequence of the following facts.

Let $-T$ be the infinitesimal generator of an analytic C_0-semigroup $(a^z)_{\Re z>0}$ in $\mathcal{L}(X)$ which satisfies condition (HG_α), with $\alpha \geq 0$. In [10], a functional calculus for T has been given in the form of a bounded algebra homomorphism $\Phi\colon AC_{2,1}^{(\nu)} \to \mathcal{L}(X)$, whenever $\nu > \alpha + (1/2)$, such that $\Phi(AC_{2,1}^{(\nu)})X$ is dense in X. Incidentally, such an operator T is sectorial: if $n \in \mathbb{N}$, $n > \nu$, then $(T - zI)^{-1} = \Phi((u-z)^{-1})$ and therefore $\|(T - zI)^{-1}\| \leq C\|(u-z)^{-1}\|_{(\nu+1/2);2,1} \leq C_{n,\nu}\int_0^\infty u^n|u-z|^{-(n+2)}\,du$ for every $z \notin [0,\infty)$, by [10, Proposition 3.7]. Moreover, the last integral is equal to $|z|^{-1}\int_0^\infty r^n|r - e^{i\arg(z)}|^{-(n+2)}dr \equiv C|z|^{-1}$, so T is sectorial of type 0.

Let Φ_0 denote the restriction map of Φ to $\mathcal{M}_{2,1}^{(\nu)}$. Set $A := \overline{\operatorname{span}}\{a^z : \Re z > 0\}$ in $\mathcal{L}(X)$ and let A_0 be the closed ideal of A generated by Ta^1, $A_0 := \overline{(Ta^1)A}$. Then Φ_0 goes from $\mathcal{M}_{2,1}^{(\nu)}$ into A_0. For $\delta, \tau > 0$, let $\mathcal{C}$ denote the (bounded) inclusion $\mathcal{A}_0^\delta(S_\tau) \hookrightarrow \mathcal{M}_{2,1}^{(\alpha)}$ given by the Cauchy formula in Proposition 4.2. Then it is readily seen that the Dunford–Riesz calculus (see the beginning of this section) factors as

$$\mathcal{A}_0^\delta(S_\tau) \overset{\mathcal{C}}{\hookrightarrow} \mathcal{M}_{2,1}^{(\nu)} \overset{\Phi_0}{\to} A_0 \hookrightarrow A.$$

Furthermore, this factorization can be extended to the corresponding algebras of quasimultipliers, so that we obtain the H^∞ functional calculus of [4, 16] (for the operator T) given by

$$H^\infty(S_\rho) \hookrightarrow \mathcal{A}_b(S_\tau) \hookrightarrow \mathcal{M}_\infty^{(\nu)} \hookrightarrow \operatorname{Mul}(\mathcal{M}_{2,1}^{(\nu)}) \hookrightarrow QM_r(\mathcal{M}_{2,1}^{(\nu)}) \to QM_r(A_0),$$

if $\rho > \tau$. Note that the inclusion $\mathcal{M}_\infty^{(\nu)} \hookrightarrow \operatorname{Mul}(\mathcal{M}_{2,1}^{(\nu)})$ is Theorem 2.6. (For definitions and properties about algebras $QM_r(A)$ of regular quasimultipliers, see [7]. For the existence of $QM_r(\mathcal{M}_{2,1}^{(\nu)})$ and $QM_r(\mathcal{A}_0^\delta(S_\tau)) = \mathcal{A}_b(S_\tau)$, see [9].)

We find the above result interesting in that it reveals a natural and consistent framework for the unbounded operators (on general Banach spaces X) obtained from Mikhlin-type conditions. Also, the algebras $QM_r(A)$ are inductive limits of certain multiplier Banach algebras. In this way, the calculus yields (many) generalized multipliers on X, defined on Banach spaces suitably associated with X. Details of these results will be given in a subsequent paper.

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