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THE INVERSE FRACTIONAL MATCHING PROBLEM

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Abstract

This paper presents a method for the inverse fractional matching problem. We show that the dual of this inverse problem can be transformed into the circulation flow problem on a directed bipartite graph which can be solved easily. We also give an algorithm to obtain the primal optimum solution of the inverse problem from its dual optimum solution by solving a shortest path problem. Furthermore, we generalize this method to solve the inverse symmetric transportation problem.

1. Introduction

The first inverse network optimization problem was proposed by D. Burton and L. Toint [2]. Since then a number of papers discussing inverse combinatorial optimization problems have appeared [4], [6–13], which investigate inverse minimum spanning trees, inverse minimum matching in bipartite graphs, inverse minimum cost flows, inverse minimum cuts and inverse maximum flow problems. Furthermore, some inverse network problems have been generalized to abstract algebraic systems, such as the inverse maximum weighted intersection of two matroids and inverse submodular function problems, see [3].

In this paper we discuss the inverse fractional matching problem, which includes the inverse matching in bipartite graphs. The paper is organized as follows: in Section 1 we introduce the model of fractional matching. In Section 2 we investigate the model of inverse fractional matching and propose a method to solve the problem. In Section 3 we show how to get the solution of the inverse fractional matching from its dual optimal solution. Finally Section 4 generalizes the fractional matching to the symmetric transportation problem and points out that its inverse problem can be solved by the same method. Throughout the paper we denote by $[ij]$ the edge between

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nodes i and j in an undirected graph, and by (i, j) the arc from i to j in a directed graph.

Let $G = (V, E; c)$ be an undirected graph in which each edge e is associated with a positive number c_e, called the weight of edge e. A subset T of E is called an odd monocycle graph of G if T is a connected subgraph of G and contains exactly one cycle which has an odd number of edges.

The fractional matching problem (PFM) can be formulated as the following:
(PFM)

$$\begin{aligned} \text{Min} \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in E_i} x_e = 1, \qquad \forall i \in V, \\ & x_e \geq 0, \qquad \forall e \in E, \end{aligned}$$

where $E_i = \{e \in E \mid e \text{ is incident to node } i\}$.

THEOREM 1 ([1, 5]). *Let x be a basic feasible solution of* (PFM), *and*

$$F_x = \{e \in E | x_e \textit{ is a basic variable of } x\},$$

then we have:

(i) *each component of F_x is an odd monocycle graph;*
(ii) $x_e \in \{0, 1/2, 1\}$;
(iii) $F^0 = \{e \in E \mid x_e = 1/2\}$ *is the union of some node-disjoint odd cycles.*

The dual of problem (PFM) is:
(DFM)

$$\begin{aligned} \text{Max} \quad & \sum_{i \in V} u_i \\ \text{s.t.} \quad & u_i + u_j \leq c_{ij}, \quad \text{for} \quad e = [ij] \in E. \end{aligned}$$

The minimum fractional matching problem can be formulated as a minimum perfect matching of a bipartite graph [5], which can be easily solved. Now we consider its inverse problem.

2. Inverse minimum fractional matching

The inverse problem of minimum fractional matching (PFM) can be stated as follows: let $x^1, x^2, \ldots, x^r$ be basic feasible solutions of (PFM). The problem is then to find a weight vector c^* such that

(i) $x^1, x^2, \dots, x^r$ are the minimum fractional matchings under the weight vector c^*;

(ii) $|c^*_e - c_e| \le \varepsilon_e$ for each $e \in E$, where ε_e is the 'permissible change' of the weight for edge e ($\varepsilon_e \le c_e$);

(iii) c^* minimizes $\sum_{e\in E} |\bar{c}_e - c_e|$ over all $\bar{c}$ which meet conditions (i) and (ii).

If we express each c^*_e as $c_e + \alpha_e - \beta_e$, where α_e and β_e are respectively the increment and decrement of c_e, and introduce sets

$$S = \{e \in E | \exists i \in \{1, 2, \dots, r\}, x^i_e > 0\}$$

and $\bar{S} = E \setminus S$, then the inverse minimum fractional matching problem can be written from the model of inverse linear programming [10] as follows:
(IFM)

$$\begin{aligned}
\text{Min} \quad & \sum_{(ij)\in E} \alpha_{ij} + \sum_{(ij)\in S} \beta_{ij} \\
\text{s.t.} \quad & u_i + u_j = c_{ij} + \alpha_{ij} - \beta_{ij}, \quad \forall e = [ij] \in S, && (1)\\
& u_i + u_j \le c_{ij} + \alpha_{ij}, \quad \forall e = [ij] \in \bar{S}, && (2)\\
& 0 \le \alpha_{ij} \le \varepsilon_{ij}, \quad \forall e = [ij] \in E, && (3)\\
& 0 \le \beta_{ij} \le \varepsilon_{ij}, \quad \forall e = [ij] \in S. && (4)
\end{aligned}$$

Note that it is unnecessary to introduce β_{ij} for $[ij] \in \bar{S}$ as they must be zero to reach the minimum value. The dual of (IFM) is the following:
(DIFM)

$$\begin{aligned}
\text{Min} \quad & \sum_{e\in E} c_e y_e + \sum_{e\in E} \varepsilon_e \lambda_e + \sum_{e\in S} \varepsilon_e \mu_e \\
\text{s.t.} \quad & \sum_{e\in E_i} y_e = 0, \quad \forall i \in V, \\
& y_e - \lambda_e \le 1, \quad \forall e \in E, \\
& -y_e - \mu_e \le 1, \quad \forall e \in S, \\
& y_e \ge 0, \quad \forall e \in \bar{S}, \\
& \lambda_e \ge 0, \quad \forall e \in E, \\
& \mu_e \ge 0, \quad \forall e \in S.
\end{aligned}$$

Since $\varepsilon_e > 0$ and the objective function is to be minimized, we know that the optimal solution must satisfy the following conditions:

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(i) For $e \in S$,

$$\lambda_e = \begin{cases} 0, & \text{if } y_e \le 1, \\ y_e - 1, & \text{if } y_e > 1, \end{cases} \quad \text{and}$$

$$\mu_e = \begin{cases} 0, & \text{if } y_e \ge -1, \\ -1 - y_e, & \text{if } y_e < -1. \end{cases}$$

(ii) For $e \in \bar{S}$,

$$\lambda_e = \begin{cases} 0, & \text{if } 0 \le y_e \le 1, \\ y_e - 1, & \text{if } y_e > 1. \end{cases}$$

In order to solve the inverse problem more effectively, or to be more specific, to find a polynomial algorithm, we first transform problem (DIFM) into a circulation flow problem in an undirected network as follows. Let $\bar{G} = (V, \bar{E}; \bar{c}, l, u)$ be a graph obtained from G by the following manner:

(a) Let each edge $e \in S$ be replaced by three parallel edges e^1, e^2, e^3, whose weights, lower bounds and upper bounds for flows are defined respectively as follows:

$$\bar{c}_{e^i} = \begin{cases} c_e, & \text{if } i = 1, \\ c_e + \varepsilon_e, & \text{if } i = 2, \\ c_e - \varepsilon_e, & \text{if } i = 3, \end{cases}$$

$$l_{e^i} = \begin{cases} -1, & \text{if } i = 1, \\ 0, & \text{if } i = 2, \\ -\infty, & \text{if } i = 3, \end{cases} \quad \text{and}$$

$$u_{e^i} = \begin{cases} 1, & \text{if } i = 1, \\ \infty, & \text{if } i = 2, \\ 0, & \text{if } i = 3. \end{cases}$$

On each edge e^i let the flow be y_{e^i}, and we can define $y_e = y_{e^5} + y_{e^2} + y_{e^3}$, $\lambda_e = y_{e^2}$, and $\mu_e = -y_{e^3}$. It is easy to see that

$$c_e y_e + \varepsilon_e \lambda_e + \varepsilon_e \mu_e = c_e(y_{e^1} + y_{e^2} + y_{e^3}) + \varepsilon_e y_{e^2} - \varepsilon_e y_{e^3} = \bar{c}_{e^1} y_{e^1} + \bar{c}_{e^2} y_{e^2} + \bar{c}_{e^3} y_{e^3}.$$

(b) Let each edge $e \in \bar{S}$ be replaced by two parallel edges e^1 and e^2, and define

$$\bar{c}_{e^i} = \begin{cases} c_e, & \text{if } i = 1, \\ c_e + \varepsilon_e, & \text{if } i = 2, \end{cases}$$

$$l_{e^i} = 0, \quad \text{for } i = 1, 2, \quad \text{and}$$

$$u_{e^i} = \begin{cases} 1, & \text{if } i = 1, \\ \infty, & \text{if } i = 2. \end{cases}$$

On each edge e^i let the flow be y_{e^i}, and define $y_e = y_{e^1} + y_{e^2}$ and $\lambda_e = y_{e^2}$. Therefore,

$$c_e y_e + \varepsilon_e \lambda_e = c_e(y_{e^1} + y_{e^2}) + \varepsilon_e \lambda_e = c_e y_{e^1} + (c_e + \varepsilon_e) y_{e^2} = \bar{c}_{e^1} y_{e^1} + \bar{c}_{e^2} y_{e^2}.$$

If $e \in S$, the set $\bar{E}$ is formed from edges e^1, e^2 and e^3. It is not difficult to see from the construction of $\bar{G}$ that problem (DIFM) is equivalent to the following circulation flow problem:
(P)

$$\begin{aligned} \text{Min} \quad & \sum_{e \in \bar{E}} \bar{c}_e \bar{y}_e \\ \text{s.t.} \quad & \sum_{e \in \bar{E}_i} \bar{y}_e = 0, \quad \forall i \in V, \\ & l_e \leq \bar{y}_e \leq u_e, \quad \forall e \in \bar{E}, \end{aligned}$$

where $\bar{E}_i$ consists of all edges of $\bar{E}$ which are incident to node i.

To solve the inverse problem more easily, we can further transform the circulation flow problem (P) in an undirected graph into a circulation flow in a directed bipartite graph. To this purpose, we construct a directed bipartite network $N = (V^1 \cup V^2, A; \hat{c}, \hat{l}, \hat{u})$ from $\bar{G}$ by splitting each node i in V into two nodes i' and i'' such that

(1) $i \in V$ if and only if $i' \in V^1$ and $i'' \in V^2$;
(2) each edge $e = [ij] \in \bar{E}$ if and only if arcs (i', j'') and (j', i'') are both in A;
(3) $\hat{l}_{i'j''} = \hat{l}_{j'i''} = l_e$ and $\hat{u}_{i'j''} = \hat{u}_{j'i''} = u_e$ for each $e = [ij] \in \bar{E}$;
(4) $\hat{c}_{i'j''} = \hat{c}_{j'i''} = \bar{c}_e$ for each $e = [ij] \in \bar{E}$.

Then we can formulate problem

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(Q)

$$\begin{aligned} \text{Min} \quad & \sum_{(i,j)\in A} \hat{c}_{ij} f_{ij} \\ \text{s.t.} \quad & \sum_{j\in N^+(i)} f_{ij} = 0, \qquad \text{for } i \in V^1, \\ & \sum_{j\in N^-(i)} f_{ji} = 0, \qquad \text{for } i \in V^2, \\ & \hat{l}_{ij} \le f_{ij} \le \hat{u}_{ij}, \qquad \text{for } (i,j) \in A, \end{aligned}$$

where $N^+(i)$ and $N^-(i)$ are respectively the arcs leaving node i' and entering node i''.

THEOREM 2. *Problem* (P) *is equivalent to the circulation flow problem* (Q) *in* N.

PROOF. Let $\{y_e\}$ be a feasible solution of (P) with objective value v. Put

$$f_{i'j''} = f_{j'i''} = y_e, \text{ for each } e = [ij] \in \bar{E}.$$

Then we have a feasible solution of $(\bar{P})$ with objective value $2v$.

Conversely let $\{f_{i'j''}, f_{j'i''}\}$ be a feasible solution of (Q) with objective value v. Then

$$\{y_{ij} = (f_{i'j''} + f_{j'i''})/2, \text{ for each } e = [ij] \in \bar{E}\}$$

is a feasible solution of (P) with objective value $v/2$. Such 1-1 correspondence shows that problems (P) and (Q) are equivalent.

3. Obtaining the solution of problem (IFM)

In this section we will show how to get an optimal solution of the inverse problem (IFM) from the optimal solution of problem (Q). Note that as our purpose in this paper is to find a strongly polynomial algorithm, we do not solve the dual problem (Q) by the simplex method, and thus obtaining the primal optimal solution from its dual optimal solution is not a trivial problem. The problem (IFM) can be solved very efficiently only if we are able to give an easy way to obtain the primal solution in strongly polynomial time.

The dual of (Q) is the following:

(DQ)

$$\begin{aligned} \text{Max} \quad & \sum_{(ij)\in A} (\hat{l}_{ij} t_{ij} - \hat{u}_{ij} r_{ij}) \\ \text{s.t.} \quad & \pi_j + \pi_i + t_{ij} - r_{ij} = \hat{c}_{ij}, \qquad \text{for } (i,j) \in A, \\ & t_{ij} \ge 0,\ r_{ij} \ge 0, \qquad \text{for } (i,j) \in A, \end{aligned}$$

in which the objective function does not contain any term with $\hat{l}_{ij} = -\infty$ or $\hat{u}_{ij} = \infty$.

By the Kuhn-Tucker optimality condition we have the following.

LEMMA 1. *Let* $\{f_{ij}\}$ *be a feasible solution of* (Q), *then* $\{f_{ij}\}$ *is an optimal solution of* (Q) *if and only if there exist potentials* $\{\pi_i\}$ *such that*

(a) $\pi_j + \pi_i \le \hat{c}_{ij}$, *if* $f_{ij} < \hat{u}_{ij}$;
(b) $\pi_j + \pi_i \ge \hat{c}_{ij}$, *if* $f_{ij} > \hat{l}_{ij}$.

We now discuss how to obtain the corresponding potentials once we have the optimal solution of (Q). Let $\{f_{ij}\}$ be a feasible solution of (Q). The residual network $\bar{N} = (V^1 \cup V^2, \bar{A}; w)$ with respect to $\{f_{ij}\}$ can be obtained from N by setting $\bar{A} = A^1 \cup A^2$, where

$$A^1 = \{(i,j) \mid f_{ij} < \hat{u}_{ij}, (i,j) \in A\} \quad \text{and} \quad A^2 = \{(j,i) \mid f_{ij} > \hat{l}_{ij}, (i,j) \in A\};$$

$$w_{ij} = \begin{cases} \hat{c}_{ij}, & \text{if } (i,j) \in A^1, \\ -\hat{c}_{ji}, & \text{if } (i,j) \in A^2. \end{cases}$$

The following result is known (see [1, Theorem 9.1]).

LEMMA 2. *A feasible solution* $\{f_{ij}\}$ *of* (Q) *is optimal if and only if the residual network with respect to* $\{f_{ij}\}$ *does not contain a negative weight directed cycle.*

Now, suppose $\{f_{ij}\}$ is the optimal solution of (Q) and $\bar{N}$ is the residual network with respect to $\{f_{ij}\}$. We construct a network $N^0 = (V^1 \cup V^2 \cup \{s\}, \bar{A} \cup S; w^0)$, where s is the new node, $S = \{(s,j) \mid j$ has no entering arc in $\bar{N}\}$, $w^0_{sj} = 0$ for $(s,j) \in S$ and $w^0_{ij} = w_{ij}$ for $(i,j) \in \bar{A}$. It is known from Lemma 2 and the definition w^0_{sj} that N^0 has no negative weight cycle. There therefore exist shortest paths from s to each node in $\bar{N}$. Let ρ_i be the length of the shortest path from s to node i in N^0, then we have the following.

LEMMA 3. *Let* $\{f_{ij}\}$ *be the optimal solution of problem* (Q), *and set*

$$\pi_i = \begin{cases} -\rho_i, & \text{if } i \in V^1, \\ \rho_i, & \text{if } i \in V^2. \end{cases}$$

Then $\{\pi_i\}$ *and* $\{f_{ij}\}$ *satisfy conditions* (a) *and* (b) *in* Lemma 1, *that is,* π *is the corresponding potential vector.*

PROOF. We know that the lengths of shortest paths have the property that $\rho_i \leq \rho_j + w^0_{ji}$. Now for any arc $(i, j) \in A$, if $f_{ij} > \hat{l}_{ij}$, then $(j, i) \in A^2$ so that $w_{ji} = -\hat{c}_{ij}$, and

$$-\pi_i = \rho_i \leq \rho_j + w_{ji} = \pi_j - \hat{c}_{ij},$$

that is, condition (b) holds. We can prove that (a) is also true similarly.

From the optimal solution of (Q) and its associated potential vector, it is easy to obtain the optimal solution and potentials for problem (P).

LEMMA 4. *Let* $\{f_{ij}\}$ *be the optimal solution of* (Q) *and let* $\{\pi_i\}$ *be the corresponding potentials satisfying conditions* (a) *and* (b) *in Lemma* 1. *If for every* $e = [ij] \in \bar{E}$ *and every* $i \in V$, *we define*

$$\bar{y}_e = (f_{i'j''} + f_{j'i''})/2, \tag{5}$$

$$\nu_i = (\pi_{i'} + \pi_{i''})/2, \tag{6}$$

then $\bar{y}_e$ *must be an optimal solution of problem* (P) *and* ν *is the associated potential vector.*

PROOF. The dual of (P) is the following problem:

$$\begin{aligned} \text{Max} \quad & \sum_{e \in \bar{E}} (l_e v_e - u_e t_e) \\ \text{s.t.} \quad & \nu_i + \nu_j + v_e - t_e = \bar{c}_e, \quad \text{for all } [ij] = e \in \bar{E}, \\ & v_e \geq 0, \quad t_e \geq 0, \quad \text{for all } e \in \bar{E}. \end{aligned}$$

Following the argument used in Lemma 1, we know that if $\{\bar{y}_e\}$ is a feasible solution of (P), then it is an optimal solution if and only if there exist $\{\nu_i\}$ such that for all $e = [ij] \in \bar{E}$,

(a') $\nu_i + \nu_j \leq \bar{c}_e$, if $\bar{y}_e < u_e$;
(b') $\nu_i + \nu_j \geq \bar{c}_e$, if $\bar{y}_e > l_e$.

Now it is very easy to verify that if $\{f_{ij}\}$ is the optimal solution of (Q), which together with $\{\pi_i\}$ satisfies (a) and (b) in Lemma 1, then the $\bar{y}_e$ defined by (5) must be a feasible solution of (P), and $\bar{y}_e$ and ν, defined by (6), must satisfy conditions (a') and (b'). So, $\bar{y}_e$ is the optimal solution of (P) and ν is the associated potential vector.

Now we are ready to get the solutions of (DIFM) and (IFM) from $\{\bar{y}_e\}$ and $\{\nu_i\}$. Let

$$y_e = \begin{cases} \bar{y}_{e^1} + \bar{y}_{e^2} + \bar{y}_{e^3}, & \text{if } e \in S, \\ \bar{y}_{e^1} + \bar{y}_{e^2}, & \text{if } e \in \bar{S}, \end{cases}$$

$$\lambda_e = \bar{y}_{e^2}, \quad \text{for all } e \in E, \quad \text{and}$$
$$\mu_e = -\bar{y}_{e^3}, \quad \text{for all } e \in S.$$

It is easy to verify that $\{y_e, \lambda_e, \mu_e\}$, defined above, is a feasible solution of (DIFM).

We further define that

$$\alpha_e = \max\{\nu_i + \nu_j - c_e, 0\}, \quad \forall e = [ij] \in E, \tag{7}$$
$$\beta_e = \max\{c_e - \nu_i - \nu_j, 0\}, \quad \forall e = [ij] \in S. \tag{8}$$

LEMMA 5. $\{(\nu_i, \alpha_e, \beta_e)\}$, *defined by* (6)–(8), *is a feasible solution of problem* (IFM).

PROOF. Obviously $\alpha_e, \beta_e \geq 0$ and for each $e \in S$, only one of them is nonzero. It is easy to see that $\{\nu_i, \alpha_e, \beta_e\}$ satisfies constraints (1) and (2) of (IFM). Now we only need to prove that $\alpha_e \leq \varepsilon_e$ and $\beta_e \leq \varepsilon_e$ as required by constraints (3) and (4).

For any $e = [ij] \in E$, if $\alpha_e > 0$, then $\alpha_e = \nu_i + \nu_j - c_e$. Since $\bar{y}_{e^2} < u_{e^2} = +\infty$, from condition (a') in Lemma 4, $\nu_i + \nu_j \leq \bar{c}_{e^2} = c_e + \varepsilon_e$ which ensures that $\alpha_e \leq \varepsilon_e$.

Similarly, if $\beta_e > 0$ for some $e = [ij] \in S$, then $\beta_e = c_e - \nu_i - \nu_j$. Since $\bar{y}_{e^3} > l_{e^3} = -\infty$, from Lemma 4, we know that $\nu_i + \nu_j \geq \bar{c}_{e^3} = c_e - \varepsilon_e$. So $\beta_e \leq \varepsilon_e$.

Furthermore, we can prove the following.

THEOREM 3. *The sets* $\{y_e, \lambda_e, \mu_e\}$ *and* $\{\nu_i, \alpha_e, \beta_e\}$ *defined above are the optimal solutions of* (DIFM) *and* (IFM) *respectively.*

PROOF. As sets $\{y_e, \lambda_e, \mu_e\}$ and $\{\nu_i, \alpha_e, \beta_e\}$ are feasible solutions to problems (DIFM) and (IFM) respectively, in order to prove their optimality, it will suffice to be able to show that the complementary slackness conditions hold. That is, we only need to prove that

(1) $\alpha_e > 0 \implies y_e - \lambda_e = 1$, and $\beta_e > 0 \implies -y_e - \mu_e = 1$;
(2) $\lambda_e > 0 \implies \alpha_e = \varepsilon_e$, and $\mu_e > 0 \implies \beta_e = \varepsilon_e$;
(3) for $e = [ij] \in \bar{S}$, $y_e > 0 \implies \nu_i + \nu_j = c_e + \alpha_e$.

To prove (1), $\alpha_e > 0$ implies $\nu_i + \nu_j > c_e = \bar{c}_e^1$. Thus for any $e \in E$, from Lemma 4 we know that $\bar{y}_{e^1} = u_{e^1} = 1$. If $e \in S$, we also have $\nu_i + \nu_j > c_e - \varepsilon_e = \bar{c}_{e^3}$, and hence $\bar{y}_{e^3} = u_e = 0$. Now by the definition of y_e and λ_e,

$$y_e - \lambda_e = \bar{y}_{e^1} + \bar{y}_{e^2} + \bar{y}_{e^3} - \bar{y}_{e^2} = \bar{y}_{e^1} + \bar{y}_{e^3} = 1, \quad \text{for } e \in S,$$
$$y_e - \lambda_e = \bar{y}_{e^1} + \bar{y}_{e^2} - \bar{y}_{e^2} = 1, \quad \text{for } e \in \bar{S},$$

which proves the first half of (1). Similarly, for any $e \in S$, $\beta_e > 0$ implies $u_i + u_j < c_e = \bar{c}_{e^1} < c_e + \varepsilon_e = \bar{c}_{e^2}$, and thus $\bar{y}_{e^1} = l_e = -1$, and $\bar{y}_{e^2} = l_{e^2} = 0$. So for any $e \in S$,

$$-y_e - \mu_e = -(\bar{y}_{e^1} + \bar{y}_{e^2} + \bar{y}_{e^3}) + \bar{y}_{e^3} = -\bar{y}_{e^1} - \bar{y}_{e^2} = 1,$$

which proves the second part of (1).

To prove (2), $\bar{y}_{e^2} = \lambda_e > 0 = l_{e^2}$ implies that $\nu_i + \nu_j \geq \bar{c}_{e^2} = c_e + \varepsilon_e$ and hence by the definition of α_e, $\alpha_e \geq \varepsilon_e$ which means that $\alpha_e = \varepsilon_e$. The second assertion of (2) can be proved similarly.

To prove (3), if $y_e > 0$ for $e \in \bar{S}$, then at least one of

$$\bar{y}_{e^1} > l_{e^1} \quad \text{or} \quad \bar{y}_{e^2} > l_{e^2}$$

must be true, for otherwise we would have $y_e = \bar{y}_{e^1} + \bar{y}_{e^2} = l_{e^1} + l_{e^2} = -1$, which contradicts the assumption. So from (b') of Lemma 4 we have either

$$\nu_i + \nu_j \geq \bar{c}_{e^1} = c_e,$$

or

$$\nu_i + \nu_j \geq \bar{c}_{e^2} = c_e + \varepsilon_e,$$

which means that $\alpha_e = \nu_i + \nu_j - c_e$.

To summarize, our method for solving the IFM problems consists of the following four main steps:

Step1. Solve the circulation flow problem (Q) in network N to obtain $\{f_{ij}\}$.
Step 2. Solve the shortest path problems in network N^0, from s to other nodes, to obtain potential vector π.
Step 3. Use formula (6) to obtain the potential vector ν for problem (P).
Step 4. Use formulae (7)–(8) to obtain α and β.

Then (ν, α, β) provides the optimal solution to the inverse problem (IFM).

As there are strongly polynomial algorithms for solving minimum cost circulation flow problems and shortest path problems, (see, for example, [1]) the proposed method is a strongly polynomial method.

4. Symmetric transportation problem

In this section we will generalize the results of fractional matching. In transportation problems it is often the case that the number of vehicles travelling from city i to city j is equal to the number of vehicles returning from city j to city i. Such a problem is called a symmetric transportation problem. In this case, we can describe the model by an undirected network and use variable x_e to represent the total flow on the edge e, regardless of direction. Mathematically, this can be formulated as the following problem:

(ST)

$$\begin{aligned} \text{Min} \quad & \sum_{e\in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e\in E_i} x_e = a_i, \quad \forall i \in V, \\ & x_e \geq 0, \quad \forall e \in E, \end{aligned}$$

where V is a node set, E is an edge set, E_i is the set of edges incident to node i, a_i is the amount of commodities (requirement or supply) flowing into or out of node i and c_e is the unit cost for passing through edge e. The corresponding network is denoted by $G = (V, E; c, a)$.

Problem (ST) can be solved in a manner similar to that used for the fractional matching problem, using a minimum cost flow algorithm in a directed bipartite network. The directed bipartite network $N = (V' \cup V'', A; c', a')$ is constructed from G as follows: $V' = \{i' \mid i \in V\} \cup \{s\}$, $V'' = \{i'' \mid i \in V\} \cup \{t\}$, $A = A_1 \cup A_2 \cup A_3$, where $A_1 = \{(s, i') \mid i' \in V' \setminus \{s\}\}$, $A_2 = \{(i', j''), (j', i'') \mid i', j' \in V';\ i'', j'' \in V''$ and $[ij] \in E\}$, $A_3 = \{(j'', t) \mid j'' \in V'' \setminus \{t\}$, and

$$c'_{i'j''} = \begin{cases} c_{ij}, & \text{if } (i', j'') \in A_2, \\ 0, & \text{if } (i', j'') \in A_1 \cup A_3, \end{cases}$$

$a'_{si'} = a_i$ for $i' \in V' \setminus \{s\}$, $a'_{j''t} = a_j$ for $j'' \in V'' \setminus \{t\}$, and $a'_{ij} = \infty$ for all $(i, j) \in A_2$.

Let $v = \sum_{i\in V} a_i$ and consider the following minimum cost flow problem:

(MCF)

$$\begin{aligned} \text{Min} \quad & \sum_{(ij)\in A} c'_{ij} f_{ij} \\ \text{s.t.} \quad & -\sum_{j\in N^+(i)} f_{ij} + \sum_{j\in N^-(i)} f_{ij} = 0, \quad \text{for } i \in (V' \cup V'') \setminus \{s, t\}, \\ & \sum_{j\in N^+(i)} f_{ij} - \sum_{j\in N^-(i)} f_{ij} = v, \quad \text{for } i = s, \\ & -\sum_{j\in N^+(i)} f_{ij} + \sum_{j\in N^-(i)} f_{ij} = v, \quad \text{for } i = t, \\ & 0 \leq f_{ij} \leq a'_{ij}, \quad \text{for all } (i, j) \in A. \end{aligned}$$

We can prove the equivalence of the two problems.

THEOREM 4. *Let* $\{f_{ij}\}$ *be an optimal solution of* (MCF) *with flow value* $v = \sum_{i\in V} a_i$. *Then*

$$x_{ij} = (f_{i'j''} + f_{j'i''})/2, \quad \forall [ij] \in E \tag{9}$$

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is an optimal solution of problem (ST). *Conversely, if* $\{x_{ij}\}$ *is an optimal solution of problem* (ST), *then*

$$f_{i'j''} = f_{j'i''} = x_{ij}, \quad f_{si'} = a_i, \quad f_{j''t} = a_j \tag{10}$$

is an optimal solution of problem (MCF) *with flow value* $\sum_{i\in V} a_i$.

PROOF. Let $\{f_{ij}\}$ be a feasible solution of (MCF) with flow value $\sum_{i\in V} a_i$, then the $\{x_{ij}\}$ defined by (9) is a feasible solution of (ST) because for each $i \in V$,

$$\sum_{e\in E_i} x_e = \left(\sum_{j''\in A^+(i')} f_{i'j''} + \sum_{j'\in A^-(i'')} f_{j'i''}\right)/2 = (f_{si'} + f_{i''t})/2 = \left(a'_{i'} + a'_{i''}\right)/2 = a_i.$$

Conversely, a feasible solution $\{f_{ij}\}$ of (MCF) can be constructed from a feasible solution $\{x_e\}$ of (ST) by (10).

Since the objective values corresponding to such a pair $\{x_e\}$ and $\{f_{ij}\}$ have the following relation:

$$\begin{aligned}\sum_{e\in E} c_e y_e &= \sum_{[ij]\in E} c_{ij}\left(f_{i'j''} + f_{j'i''}\right)/2\\ &= \left(\sum_{j''\in A^+(i')} c'_{i'j''} f_{i'j''} + \sum_{j'\in A^-(i'')} c'_{j'i''} f_{j'i''}\right)/2\\ &= \left(\sum_{(ij)\in A} c'_{ij} f_{ij}\right)/2,\end{aligned}$$

we know that $\{x_e\}$ is an optimal solution of (ST) if and only if $\{f_{ij}\}$ is an optimal solution of (MCF).

It is easy to see that the inverse problem of (ST) is still (IFM), that is, it is the same as the inverse minimum fractional matching problem, and thus can be solved by the method given in Sections 2 and 3.

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References

[1] R. K. Ahuja, T. L.Magnanti and J. B. Orlin, *Network Flows* (Prentice Hall, 1993).

[2] M. Cai and Y. Li, "Inverse matroid intersection problem", *ZOR Mathematical Methods of Operations Research* **45** (1997) 235–243.

[3] D.Burton and Ph. L. Toint, "On an instance of the inverse shortest paths problem", *Mathematical Programming* **53** (1992) 45–61.

[4] S. P. Fekete et al, "The complexity of an inverse shortest paths problem, working report", Technical report, Center for Parallel Computing, Universität zu Köln, Germany, 1996.

[5] M. Gondran, M. Minoux and S. Vajda, *Graphs and Algorithms* (John Wiley & Sons, 1984).

[6] Z. Ma, S. Xu and J. Zhang, "An algorithm for inverse minimum spanning tree problem", Optimization Methods and Software, to appear.

[7] P. T. Sokkalingam et al, "Inverse spanning tree problem: formulations and algorithm", Working report 3890–96, Sloan School of Management, MIT, USA, 1996.

[8] S. Xu and J. Zhang, "An inverse problem of the weighted shortest path problem", *Japan J. Industr. Appl. Math.* **12** (1995) 47–59.

[9] C. Yang and J. Zhang, "Inverse maximum flow and minimum cut problems", *Optimization* **40** (1997) 147–170.

[10] J. Zhang and Z. Liu, "Calculating some inverse linear programming problem", *J. Comput. Appl. Math.* **72** (1996) 261–273.

[11] J. Zhang, Z. Liu and Z. Ma, "On inverse problem of minimum spanning tree with partition constraints", *ZOR Mathematical Methods of Operations Research* **44** (1996) 347–358.

[12] J. Zhang and Z. Ma, "A network flow method for solving some inverse combinatorial optimization problems", *Optimization* **37** (1996) 59–72.

[13] J. Zhang, Z. Ma and C. Yang, "A column generation method for inverse shortest path problem", *ZOR Mathematical Methods of Operations Research* **41** (1995) 347–358.