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Smooth Finite Dimensional Embeddings

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Abstract. We give necessary and sufficient conditions for a norm-compact subset of a Hilbert space to admit a C^1 embedding into a finite dimensional Euclidean space. Using quasibundles, we prove a structure theorem saying that the stratum of n-dimensional points is contained in an n-dimensional C^1 submanifold of the ambient Hilbert space. This work sharpens and extends earlier results of G. Glaeser on paratingents. As byproducts we obtain smoothing theorems for compact subsets of Hilbert space and disjunction theorems for locally compact subsets of Euclidean space.

1 Introduction

The principal purpose of this paper is to characterize those compact subsets of a Hilbert space which admit C^1 embeddings into finite dimensional Euclidean spaces. We define a C^1 map from a subset X of a Banach space to a subset Y of another to be a map which extends to a C^1 map (in the familiar sense) from the first ambient Banach space to the other. We thus obtain the C^1 category whose objects are the subsets of all the Banach spaces and whose morphisms are the C^1 maps as just defined. A function from X to Y is a diffeomorphism if its inverse is also in the category. For example, it is an easy exercise to show that the Cantor set and the fat Cantor set are not diffeomorphic. The definition we adopt for the tangent space T_pX of X at a point $p \in X$ is that of G. Glaeser [4], which is built on the notion of paratingent introduced by G. Bouligand [2]. This definition allows the dimension of T_pX to vary with p when X is not a C^1 submanifold, but does have the usual functorial properties. Our characterization is given by the following theorem.

Generalized Whitney Embedding Theorem *A compact subset X of a Hilbert space admits a C^1 embedding into a finite dimensional Euclidean space if and only if it satisfies the following three conditions:*

(1) Every T_pX is finite dimensional.
(2) T_pX depends continuously on p (in the sense of Section 2).
(3) The set of normalized secants of X has norm-compact closure.

Later we express conditions (1) and (2) by saying that TX is a quasibundle.

In [4, Chapter 2, Theorem 1], Glaeser presents a version of the Inverse Function Theorem for the case when the ambient Banach space is finite dimensional, and below we obtain a useable version of that theorem for suitable norm-compact subsets of a Hilbert space. It is interesting to note that Glaeser's extremely elegant proof of the Inverse Function Theorem

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hinges crucially on a 'bundle extension theorem' [4, Chapter 2, Proposition III]; we are able to obtain a generalization of this extension theorem [21, Theorem 3.4] in Euclidean space, but our proof breaks down in Hilbert space. Hence, we are not able to apply Glaeser's proof to the Hilbert space case. Instead we have to give a somewhat laborious proof that does not require any bundle extension theorem.

We use Glaeser's definition to stratify locally compact subsets of a Banach space in a manner reminiscent of the standard stratification of affine varieties. This stratification produces locally compact strata (Lemma 2.4). In addition, we show (Theorem 4.3) that under suitable hypotheses, the stratum of n-dimensional points has a relative neighborhood contained in a C^1 manifold with the same tangent spaces.

Our other main result, the Stopping Theorem (Theorem 3.1), addresses the definition of the tangent space T_pX as the union of a chain of closed linear subspaces $T^n{}_pX$ ordered by inclusion and defined for every *ordinal* n. This result extends and improves a similar result of Glaeser [4, Chapter 2, Proposition VIII]. Glaeser proves that for X a closed subset of R^N, we have $T^n{}_xX = T_xX$ for $2N \leq n$. Our result improves Glaeser's to $T^n{}_xX = T_xX$ for $\dim T_xX \leq n$. Further, our result extends to subsets X of Hilbert space, where N is infinite, which are compact and satisfy the three conditions in the theorem above. Our proof is independent of the Inverse Function Theorem, though Glaeser's result is a consequence of that theorem.

In Section 7 we prove smoothing theorems (Theorems 7.1 and 7.2) for compact subsets of Hilbert space and disjunction theorems (Theorems 7.3 and 7.4) for locally compact subsets of Euclidean space. In the final section, we present examples in order to make some of the ideas more concrete. Some other works discussing these matters are [5], [6], and [8]. Also [16], [17] and [7] are good background references for the work done here.

Our original motivation for this work was the following: In [25], F. Takens proves a smooth embedding theorem for a generic smooth dynamical system on a smooth finite dimensional manifold. This embedding restricts to a smooth embedding of any (possibly nonmanifold) invariant subset. As far as we know, there is no analogous result for a suitably finite dimensional invariant subset associated with a (generic) smooth dynamical system on a Hilbert space. However, there do exist some counterexamples [19], [1], [20] that serve to limit proposed extensions of the Takens Theorem to Hilbert space. Because the invariant subsets of such a dynamical system are not necessarily subsets of invariant smooth finite dimensional submanifolds, it seems necessary to extend to their case enough of the fundamental machinery of smooth topology to prove smooth embedding theorems. As a first step in this direction, we present our generalization of the Whitney Embedding Theorem [7].

Throughout this paper, $\mathbb{H}$ denotes a Hilbert space.

2 The Tangent Space and Quasibundles

In this section, we recall from [4] and [26] two definitions of a tangent space and prove some basic properties for these definitions. The principal theorem of this paper (proved in later sections) may be interpreted as a statement that the two definitions are equivalent for compact and spherically compact subsets X of a Hilbert space $\mathbb{H}$ with TX a quasibundle. (See below for definitions and the Projection Theorem 5.1.) The first definition is:

Let X be a set and p be a member of X. The *tangent space* of X at p, denoted T_pX is the intersection of all the T_pM such that M is a C^1 manifold containing a neighborhood of p in X.

The second definition, the one we will use, is closer to the internal structure of X and is based on the notion that tangent vectors ought to be limits of secants. Let us begin with some examples which show the need for some care. Consider two tangent circles. Suppose we were to define the tangent space of a set X at a point p in X to be the linear span of the set of limits,

$$\lim_{n\to\infty}(p_n - p)/\|p_n - p\|$$

as $\{p_n\}$ ranges over all sequences of points in X which converge to p. Then at the point of tangency, the tangent space would be the common tangent line and in all neighborhoods the projection onto the tangent space would not even be one-to-one, contradicting a desirable basic property of tangent spaces.

We can overcome this difficulty by using "two sided" secants. Define the 0-th level tangent space of X at p, in symbols T_p^0X as follows:

Let $C_p^0X =$ the set of (norm) limits of the of the form,

$$\lim_{n\to\infty}(p_n - q_n)/\|p_n - q_n\|$$

where $\{p_n\}$ and $\{q_n\}$ are both sequences from X converging to p with $p_n \neq q_n$. In other words, C_p^0X consists of all limits of sequences of normalized secants converging to p. Let T_p^0X be the closed linear span of C_p^0X.

With this definition, we see that the tangent space T_p^0X at the point of tangency discussed above is the whole plane so that projection onto the tangent space is just the identity. At all other points on the circles the tangent space is a line. Because in an infinite dimensional Hilbert space weak convergence plays an important role, we must supplement the definition of C_p^0X with the following:

WC_p^0X is the set of *weak* limits of normalized secants.

Of course the two concepts of C_p^0 and WC_p^0 do not coincide, but they are linked by a third concept, that of *spherical compactness*:

We say that a subset X of a Hilbert space $\mathbb{H}$ is *spherically compact* if and only if the image of the map

$$J\colon X \times X \setminus \Delta \to \mathbb{H}$$

given by

$$J(x, y) = \frac{x - y}{\|x - y\|}$$

has compact closure in the norm topology of $\mathbb{H}$. Here Δ denotes the diagonal subspace of $X \times X$.

It is clear that in a finite dimensional $\mathbb{H}$, every subset is spherically compact. An alternate characterization of spherical compactness linking the weak and the strong limiting secant spaces is given by the following lemma [18].

Lemma 2.1 *Let X be a compact subset of $\mathbb{H}$. Then X is spherically compact if and only if $C_p^0 X = WC_p^0 X$ for all $p \in X$.*

The proof is a straightforward application of the definitions.

These definitions are still not good enough for our purposes. To begin to see why, let $d_p^0 X$ be the dimension of $T_p^0 X$. If there were points q arbitrarily close to p with $d_q^0 X = n$, then we might hope that $d_p^0 \geq n$. This is because part of the standard theory of tangent spaces which we wish to preserve is that the projection onto the tangent space at p would be a local diffeomorphism and thus contain a diffeomorphic copy of a neighborhood of all sufficiently nearby q. Also, diffeomorphisms should preserve the dimension of the tangent space.

The next example gives a case where a one dimensional T^0 is a limit of two dimensional T^0's. Pick a point p on the positive x-axis and pick an angle θ. At each point at distance $1/n$ further out the axis from p draw the line segment in the xy-plane at angle θ with the x-axis and length $1/n^3$. Define the θ-feather at p to be the union of these lines together with the point p. When X is the θ-feather at p we see that $C_p^0 X$ consists of two lines, the x-axis and the line at angle θ. Thus $T_p^0 X$ is the whole xy-plane.

Now consider the set X contained in the Euclidean plane, $\mathbb{R}^2$, which at the point $p + 1/n$ has a whole $1/n$-feather scaled down by a factor of n^3 along with the point p. (We call this set a 2D arrow.) Then, since the feather angles are converging to 0, $C_p^0 X$ consists only of the unit vector along the x-axis. At level zero, p has dimension one while each of the $p + 1/n$ has dimension two.

We can overcome this problem with the following definition. For each ordinal number α, we define the sets $C_p^\alpha X$ and $T_p^\alpha X$:

> $C_p^0 X$ and $T_p^0 X$ are as above. For $\alpha > 0$, $C_p^\alpha X$ is the set of limits of convergent sequences $\{v_n\}$ where each v_n is in the union of the $T_{p_n}^{\beta_n} X$ for $\beta_n < \alpha$ and the sequence $\{p_n\}$ converges to p. $T_p^\alpha X$ is the closed linear span of $C_p^\alpha X$. If p is an isolated point of X, then $T_p^\alpha X$ is the 0-dimensional space.

Observe that one of the sequences converging to p is the constant sequence. Therefore $\alpha < \beta$ implies that $T_p^\alpha X$ is a subset of $T_p^\beta X$. Consequently, by the usual arguments from axiomatic set theory (See the proof of Zermelo's theorem in [9, p. 20]), as a function of α the chain $T_p^\alpha X$ is eventually constant. Let $T_p X$ be its limiting value and let $d_p X$ be its limiting dimension; we refer to $T_p X$ as the *tangent space at p*. From this point forth, we shall also omit the explicit mention of X in these notations when it can be determined from context.

A pithier definition of the T_p is that it is the smallest system of vector spaces containing the C_p^0 and closed under the above limits. That is, if $\lim p_n = p$ and v_n is in $T_{p_n} X$ and $\lim v_n = v$, then v is in $T_p X$. (See Lemma 2.3 for a precise statement.) Finally, we may make the same definitions using weak limits and beginning with WC_p^0. As usual, we may define the *tangent set*, $|TX|$, by setting

$$|TX| = \{(p, v) : p \in X, v \in T_pX\} \subset X \times \mathbb{H}$$

with the inherited topology. We may also define the associated projection, $\rho\colon |TX| \to X$, by setting $\rho(p, v) = p$.

Next we define the differential df of a differentiable map f in our category. Suppose f has domain $X \subseteq \mathbb{B}_1$ and range $Y \subseteq \mathbb{B}_2$, where $\mathbb{B}_1$ and $\mathbb{B}_2$ are Banach spaces. Recall that for each p in X, $df(p)$ should be a linear map from T_pX to $T_{f(p)}Y$. Let F be any C^1 extension of f to the ambient spaces and let dF be its differential. Suppose $v \in C_p^0X$. That is,

$$v = \lim_{n\to\infty} (p_n - q_n)/\|p_n - q_n\|$$

where p_n and q_n are sequences from X converging to p. Then $F(p_n) = f(p_n)$ and likewise for q_n. Therefore,

$$dF(p)v = \lim_{n\to\infty} \big(f(p_n) - f(q_n)\big)/\|p_n - q_n\|$$

and therefore the vector $dF(p)v$ is independent of the choice of the C^1 extension F and is a member of $T_{f(p)}Y$. By linearity, this property extends to all of T_p^0X. Define the differential df by setting $df(p)v = dF(p)v$ so that $df(p)\colon T_p^0X \to T_{f(p)}^0Y$. It follows from an easy induction argument that for any ordinal α and any $v \in T_p^\alpha X$, $dF(p)v$ is independent of F and is a member of $T_{f(p)}^\alpha Y$.

It is also straightforward to check the identity rule and the chain rule. That is,

$$d(\mathrm{id})(p) = \mathrm{id}$$
$$d(f \circ g)(p) = df\big(g(p)\big) \circ dg(p).$$

Therefore the usual functorial properties hold. For example, if f is a diffeomorphism, $df(p)$ is an isomorphism.

Returning to our 2D arrow, we see that d_p^0 is one, but d_p^1 is two. If we add to this set another similar construction coming into p from the negative direction and aligned in the xz-plane instead of the xy-plane, we get that d_p^0 is still one, but d_p^1 is three.

This example can be generalized to get a set in $\mathbb{R}^N$ with a point p such that d_p^{N-1} is N, but d_p^{N-2} is only $N-1$. Let us sketch the construction and leave the details to the privacy of the reader's mind. We begin with the 3D arrow. Start with a system of scaled down 2D arrows converging to p along the x-axis. In the n-th arrow, make sure that the maximum angle used in any feather is less that $1/n$. Between these arrows add short parallel lines sticking out into the z direction. Let the length of these lines converge to 0 quickly enough so that at the nose p, T_p^0 is just the plane spanned by the x-axis and these lines. T_p^1 also contains the xy-plane since it is the limit of the noses of the feathers of the 2D arrows. Thus it is all of $\mathbb{R}^3$. Let φ be the angle between the quills and the positive x-axis and let θ be the angle between the xy-plane and the plane generated by the new quills and the x-axis.

Now consider the set which has a scaled down version of this whole system at each of the points $p + 1/n$. Let both the angles θ_n and φ_n converge to 0. This is the 3D arrow. Since the new planes generated by our new quills all converge to the xy-plane, we see that at the nose p, T_p^1 is just the xy-plane, but T_p^2 is a limit of 3-spaces and hence 3-space itself. To build a 4D arrow *etc.*, repeat the construction of the 3D arrow, starting with 3D arrows rather than 2D arrows. The new quills should satisfy $z = 0$, but stick out into the w direction just as the previous quills stuck out into the z direction. In Section 3 we will prove the Stopping Theorem (Theorem 3.1) to show that these examples are optimal.

In classical differential topology [16], [17], and [7], if M is a (finite dimensional) smooth submanifold of a Hilbert space $\mathbb{H}$, a fundamental property is that the map $p \mapsto T_pM$ is continuous. In our context, where X is a compact subset of $\mathbb{H}$, the tangent space T_pX may *change dimension* as p varies over X, but we still require that the map $TX\colon p \mapsto T_pX$ be continuous in some sense. In the case that T_pX is finite dimensional for all $p \in X$, this sense is determined by imposing a suitable topology on the set $\mathbb{G}(\mathbb{H})$ of all finite-dimensional linear subspaces of $\mathbb{H}$. In the terminology we adopt below from [21], this is equivalent to the condition that the map $TX\colon X \to \mathbb{G}(\mathbb{H})$ is a *quasibundle*. The condition that TX be a quasibundle is automatic for a k-dimensional submanifold $M = X$ because the set $\mathbb{G}_k(\mathbb{H})$ of all k-dimensional linear subspaces of $\mathbb{H}$ inherits its usual topology from $\mathbb{G}(\mathbb{H})$. Also, the condition that TX be a quasibundle is automatic for $X \subset \mathbb{R}^N \subset \mathbb{H}$. Unfortunately, this condition is not always automatic: Example 8.7 (due to an anonymous reader) shows a compact subset $X \subset \mathbb{H}$ with all T_pX finite dimensional but $p \mapsto T_pX$ *not* continuous. In this example $\dim T_pX$ is not uniformly bounded over X as it would be (Corollary 2.1) for TX a quasibundle.

We topologize $\mathbb{G}(\mathbb{H})$ by using the one-sided (and unsymmetric) Hausdorff distance, defined as follows:

> If ξ and η are lines in $\mathbb{H}$ then we denote the acute angle between them by $\theta(\xi, \eta)$. We note that θ is a metric for the set of all lines. If ξ is a line and Q a finite dimensional linear subspace of $\mathbb{H}$, then we write
>
> $$\theta(\xi, Q) = \inf\{\theta(\xi, \eta) : \eta \text{ is a line in } Q\}$$
>
> and finally we define the one-sided Hausdorff distance $d(P, Q)$ from one finite dimensional linear subspace P to another Q by writing
>
> $$d(P, Q) = \sup\{\theta(\xi, Q) : \xi \text{ is a line in } P\}.$$

This distance function has the following properties analogous to those of a metric:

(i) $0 \le d(P, Q) \le \frac{\pi}{2}$.
(ii) $d(P, Q) = 0$ if and only if $P \subset Q$.
(iii) $d(P, Q) \le d(P, R) + d(R, Q)$.

Accordingly we define the *r-neighborhood* of Q by setting

$$N_r(Q) = \{P : d(P, Q) < r\}.$$

Then the family $\{N_r(Q) : 0 < r, Q \in \mathbb{G}(\mathbb{H})\}$ is a basis for a non-Hausdorff topology on $\mathbb{G}(\mathbb{H})$.

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We define a (*right*) *quasibundle* η over a space X to be simply a continuous map $\eta\colon X \to \mathbb{G}(\mathbb{H})$.

In particular, if all T_pX are finite dimensional, the assignment $p \mapsto T_pX$ is a map $X \to \mathbb{G}(\mathbb{H})$, which may or may not be continuous. We denote this map by TX and call it the *tangent map*. Now, we introduce some simple properties of the one-sided distance. For $Q \in \mathbb{G}(\mathbb{H})$ we let $\pi_Q\colon \mathbb{H} \to Q$ be orthogonal projection. Then for $x \in P$ we have $\|\pi_Q x\| \geq \cos d(P, Q)\|x\|$ which proves the following lemma.

Lemma 2.2 *If $d(P, Q) < \frac{\pi}{2}$, then the restriction $\pi_Q \restriction P$ is an injection. Moreover, if $0 \leq \alpha < 1$ then there is $\delta > 0$ so that $d(P, Q) < \delta$ implies that $\|\pi_Q x\| \geq \alpha\|x\|$ for all $x \in P$.*

For $\eta\colon X \to \mathbb{G}(\mathbb{H})$ any map we define

$$|\eta| = \{(p, v) : p \in X, v \in \eta(p)\} \subset X \times \mathbb{H}$$

and

$$\sigma(\eta) = \{(p, v) : (p, v) \in |\eta|, \|v\| = 1\}$$

with the inherited topologies. Then we have the following characterization of quasibundles.

Lemma 2.3 *Let X be compact and first countable. Then $\eta\colon X \to \mathbb{G}(\mathbb{H})$ is a quasibundle if and only if $\sigma(\eta)$ is compact in $X \times \mathbb{H}$.*

Proof Clearly it suffices to prove that sequential compactness of $\sigma(\eta)$ is equivalent to sequential continuity of η.

First we assume that η is continuous and let $\{(p_n, v_n)\}_{n\geq 1}$ be a sequence in $\sigma(\eta)$. By selecting a subsequence, we may assume that the subsequence $\{p_n\}_{n\geq 1}$ converges to some p. Then, because η is continuous, we have

$$\lim_{n\to\infty} d\big(\eta(p_n), \eta(p)\big) = 0$$

which implies that for for n sufficiently large, $\pi_{\eta(p)} \mid \eta(p_n)$ is an injection. Let $\pi_{\eta(p)}(v_n) = v_n'$. Again, we may assume that the sequence $\{v_n'\}_{n\geq 1}$ converges to some $v' \in \eta(p)$ (recall that $\eta(p)$ is finite dimensional). We note that we have

$$\|v_n - v_n'\| \leq d\big(\eta(p_n), \eta(p)\big)$$

so that the sequence $\{v_n\}_{n\geq 1}$ converges to v'. Therefore $\{(p_n, v_n)\}_{n\geq 1}$ converges to (p', v') and $\sigma(\eta)$ is sequentially compact.

Conversely, we assume that $\sigma(\eta)$ is sequentially compact. If η is not sequentially continuous, there is some $\varepsilon > 0$ and a sequence $\{p_n\}_{n\geq 1}$ with limit $p \in X$ so that $d\big(\eta(p_n), \eta(p)\big) \geq \varepsilon$. Consequently, we may select $v_n \in \eta(p_n)$ with $\|v_n\| = 1$ so that $\theta\big(v_n, \eta(p)\big) \geq \varepsilon$ holds. However, because $\sigma(\eta)$ is sequentially compact, we may assume that the sequence $\{(p_n, v_n)\}_{n\geq 1}$ has a limit $(p, v) \in \sigma(\eta)$. Thus $v \in \eta(p)$ and then $\theta\big(v_n, \eta(p)\big) \leq \theta(v_n, v)$. But the right side of this inequality converges to 0, contradicting the earlier inequality. ■

Corollary 2.1 *Let X be first countable and let $\eta\colon X \to \mathbb{G}(\mathbb{H})$ be a quasibundle. Then the map $p \mapsto \dim \eta(p)$ is upper semi-continuous. Consequently, if X is compact, then $\dim \eta(p)$ has a uniform finite upper bound.*

Finally, we use Lemma 2.3 above to introduce a construction of new quasibundles from old ones and to present a criterion for the map TX to be a quasibundle: For the construction, we let $\mathbb{F}$ be a nonempty family of quasibundles over a space X, and we define

$$\bigcap \mathbb{F}\colon X \to \mathbb{G}(\mathbb{H})$$

by setting

$$\bigcap \mathbb{F}(p) = \bigcap \{\eta(p) : \eta \in \mathbb{F}\}.$$

Of course, $\bigcap \mathbb{F}$ need not be continuous, *i.e.*, a quasibundle. However, we have the following result.

Corollary 2.2 *If $\mathbb{F}$ is a nonempty family of quasibundles over X, with X compact and first countable, then $\bigcap \mathbb{F}\colon X \to \mathbb{G}(\mathbb{H})$ is a quasibundle.*

For the criterion, we state the following theorem and refer the reader to Example 8.7.

Theorem 2.1 *Let X be a compact subset of a Hilbert space $\mathbb{H}$ with $\dim T_pX < \infty$ for all $p \in X$. Then there exists a quasibundle $\eta\colon X \to \mathbb{G}(\mathbb{H})$ such that $C_p^0 X \subset \eta(p)$ for all $p \in X$ if and only if the map $TX\colon X \to \mathbb{G}(\mathbb{H})$ defined by setting $TX\colon p \mapsto T_pX$ is a quasibundle. That is, in this case TX is the smallest quasibundle containing C_p^0X for all $p \in X$.*

Proof One direction is trivial: If TX is a quasibundle, then $\eta = TX$ will do.

For the other direction, we let

$$\mathbb{F} = \{\eta : \eta\colon X \to \mathbb{G}(\mathbb{H}) \text{ continuous with } C_p^0X \subset \eta(p) \text{ for all } p \in X\}$$

and note that $\mathbb{F} \neq \varnothing$ by hypothesis. Then we define $\tau(X) = \bigcap \mathbb{F}$ and note that it is a quasibundle. We wish to show that $TX = \tau(X)$. We begin by showing that for any $\eta \in \mathbb{F}$, we have $|TX| \subset |\eta|$ so that $|TX| \subset |\tau(X)|$. To do so, we show that $T_p^kX \subset \eta(p)$ for all $p \in X$. We begin with the inclusion $T_p^0X \subset \eta(p)$, which is valid because $C_p^0X \subset \eta(p)$ by definition. We suppose inductively that we have $T_p^kX \subset \eta(p)$ for all ordinals $k < m$, and we let $v \in C_p^mX$; we wish to show that $v \in \eta(p)$. To this end, let $T^kX\colon X \to \mathbb{G}(\mathbb{H})$ by setting $T^kX\colon q \mapsto T_q^kX$. By definition, there exists a sequence $\{(p_n, v_n)\}_{n\geq 1}$ in $\bigcup\{|T^kX| : k < m\}$ converging to (p, v). If $v = 0$ there is nothing to prove. If $v \neq 0$ we may as well assume that $\|v\| = 1$ and $\|v_n\| = 1$ for all n. Because of our inductive hypothesis,

$$\bigcup\{|T^kX| : k < m\} \subset |\eta|,$$

our sequence $\{(p_n, v_n)\}_{n\geq 1}$ is in $\sigma(\eta)$; because $\sigma(\eta)$ is compact we must have $v = \lim_{n\to\infty} v_n \in \sigma(\eta)(p)$ so that $C_p^mX \subset \eta(p)$ and the induction is complete, thus showing that $|TX| \subset |\tau(X)|$.

Conversely, because the sequence $\{T_p^k X\}_{k\geq 1}$ is eventually constant, there is an ordinal α such that $T^\alpha X = T^{\alpha+1}X$. Then it is easy to see that $|T^\alpha X|$ is a closed subset of $|\tau(X)|$ and hence that $\sigma(T^\alpha X)$ is a closed subset of $\sigma\bigl(\tau(X)\bigr)$. Thus $\sigma(T^\alpha X)$ is compact and $T^\alpha X$ is a quasibundle. But then $T^\alpha X \in \mathbb{F}$ so that

$$|\tau(X)| \subset |T^\alpha X| \subset |TX| \subset |\tau(X)|$$

and our proof is complete; that is, $TX = \tau(X)$ and $\tau(X)$ is a quasibundle. ■

The following proposition will be useful.

Proposition 2.1 **(Grassmann Convergence Proposition)** *Let X be a compact subset of a Hilbert space $\mathbb{H}$ with TX a quasibundle. Let $\{p_n\}_{n\geq 1}$ be a sequence in X converging to $p \in X$. Then for some m, the sequence $\{T_{p_n}X\}_{n\geq 1}$ contains a constantly m-dimensional subsequence which converges in the Grassmannian $\mathbb{G}_m(\mathbb{H})$ to an m-plane $V \subset T_pX$.*

Proof Suppose that p_n, $n \geq 1$, are as stated and $u_{1,n}, \ldots, u_{m,n}$ is an orthonormal set in $T_{p_n}^k X$. Here we call on Lemma 2.3; it allows us to choose convergent subsequences. Consequently, by extracting subsequences, we may assume that $\{u_{i,n}\}_{n\geq 1}$ converges to a limit u_i. Then $u_1, \ldots, u_m$ is an orthonormal set contained in $C_p^{k+1}X$. Furthermore, we see that any linear combination of the u_i is the limit of the same linear combination of the $u_{i,n}$ and so is also in $C_p^{k+1}X$. ■

Let us introduce the notation:

$$X_k = \{p \in X : d_pX = k\} \tag{2.1}$$
$$X^l = \bigcup_{k\geq l} X_k.$$

When TX is a quasibundle, this provides us with a filtration of X by sets closed in X,

$$\cdots \subseteq X^N \subseteq X^{N-1} \subseteq \cdots \subseteq X^0 = X. \tag{2.2}$$

Lemma 2.4 *If X is locally compact and TX is a quasibundle, then the sets X_k and X^l are locally compact.*

Proof X^l is closed in X and therefore it is locally compact. Also, X^{l+1} is closed in the locally compact space X^l. Hence $X_l = X^l \setminus X^{l+1}$ is locally compact. ■

We refer to the given filtration (2.2) as the C^1 *filtration of* X, and associated stratification (2.1) as the C^1 *stratification of* X. One of our major goals is to see how the different strata fit together at the tangent level.

3 The Stopping Theorem

In this section we prove the following theorem.

Theorem 3.1 **(The Stopping Theorem)** *Let X be a compact and spherically compact subset of a Hilbert space $\mathbb{H}$ with TX a quasibundle. If p is a point in X with* $\dim T_pX = n$, *then* $T_pX = T_p^{n-1}X$.

We note that our proof does not use the projection or embedding theorems of Sections 4, 5, and 6.

For X a compact subset of $\mathbb{H}$, we recall the notation $d_p^k = \dim T_p^kX$ and $d_p = \dim T_pX$. Let $D_{n,k} = \text{closure}\{p \in X : d_p^k = n\}$. We show that if $D_{n,k} \setminus D_{n,k-1}$ is non-empty, then $k \leq n-2$. We consider the cases $n = 1, 2$ in the following lemma.

Lemma 3.1 *Let X be a compact and spherically compact subset of $\mathbb{H}$. Then*

(i) If $T_p^\alpha X$ is a line, then its unit vectors are both in C_p^0X.
(ii) $D_{2,k} = D_{2,0}$.

Proof To prove (i), we note that spherical compactness implies that, if p is not an isolated point, C_p^0X has at least one non-zero vector. Since $T_p^\alpha X$ increases with α this completes the proof.

To prove (ii), suppose by way of contradiction that $D_{2,k} \neq D_{2,0}$. Pick a point p such that $d_p^k = 2$ but p is not in $D_{2,0}$. Each vector in C_p^kX must be a limit vectors in T_q^nX for $n < k$ and $d_q^n = 1$. By (i), this means that each unit vector in T_p^kX is actually a limit of vectors in $C^0X = \bigcup\{C_p^0X : p \in X\}$. Since everything in C^0X is a limit of secants, this means that all vectors in C_p^kX are actually limits of secants. Therefore C_p^kX is contained in C_p^0X. Contradiction. ■

Lemma 3.2 *Let X be a compact and spherically compact subset of $\mathbb{H}$ with TX a quasibundle. If $p = \lim_{n\to\infty} p_n$ and $d_{p_n}^k \geq m$, then $C_p^{k+1}X$ contains a vector space of dimension m.*

Proof Exactly like Proposition 2.1. ■

Lemma 3.3 *Let X be a compact and spherically compact subset of $\mathbb{H}$ with TX a quasibundle. If $D_{n,k} \setminus D_{n,k-1}$ is non-empty, then $k \leq \max\{n-2, 0\}$.*

Proof We proceed by induction on n. By Lemma 3.1, this is true for $n = 1$ or 2. So suppose that p is in $D_{n,k} \setminus D_{n,k-1}$. We may as well assume that $d_p^k = n$. Choose a vector v in $C_p^kX \setminus T_p^{k-1}X$. This vector must be a limit of vectors v_i in $T_{q_i}^{k-1}X$, where q_i converges to p. Since p is not in $D_{n,k-1}$, we may as well assume that all of the $d_{q_i}^{k-1} = m < n$.

At least one of the q_i is not in $D_{m,k-2}$. For suppose otherwise. Then by Lemma 3.2, $C_{q_i}^{k-1}X$ contains a vector space of dimension m. Since $d_{q_i}^{k-1} = m$ this means that $T_{q_i}^{k-1}X = C_{q_i}^{k-1}X$ and consequently, $v_i \in C_{q_i}^{k-1}X$. According to the definition of $C_{q_i}^{k-1}X$, this means that there are sequences $\{q_{i,j}\}_{i\geq 1, j\geq 1}$ and $\{v_{i,j}\}_{i\geq 1, j\geq 1}$ in $T_{q_{i,j}}^{k-2}X$ such that $q_i = \lim_{j\to\infty} p_{i,j}$ and $v_i = \lim_{j\to\infty} v_{i,j}$. By taking double limits, we see that there is a function f such that

$p = \lim_{i\to\infty} p_{i,f(i)}$ and $v = \lim_{i\to\infty} v_{i,f(i)}$. Therefore $v \in C_p^{k-1}X$. This contradiction proves the assertion in the first sentence of this paragraph.

It is now an easy matter to apply the induction hypothesis to any q_i not in $D_{m,k-2}$. We get that $k-1 \le m-2$ and $k \le n-2$. ■

Lemma 3.4 *Let X be a compact and spherically compact subset of $\mathbb{H}$ with TX a quasibundle. If $T_p^kX \setminus T_p^{k-1}X$ is non-empty, then $k \le d_p^k - 1$.*

Proof If $n = d_p^k > d_p^{k-1}$ and p is not in $D_{n,k-1}$, then $k \le n-2$. Otherwise, p is a limit of points q_i in $D_{n,k-1}$. As in the previous lemma, not all of the q_i can be in $D_{n,k-2}$ for otherwise $T_p^{k-1}X$ would be a limit of spaces of dimension n and hence (by Lemma 3.2) of dimension n itself. Therefore, by Lemma 3.3, $k-1 \le n-2$. ■

We can now finish the proof of Theorem 3.1.

Proof We begin by observing that some neighborhood of p must contain only points of dimension $\le n$. By Lemma 3.4, for every q in this neighborhood, $T_q^nX = T_q^{n-1}X$. From this it follows easily that $T_q^\alpha X = T_q^{n-1}X$ for any ordinal $\alpha > n-1$ and any q in the neighborhood. ■

4 Projection on the Tangent Space

Let $X \subset \mathbb{H}$ and let $p \in X$. In this section, we study the effect of a continuous linear projection, $\pi\colon \mathbb{H} \to T_pX$ on the set X. We will prove the Weak Projection Theorem as well as the Structure Theorem in this section. We begin by noting the following lemma whose proof is left to the reader.

Lemma 4.1 *If $\{p_i\}_{i\ge1}$ is a sequence in $X \subseteq \mathbb{H}$ converging to the point $p \in X$, and if $\xi_i \in T_{p_i}X$, $i \ge 1$, are vectors converging to ξ, then $\xi \in T_pX$.*

Lemma 4.2 *Let X be a compact and spherically compact subset of $\mathbb{H}$ with TX a quasibundle. Let $p \in X$ and let $\pi\colon \mathbb{H} \to T_pX$ be a linear projection. Then there exists an open neighborhood U of p (in X) and a positive number c such that for every $q \in U$ the restriction $\pi \restriction T_qX$ is bounded below by c (i.e., $\|\pi(\xi)\| > c\|\xi\|$ for all ξ in T_qX.)*

Proof If no such U and $c > 0$ exist, we may find a sequence of *unit* vectors $\{\xi_n\}_{n\ge1}$ with $\xi_n \in T_{p_n}X$, $p_n \to p$, and $\|\pi(\xi_n)\| < 1/n$. By Lemmas 4.1 and 2.3 (choosing a subsequence if necessary), we see that the $\{\xi_n\}_{n\ge1}$ converge to a *unit* vector $\xi \in T_pX$ with $\|\pi(\xi)\| = 0$. But then, since π is a projection, $\xi = \pi(\xi)$. This contradiction proves the lemma. ■

As an immediate consequence of this lemma we see that $\pi \restriction U$ is an immersion in the very weak sense that $d\pi(q)$ is injective for all $q \in U$. To see that $\pi\colon U \to \pi(U)$ is a diffeomorphism requires all of Sections 4 and 5. However, it is straightforward to see that $\pi\colon U \to \pi(U)$ is a bi-Lipschitz homeomorphism [18].

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Lemma 4.3 *Let X be a compact and spherically compact subset of $\mathbb{H}$ with TX a quasibundle. Let $p \in X$ and let $\pi \colon \mathbb{H} \to T_pX$ be any linear projection. Then there exists an open neighborhood U of p (in X) such that $\pi \restriction U$ is a bi-Lipschitz homeomorphism onto $\pi(U)$. (For a more general version, see Lemma 6.1.)*

It follows immediately from the next proposition that for $q \in U$, π carries the tangent space T_qX isomorphically onto the corresponding tangent space $T_{\pi(q)}\pi(U)$.

Lemma 4.4 *If U is a neighborhood of p (in X) as in Lemma 4.2 and $q \in U$, then π is a linear isomorphism from T_qX to $T_{\pi(q)}\pi(U)$.*

Proof Lemma 4.2 says that the map has no kernel. Therefore we need to show that its range is $T_{\pi(q)}\pi(U)$. To begin this task, note that the linearity and lower boundedness of π easily imply that

$$\pi\big((0,\infty)C^0_qX\big) = d\pi(q)(0,\infty)C^0_{\pi(q)}\pi(U).$$

By linearity, this equality extends from C^0 to T^0. Now an easy induction shows that for any ordinal α,

$$\pi(T^\alpha_qX) = d\pi(q)T^\alpha_{\pi(q)}\pi(U).$$

Therefore, to show that π is onto, it suffices to prove that $d\pi(q)(T_qX) = T_{\pi(q)}\pi(U)$. It is clear that for each ordinal α

$$d\pi(q)T^\alpha_qX \subseteq T^\alpha_{\pi(q)}\pi(U).$$

Hence $d\pi(q)(T_qX) \subseteq T_{\pi(q)}\pi(U)$.

To get the reverse inclusion, we use Theorem 2.1 of Section 2. We claim that $d\pi(q)(T_qX)$ contains $C^0_{\pi(q)}\pi(U)$ and that the correspondence $\pi(q) \mapsto d\pi(T_qX)$ is a quasibundle. Both these claims are easily verified and are left to the reader. ∎

If U is a neighborhood of p (in X) satisfying both Lemma 4.2 and Lemma 4.3, then $g = (\pi \restriction U)^{-1}$ is a bi-Lipschitz homeomorphism. Our eventual goal is to prove that g is C^1 (*i.e.*, that it can be extended to an actual C^1 map between the ambient spaces). At the moment we have a candidate for the differential of g. By Lemma 4.4, for any point $x \in \pi(U)$, the map π is a linear isomorphism from $T_{g(x)}X$ to $T_x\pi(U)$. Let $\lambda(x)$ be its inverse. We begin the task of showing that g is C^1 by showing that λ is continuous as a function of the two variables, $x \in \pi(U)$ and $v \in T_x\pi(U)$.

Lemma 4.5 *The function λ just defined is continuous.*

Proof Suppose $\{x_n\}$ is a sequence in $\pi(U)$ converging to x and $\{\xi_n\}$ a sequence converging to ξ with $\xi_n \in T_{x_n}\pi(U)$. We must show that $\lim_{n\to\infty} \lambda(x_n)(\xi_n) = \lambda(x)(\xi)$. For each $n \geq 1$, there exists (by Lemma 4.4) a unique $\eta_n = \lambda(x_n)(\xi_n) \in T_{g(x_n)}X$ such that $\pi(\eta_n) = \xi_n$. By Lemma 4.2, the sequence $\{\eta_n\}$ is bounded. In addition, if η is any limit point of $\{\eta_n\}$, then

$\pi(\eta) = \xi$ and $\eta \in T_{g(x)}X$ by Lemmas 2.3 and 4.1. Therefore, $\eta = \lambda(x)(\xi)$; that is, the sequence $\{\eta_n\}$ is bounded and its only limit point is $\lambda(x)(\xi)$. ■

We are now ready to take the first step towards proving that when $X \subset \mathbb{H}$ is compact and spherically compact with TX a quasibundle, the map g defined above is actually in our C^1 category with derivative $dg(x) = \lambda(x)$. To this end, we consider the restriction g_k of g to the set $\pi(U_k)$, where U_k is defined as in Equation 2.1. Clearly, $\pi(U)_k = \pi(U_k)$ and $U_k = X_k \cap U$. By replacing the open neighborhood U with a smaller one, we may assume that the above four lemmas hold for the compact set $\bar{U}$. We will use the Whitney Extension Theorem to show that g_k (and eventually g) is C^1. We may formulate this theorem as follows:

Theorem 4.1 **(The Whitney Extension Theorem)** *Let $L(\mathbb{R}^n : \mathbb{H})$ denote the space of continuous linear maps from $\mathbb{R}^n$ to $\mathbb{H}$. Let C be a compact subset of $\mathbb{R}^n$. If $G\colon C \to \mathbb{H}$ and $\Lambda\colon C \to L(\mathbb{R}^n : \mathbb{H})$ are continuous maps with the property that for $x, z \in C$,*

$$G(z) - G(x) - \Lambda(x)(z - x) \in o(\|z - x\|),$$

then there is a C^1 map $F\colon \mathbb{R}^n \to \mathbb{H}$ such that $F \restriction C = G$ and $dF \restriction C = \Lambda$.

For a proof of this theorem see [5] or [27]. It is also proven in [14, Chapter 1]. S. Bromberg [3] gives another version.

Note that an easy partition of unity argument extends this theorem to the case of locally compact sets. We recall that the set U_k is locally compact because TX is a quasibundle. Now we are ready to state our Weak Projection Theorem.

Theorem 4.2 **(The Weak Projection Theorem)** *Let $\mathbb{H}$ be a Hilbert space and let $X \subset \mathbb{H}$ be compact and spherically compact with TX a quasibundle. Let $p \in X$ and let U be an open neighborhood of p (in X) satisfying Lemmas 4.2 and 4.3. Then the map $g_k = (\pi \restriction U_k)^{-1}$ has a C^1 extension whose derivative is an extension of λ.*

Proof Since π is a projection, there is a unique continuous function $G_k\colon \pi(U) \to \ker(\pi)$ with $g_k(x) = x + G_k(x)$. That is, $\pi\big(x + G_k(x)\big) = x$. We would like to do the same thing for λ; *i.e.*, define the function Λ with domain, $\pi(U)_k \times T_pX$ via the equation, $\lambda(x)(\xi) = \xi + \Lambda(x)\xi$. The problem is that domain of λ is just the quasibundle $\{(x, \xi) : \xi \in T_x\pi(U)\}$. If we let $(x, \xi) \mapsto \phi(x)(\xi)$ be the orthogonal projection of T_pX onto $T_x\pi(U)$, then the composition, $\lambda(x)\big(\phi(x)(\xi)\big)$ has the right domain. We must show that it is continuous. This is where we use the fact that we are restricting ourselves to those x's such that T_x is of dimension k. If $\{x_n\}$ converges to x, then by our definition of tangent space, $\lim_{n\to\infty} T_{x_n}\pi(U) \subseteq T_x\pi(U)$. But since all these spaces have the same dimension this must actually be an equality, with the limit in $\mathbb{G}_k(\mathbb{H})$. Therefore ϕ is continuous and the equation $\Lambda(x)\xi = \lambda(x)\big(\phi(x)(\xi)\big) - \phi(x)(\xi)$ defines a continuous map Λ from $\pi(U)_k$ into $L\big(T_pX : \ker(\pi)\big)$ such that for $\xi \in T_x\pi(U)$, $\lambda(x)(\xi) = \xi + \Lambda(x)(\xi)$.

With these definitions, our goal is to find a C^1 function F extending G_k with dF extending Λ. To this end, we must show that for any $x, z \in \pi(U)_k$, we have

$$G_k(z) - G_k(x) - \Lambda(x)(z - x) \in o(\|z - x\|).$$

Assume on the contrary that this is false. Then there are sequences $\{x_n\}$ and $\{z_n\}$ in $\pi(U)_k$ both converging to some $x \in \pi(U)$ such that

$$\frac{\|G_k(z_n) - G_k(x_n) - \Lambda(x_n)(z_n - x_n)\|}{\|z_n - x_n\|} \geq \alpha > 0 \tag{4.1}$$

for some α and all n. We will derive a contradiction by showing that this sequence has a subsequence converging to zero. Since $(z_n - x_n)/\|z_n - x_n\|$ is a unit vector and the unit sphere in T_pX is compact, we may choose a subsequence so that it is convergent with limit, say, ξ. Note that ξ is a limit of secants and so is by definition in $C_x^0\pi(U)$ which is a subset of $T_x\pi(U)$. Since g and therefore G_k as well is Lipschitz, the sequence,

$$\frac{\|z_n + G_k(z_n) - \big(x_n + G_k(x_n)\big)\|}{\|z_n - x_n\|}$$

is bounded. In addition, because X is compact and spherically compact, we may assume that the sequence

$$\frac{z_n + G_k(z_n) - \big(x_n + G_k(x_n)\big)}{\|z_n + G_k(z_n) - \big(x_n + G_k(x_n)\big)\|}$$

has a limit. Consequently the sequence

$$\frac{z_n + G_k(z_n) - \big(x_n + G_k(x_n)\big)}{\|z_n - x_n\|} \tag{4.2}$$

has a limit ζ. Clearly $\pi(\zeta) = \xi$ and so $\lambda(x)(\xi) = \zeta$. Hence, by taking limits in (4.2), we obtain

$$\xi + \lim_{n\to\infty} \frac{G_k(z_n) - G_k(x_n)}{\|z_n - x_n\|} = \lambda(x)\xi.$$

Then, applying the continuity of Λ,

$$\lim_{n\to\infty} \frac{G_k(z_n) - G_k(x_n)}{\|z_n - x_n\|} = \Lambda(x)\xi = \lim_{n\to\infty} \Lambda(x_n) \frac{z_n - x_n}{\|z_n - x_n\|}.$$

This contradicts (4.1). ∎

As a consequence of this theorem, we obtain our first structure result for compact and spherically compact subsets of Hilbert space.

Lemma 4.6 *Let X be a compact and spherically compact subset of $\mathbb{H}$ with TX a quasibundle. Let X_k denote the set of all points $p \in X$ such that T_pX has dimension k. Then there is a C^1 submanifold M_k of dimension k in $\mathbb{H}$ such that $X_k \subseteq M_k$ and $T_pX = T_pM_k$ for all $p \in X_k$.*

Proof According to the theory so far presented, there is a locally finite open cover $\{Q_i\}_{i\geq 1}$ of X_k such that for each $i \geq 1$,

1. Q_i is convex and open in $\mathbb{H}$.

2. There is a point $p_i \in U_i \cap X_k$, where $U_i = Q_i \cap X$, and a number $c_i > 0$ so that orthogonal projection π_i of $\mathbb{H}$ onto $T_{p_i}X$ satisfies the inequality, $\|\pi_i(\xi)\| \geq c_i\|\xi\|$ for $\xi \in T_qX$ and $q \in U_i$. (See Lemma 4.2.) We may assume that $\overline{U_i} \cap X_k$ is compact.

3. For π_i and U_i as defined in (2), the restriction of $\pi_i \restriction U_i$ is a bi-Lipschitz homeomorphism.

4. There is an open neighborhood V_i of $\pi_i(U_i)$ in $T_{p_i}X$ and a C^1 function $g_i \colon V_i \to (T_{p_i}X)^{\perp}$ so that the function $x + g_i(x)$ is an extension of $(\pi_i \restriction U_i)^{-1}$.

We can now define a C^1 manifold N_i containing a piece of X_k. Namely, $N_i = \{x + g_i(x) : x \in V_i\}$. Note that the dimension of N_i is k. Also $T_qN_i = T_qX$ for every point q in $U_i \cap X_k$. To see that this equality holds, first note that by definition, $U_i \subseteq N_i$. Consequently, $T_qX \subseteq T_qN_i$. But now equality must hold because both spaces have the same dimension.

Our goal is to piece together the N_i to form a single manifold M_k satisfying the requirements of the lemma. To do this we shrink and isotopically deform the N_i, holding X_k fixed, so that the intersection of any two deformed manifolds is always an open subset of both.

The remainder of the proof results from an application of classically familiar techniques in our context. Let $\{P_i\}_{i\geq 1}$ be a shrinking of $\{Q_i\}_{i\geq 1}$. That is, $\{P_i\}_{i\geq 1}$ is an open cover of X_k with $\overline{P_i} \subseteq Q_i$ and $\overline{P_i} \cap X_k$ compact.

The shrinking and deforming will be done in stages. At the i-th stage, we will deform N_i and shrink some of the N_j for $j < i$. All the shrinking will be done within P_i. Since $\{P_i\}_{i\geq 1}$ is locally finite, this means that each N_j will be shrunk only finitely many times. Let $N_j^{(i)}$ be the stage i version of N_j. In order to ensure that M_k contains X_k, we will maintain the conditions

$$X_k \cap \bigcup_{j=1}^{i} \overline{P_j} \subseteq \bigcup_{j=1}^{i} N_j^{(i)}$$

and

$$T_qX = T_q \bigcup_{j=1}^{i} N_j^{(i)} \quad \text{for all } q \in X_k \cap \bigcup_{j=1}^{i} \overline{P_j}.$$

For $j > i$, let $N_j^{(i)} = N_j$ and let $N_i^{(1)} = N_i$. Plainly, the conditions hold for $i = 1$. Define $M_k^{(i)} = \bigcup_{j=1}^{i} N_j^{(i)}$.

Now assume by induction, that $i > 1$ and $M_k^{(i-1)}$ is given and that it is a C^1 manifold. We proceed to define the $N_j^{(i)}$ for $j < i$. From item (4) above, we see that $d\pi_i(q) \colon T_qN_i \to T_{\pi(q)}V_i$ is one-to-one. Because $d\pi_i(q)$ depends continuously on q as q varies over the C^1 manifold $M_k^{(i-1)}$, we see that it is one-to-one on a neighborhood of $\overline{Q_i} \cap X_k \cap \bigcup_{j=1}^{i-1} \overline{P_j}$ in $M_k^{(i-1)}$. On the other hand, π_i is itself one-to-one on the compact set, $\overline{Q_i} \cap X_k \cap \bigcup_{j=1}^{i-1} \overline{P_j}$. It is a classical exercise in differential topology to show there is an open neighborhood W_i of $\overline{Q_i} \cap X_k \cap \bigcup_{j=1}^{i-1} \overline{P_j}$ such that $\pi_i \colon W_i \to T_{p_i}X$ is a C^1 embedding. For $j < i$, define $N_j^{(i)} = (N_j^{(i-1)} \setminus \overline{Q_i}) \cup (W_i \cap N_j^{(i-1)})$. Then $N_j^{(i)}$ is an open subset of $N_j^{(i-1)}$ and so is a C^1 manifold of the same dimension containing $X_k \cap \bigcup_{j=1}^{i-1} \overline{P_j}$. Then $\bigcup_{j=1}^{i-1} N_j^{(i)}$ is an open subset of $M_k^{(i-1)}$ and is therefore also a C^1 manifold. Our next step is to define $N_i^{(i)}$ as an isotopic deformation of N_i so that $\bigcup_{j=1}^{i} N_j^{(i)}$ is also a C^1 manifold.

Because $\pi \restriction W_i$ is a C^1 embedding, and because $\dim W_i = \dim \pi_i(W_i) = \dim T_{p_i}X = k$, we see that there is a C^1 map f_i with $W_i = \{x + f_i(x) : x \in \pi_i(W_i)\}$. Furthermore, by replacing W_i with the interior of a slightly smaller closed set, we may assume that f_i is the restriction of a C^1 function mapping $T_{p_i}X$ into its orthogonal complement. We will deform g_i, the defining map for N_i, so that $f_i(x) = g_i(x)$ for $x \in W_i \cap \overline{P_i}$.

To this end, let $\alpha_i \colon T_{p_i}X \to [0,1]$ be a C^1 function which is identically zero near $T_{p_i}X \setminus Q_i$ and identically one on a neighborhood O_i of $\overline{P_i} \cap T_{p_i}X$ in $T_{p_i}X$. Define the isotopy, $i_t \colon N_i \to \mathbb{H}$ by

$$i_t\big(x + g_i(x)\big) = x + g_i(x) + t\alpha_i(x)\big(f_i(x) - g_i(x)\big).$$

By the Isotopy Extension Theorem [22], we may suppose that i_t is a global isotopy of $\mathbb{H}$ fixed outside Q_i. Note that $x \in X_k \cap \overline{Q_i}$ implies that $f_i(x) = g_i(x)$ and so we are really only deforming N_i on the points that don't count, the ones not in X_k. Let $N_i^{(i)} = i_1\big(\pi_i^{-1}(O_i) \cap N_i\big)$. Then $T_qX = T_q i_1(N_i)$ for every $q \in X_k \cap \overline{P_i}$. Also $N_i^{(i)} \cap \bigcup_{j=1}^{i-1} N_j^{(i)}$ is open in both $\bigcup_{j=1}^{i-1} N_j^{(i)}$ and $N_i^{(i)}$. Therefore $M_k^{(i)} = \bigcup_{j=1}^{i} N_j^{(i)}$ is a C^1 manifold satisfying our inductive requirements.

We note that $M_k^{(i-1)}$ and $M_k^{(i)}$ agree outside of Q_i. Because the cover Q_i is locally finite, we see that near any point the limit $M_k = \lim_{i\to\infty} M_k^{(i)}$ ceases to change as i increases. Therefore M_k is a k-dimensional C^1 manifold containing X_k satisfying the requirements of the theorem. ■

To state the next theorem, we recall that $\mathbb{G}_k(\mathbb{H})$ denotes the Grasmannian set of all k-dimensional linear subspaces of $\mathbb{H}$ with the obvious topology.

***Theorem 4.3* (The Structure Theorem)** *Let X be a compact and spherically compact subset of a Hilbert space $\mathbb{H}$ with TX a quasibundle. Then there exist C^1 submanifolds $\{M_i\}_{i\geq 1}$ of $\mathbb{H}$ with the following properties:*

(i) $\dim M_i = i$.
(ii) $X_i \subseteq M_i$.
(iii) $T_qX = T_qM_i$ *for* $q \in X_i$.
(iv) $M_i \cap X^{i+1} = \varnothing$.
(v) *For any metric d defining the topology of $\mathbb{G}_k(\mathbb{H})$, if $\{x_n\}$ is a sequence from M_k with $\lim_{n\to\infty} x_n = x \in X^{k+1}$, then there is a sequence $\{x_n'\}$ from X_k also converging to x with the property that* $\lim_{n\to\infty} d(T_{x_n}M_k, T_{x_n'}X) = 0$.

Proof By Lemma 4.6, there are manifolds $\{M_i'\}$ satisfying (i), (ii), and (iii). Let $\tilde{M}_i = M_i' \setminus (X_{i+1} \cup X_{i+2} \cup \cdots)$. Then the $\tilde{M}_i$ are C^1 manifolds satisfying (i)–(iv).

Our goal is to modify the $\tilde{M}_i$ so as to achieve property (v). To begin this process, note that X_k is the union of an increasing tower of compact sets each contained in the interior (with respect to X_k) of the next. Call this tower $C_1, C_2, \ldots$. It is straightforward to find a sequence of families of open sets, $\{\mathcal{C}_m\}_{m\geq 1}$ where each $\mathcal{C}_m$ is a collection of open subsets of $\tilde{M}_k$ and $X_k \subseteq \bigcup\{\cup\mathcal{C}_i : i \geq 1\}$ and any $O \in \mathcal{C}_m$ satisfies the following properties:

(a) $O \cap C_{m-1} \neq \varnothing$.
(b) $O \cap X_k$ is a subset of the interior (with respect to X_k) of C_{m+1}.
(c) $\operatorname{diameter}(O) < 1/m$.
(d) In $\mathbb{G}_k\mathbb{H}$, the diameter of $\{T_q\tilde{M}_k : q \in O\}$ is less than $1/m$.

Now let $M_k = \bigcup\{\cup\mathcal{C}_i : i \geq 1\}$.

We need to show that M_k satisfies property (v). Let $\{x_n\}$ and x be as in (v). For each $n \geq 1$, there is an integer $i(n)$ and an open set $O_n \in \mathcal{C}_{i(n)}$ such that $x_n \in O_n$. Our claim is that $\lim_{n\to\infty} i(n) = \infty$. If not, there is a subsequence of the $\{x_n\}$ contained in some $C_i \subseteq X_k$. This contradicts our hypothesis that $x \in X^{k+1}$. Finally let x'_n be any point in $X_k \cap O_n$. This is easily seen to satisfy property (v) and so the theorem is proved. ■

5 The Projection Theorem

Let X be a compact and spherically compact subset of a Hilbert space $\mathbb{H}$ with TX a quasibundle. We say that a C^1 manifold M is *k-canonical for X* if it satisfies (i), (ii), and (iii) of the Structure Theorem (Theorem 4.3). That is, $\dim(M) = k$, $X_k \subseteq M$, and $T_qX = T_qM$ for $q \in X_k$. Consider the following *fitting conditions*:

(i) $\mathbb{H} = \mathbb{R}^m \oplus (\mathbb{R}^m)^\perp$.
(ii) $\pi\colon \mathbb{H} \to \mathbb{R}^m$ is orthogonal projection.
(iii) U is a compact and spherically compact subset of $\mathbb{H}$ with TU a quasibundle, which is the graph of a Lipschitz map $g\colon \pi(U) \to (\mathbb{R}^m)^\perp$.
(iv) $\pi \restriction T_qU = d\pi(q)$ is bounded below by some $c > 0$ for all $q \in U$.
(v) $\pi(U_k) = \pi(U)_k$.
(vi) U_k is the graph of the restriction to $\pi(U)$ of a C^1 map $g_k\colon \mathbb{R}^m \to (\mathbb{R}^m)^\perp$.

Let us say that an isotopy i_t is vertical if $\pi\big(i_t(x)\big) = \pi(x)$.

Lemma 5.1 **(The Vertical Isotopy Lemma)** *If the fitting conditions hold with U a compact and spherically compact subset of a Hilbert space $\mathbb{H}$ with TU a quasibundle, then there exists a vertical C^1 isotopy $i_t\colon \mathbb{H} \to \mathbb{H}$ such that i_0 is the identity and $i_1(U) \subseteq \mathbb{R}^m$.*

Proof Let $k_0 = \max\{k : U_k \neq \varnothing\}$. Then U_{k_0} is closed and therefore compact and, by the Structure Theorem (Theorem 4.3), there is a k_0-canonical manifold M_{k_0} for U such that $\pi \restriction M_{k_0}$ is a diffeomorphism. Let ρ be the projection of $\mathbb{H} = \mathbb{R}^m \oplus (\mathbb{R}^m)^\perp$ onto $(\mathbb{R}^m)^\perp$. (ρ is the "complement" of π.) It is easily seen that the map $\rho \circ (\pi \restriction M_{k_0})^{-1}$ has a C^1 extension, f. We define a vertical isotopy j_t mapping $\mathbb{R}^m \oplus (\mathbb{R}^m)^\perp$ into itself by setting $j_t(x, y) = \big(x, y - tf(x)\big)$. Then $j_1(M_{k_0}) \subseteq \mathbb{R}^m$. For $k \leq k_0$, let $U^k = \bigcup_{r=k}^{k_0} U_r$. The isotopy j_t just defined satisfies the theorem with U^{k_0} in place of U. We will prove by a downward induction that such an isotopy exists for all $k \leq k_0$.

By induction, we have a vertical isotopy, i_t, such that $i_1(U^{k+1}) \subseteq \mathbb{R}^m$ and $T_qi_1(U) \subseteq \mathbb{R}^m$ for all $q \in i_1(U)^{k+1}$. In order to continue the induction, it is sufficient to assume that i_t has already moved U so that $U^{k+1} \subseteq \mathbb{R}^m$ and $T_qU \subseteq \mathbb{R}^m$ for all $q \in i_1(U)^{k+1}$ and then find a compactly supported vertical C^1 isotopy j_t such that:

(a) $j_t \restriction U^{k+1}$ is the identity.

(b) $dj_t(q)$ is the identity for $q \in U^{k+1}$.
(c) $j_t(M_k) \subseteq \mathbb{R}^m$ for some manifold k-canonical for U.

Because U^{k+1} is compact, we may find two open (with respect to $\mathbb{R}^m$) covers, $\{O_i\}_{i\in I}$ and $\{O'_i\}_{i\in I}$ of $\pi(U_k)$ with O_i open in $\mathbb{R}^m$ such that:

(i) $\{O_i \cap \pi(U_k)\}_{i\in I}$ is locally finite.
(ii) The closure of O'_i is compact and contained in O_i.
(iii) $O_i \cap U^{k+1}$ is empty.

Set $V = \bigcup\{O_i : i \in I\}$ and $V' = \bigcup\{O'_i : i \in I\}$. Then both V and V' are open in $\mathbb{R}^m$ and there are two k-canonical manifolds $M'_k \subset M_k$ for U_k with $M_k \subseteq \pi^{-1}(V)$ and $M'_k \subseteq \pi^{-1}(V')$. Consequently, if $\{q_n\}_{n\geq 1} \subseteq M'_k$ converges to some q, then either $q \in M_k$ or $q \in U^{k+1}$. Let C denote the closure of M'_k in M_k. Then the closure of M'_k in $\mathbb{H}$ (which we call $\overline{M'_k}$) is contained in $C \cup U^{k+1}$. Similarly, $\overline{\pi(M'_k)}$ in $\mathbb{R}^m$ is contained in $\pi(C) \cup U^{k+1}$. Furthermore, $\overline{\pi(C)}$ is contained in $\pi(C) \cup U^{k+1}$, which is therefore a closed set.

For each $x \in \pi(C)$ define an open neighborhood W_x of x in $\mathbb{R}^m$ by setting $W_x = \{y \in \mathbb{R}^m \mid d(x, U^{k+1}) > \|x - y\|\}$. Of course, if $y \in U^{k+1}$, then $d(x, U^{k+1}) \leq \|x - y\|$ so that $W_x \cap U^{k+1} = \varnothing$. Thus, $W = \bigcup\{W_x \mid x \in \pi(C)\}$ is an open neighborhood of $\pi(C)$ in $\mathbb{R}^m$, disjoint from U^{k+1}. Then we have the inclusion

$$\pi(C) \cup U^{k+1} \subset \pi(C) \cup (\mathbb{R}^m \setminus W)$$

with both unions closed.

Next, define a Whitney 1-jet, $(\psi, \delta\psi)$ on $\pi(C) \cup (\mathbb{R}^m \setminus W)$ as follows: Let $\hat{\psi}\colon W \to (\mathbb{R}^m)^\perp$ be the C^1 map defined by setting

$$(\pi \restriction M_k)^{-1}(x) = \big(x, \hat{\psi}(x)\big) \in \mathbb{R}^m \oplus (\mathbb{R}^m)^\perp$$

for $x \in \pi(M_k)$. Set $\psi(x) = \hat{\psi}(x)$ if $x \in \pi(C)$ and $\psi(x) = 0$ otherwise. Define $\delta\psi$ by setting $\delta\psi(x) = d\hat{\psi}(x)$ if $x \in \pi(C)$ and 0 otherwise. To check that $(\psi, \delta\psi)$ really is a Whitney 1-jet, we must show that ψ and $\delta\psi$ are continuous and that $(\psi, \delta\psi)$ satisfies the Whitney Extension condition in Theorem 4.1.

To see that ψ is continuous, observe that $\psi \restriction \pi(U_k) \cup U^{k+1} = \rho \circ (\pi \restriction U^k)^{-1}$, which is continuous. Then $\psi \restriction \pi(U^k) \cup (\mathbb{R}^m \setminus W) = \big(\psi \restriction \pi(U_k) \cup U^{k+1}\big) \cup \big(\text{ZERO} \restriction (\mathbb{R}^m \setminus W)\big)$ (where ZERO is the identically zero function) is continuous because it is the union of two continuous maps with closed common domain. Finally, for the same reason $\psi = \big(\psi \restriction \pi(C)\big) \cup \big(\psi \restriction \pi(U^k) \cup (\mathbb{R}^m \setminus W)\big)$ continuous.

To prove that $\delta\psi$ is continuous, we show that if $\{x_n\}_{n\geq 1} \subseteq \pi(C)$ converges to a point $x = \pi(x) \in \mathbb{R}^m \setminus V$, then $\lim_{n\to\infty} d\hat{\psi}(x_n) = 0$. To this end, observe that we must have $x \in U^{k+1}$ so that $T_x U \subseteq \mathbb{R}^m$. On the other hand, if $q_n = \big(x_n, \hat{\psi}(x_n)\big)$, then there is a corresponding sequence $\{q'_n\}_{n\geq 1} \subseteq U_k$ with $q'_n = \big(x'_n, \hat{\psi}(x'_n)\big)$ and $\lim_{n\to\infty} d(T_{q_n} M_k, T_{q'_n} U) = 0$ with respect to a compatible metric on the Grassmannian $G_k(\mathbb{H})$. (See the proof of the Structure Theorem 4.3.) In addition, by definition of TU any limit point of the sequence $\{T_{q'_n}\}_{n\geq 1}$ lies in $T_q(U) \subset \mathbb{R}^m$. Consequently, we have $\lim_{n\to\infty} \theta(T_{q_n}, \mathbb{R}^m) = 0$, where θ denotes the one-sided Hausdorff distance. Therefore, if $v_n \in T_{x_n} V$ is a sequence of vectors such that

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$u_n = \big(v_n, d\hat{\psi}(x_n)v_n\big)$ is a unit vector in $T_{p_n}M_k$, we have $\lim_{n\to\infty} d\hat{\psi}(x_n)v_n = 0$. It follows that $\lim_{n\to\infty} d\hat{\psi}(q_n) = 0$, so that $\delta\psi$ is continuous.

Finally, in order to show that the 1-jet $(\psi, \delta\psi)$ satisfies the Whitney Extension condition one must consider several cases. We wish to show that for any pair of sequences $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ in $\pi(C)\cup(\mathbb{R}^m\setminus W)$ with $x_n \neq y_n$, for all n, and $\lim_{n\to\infty} x_n = x = \lim_{n\to\infty} y_n$, we have

$$\lim_{n\to\infty} \frac{\psi(x_n) - \psi(y_n) - \delta\psi(y_n)(x_n - y_n)}{\|x_n - y_n\|} = 0. \tag{5.1}$$

If the limit point x lies in either of the sets $\pi(C)$ or $\mathbb{R}^m \setminus (W \cup U^{k+1})$, the fact that $\hat{\psi}$ and ZERO are C^1 leaves us with nothing to prove. In the remaining case, we have $x \in U^{k+1}$.

By taking subsequences, we may assume that both sequences lie in $\pi(C)$, both lie in $\mathbb{R}^m \setminus W$, or one in each. When both lie in $\mathbb{R}^m \setminus W$, the case is trivial. When both lie in $\pi(C)$, we note that we have already shown that $\lim_{n\to\infty} d\hat{\psi}(x_n) = 0$ so that we need only show that

$$\lim_{n\to\infty} \frac{\hat{\psi}(x_n) - \hat{\psi}(y_n)}{\|x_n - y_n\|} = 0. \tag{5.2}$$

We may assume that the normalized vectors

$$\xi_n = \frac{\big(x_n - y_n, \hat{\psi}(x_n) - \hat{\psi}(y_n)\big)}{\big(\|x_n - y_n\|^2 + \|\hat{\psi}(x_n) - \hat{\psi}(y_n)\|^2\big)^{1/2}}$$

converges to some $\xi \in T_xU \subset \mathbb{R}^m$. Consequently, using the fact that $\hat{\psi}$ is Lipschitz, we see that the limit equation (5.2) is valid.

When $\{x_n\}_{n\geq 1} \subset \pi(C)$ and $\{y_n\}_{n\geq 1} \subset \mathbb{R}^m \setminus W$, the limit equation (5.1) becomes

$$\lim_{n\to\infty} \frac{\hat{\psi}(x_n)}{\|x_n - y_n\|} = 0. \tag{5.3}$$

For each x_n, there exists $z_n \in U^{k+1}$ such that $\|x_n - z_n\| \leq (1 + 1/n)d(x_n, U^{k+1})$. Using the definition of W and the fact that $y_n \in \mathbb{R}^m \setminus W$, we arrive at the inequality

$$\|x_n - z_n\| \leq (1 + 1/n)\|x_n - y_n\| \tag{5.4}$$

so that, using $\psi(z_n) = 0$, we obtain

$$\frac{\|\hat{\psi}(x_n)\|}{\|x_n - y_n\|} \leq \Big(1 + \frac{1}{n}\Big)\frac{\|\psi(x_n) - \psi(z_n)\|}{\|x_n - y_n\|}$$

and equation (5.3) follows from an argument like that establishing equation (5.2).

Finally, we consider the case when $\{x_n\}_{n\geq 1} \subset \mathbb{R}^m \setminus W$ and $\{y_n\}_{n\geq 1} \subset \pi(C)$. Then our limit equation (5.1) becomes

$$\lim_{n\to\infty} \frac{-\hat{\psi}(y_n) - d\hat{\psi}(y_n)(x_n - y_n)}{\|x_n - y_n\|} = 0.$$

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We argue as we did for equation (5.3) to show that

$$\lim_{n\to\infty} \frac{\hat{\psi}(y_n)}{\|x_n - y_n\|} = 0$$

and we use the continuity of $\delta\psi$ to establish that

$$\lim_{n\to\infty} \frac{d\hat{\psi}(y_n)(x_n - y_n)}{\|x_n - y_n\|} = 0.$$

The proof is now complete. An obvious alarm bell is that nowhere in our proof do we appear to involve the expected interaction between the secant difference $\psi(x_n) - \psi(y_n)$ and its differential approximant $\delta\psi(y_n)(x_n - y_n)$. The only place this interaction appears non-trivially is in the case $x \in \pi(C)$, in which no argument is necessary because $\hat{\psi}$ is already C^1. In all other cases, the approximant $\delta\psi(x)$ turns out to vanish, and so does not contribute. ∎

Theorem 5.1 **(The Projection Theorem)** *Let $\mathbb{H}$ be a Hilbert space and let $X \subset \mathbb{H}$ be compact and spherically compact with TX a quasibundle. Let $p \in X$ and let π be a projection from $\mathbb{H}$ onto T_pX. If U is a sufficiently small neighborhood of p (in X), then the projection $\pi \restriction U$ is a C^1 diffeomorphism. By "sufficiently small", we mean that U satisfies Lemmas 4.2 and 4.3.*

Proof Apply Lemma 5.1. Because i_t is a vertical isotopy, we have $i_1 \restriction U = \pi \restriction U$. ∎

The Projection Theorem can be used to give an easy proof of the Inverse Function Theorem.

Theorem 5.2 **(The Inverse Function Theorem)** *Let $\mathbb{H}$ be a Hilbert space and let $X \subset \mathbb{H}$ and $Y \subset \mathbb{H}$ be compact and spherically compact with TX and TY quasibundles. Let f be a differentiable map from X onto Y. If for some point $p \in X$, the differential $df(p)$ is a linear isomorphism, then there is a neighborhood U of p (in X) such that $f \restriction U$ is a diffeomorphism.*

We now arrive at our other main result.

Theorem 5.3 **(Generalized Whitney Embedding Theorem)** *Let $\mathbb{H}$ be a Hilbert space and let X be a compact and spherically compact subset of $\mathbb{H}$ with TX a quasibundle. Then there is a C^1 embedding of X into $\mathbb{R}^N$ for some N finite.*

The usual proof [7], using the Projection Theorem, works. Furthermore, by using the Structure Theorem, we see, in the usual way, that $N = 2\dim_S X + 1$ will do, where $\dim_S X$ denotes the the *smooth dimension* of X given by $\dim_S X = \max\{\dim T_pX : p \in X\}$.

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6 An Alternative Form of the Embedding Theorem

In this section we present a cleaner formulation of our embedding theorem (Theorem 5.3) by using the concept of *tractability* introduced in [18]. For simplicity, we present our definition for a Hilbert space, though it and many of its elementary consequences generalize to Banach spaces in which C^1 functions separate disjoint compact sets.

We say that a subset X of a Hilbert space $\mathbb{H}$ is *tractable at a point* $p \in X$ if and only if for any continuous projection $\pi \colon \mathbb{H} \to T_p^0 X$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $y, z \in X$ with $\|x - y\| < \delta$ and $\|x - z\| < \delta$ we have $\|(1 - \pi)\big((y - z)/\|y - z\|\big)\| < \varepsilon$. We say that X is *tractable* if it is tractable at every point.

Of course, it is clear that every subset of a finite dimensional Hilbert space is tractable. As an illustration of the content of this definition, we have the following result.

Lemma 6.1 *Let X be a compact subset of a Hilbert space $\mathbb{H}$. Suppose that X is tractable and that $T_p^0 X$ is finite dimensional for every $p \in X$. Then there is a C^1 and bi-Lipschitz homeomorphism of X with a subset of $\mathbb{R}^N$ for some N finite (cf. Lemma 4.3).*

The proof is straightforward; it is getting the inverse to be C^1 that requires the labor of Sections 4 and 5.

It is clear that spherical compactness implies tractability. The converse fails in general but does hold when the space $T_p^0 X$ is finite dimensional for every $p \in X$.

Lemma 6.2 *Let X be a compact subset of a Hilbert space $\mathbb{H}$. Suppose that X is tractable and that $T_p^0 X$ is finite dimensional for every $p \in X$. Then X is spherically compact.*

With this preamble, we may state an alternative form of the embedding theorem (Theorem 5.3).

***Theorem 6.1* (Generalized Whitney Embedding Theorem)** *A compact subset X of a Hilbert space $\mathbb{H}$ is C^1 embeddable into a finite dimensional Euclidean space if and only if X is tractable and TX is a quasibundle.*

One may be tempted to hope from Lemma 6.1 and Theorem 6.1 that yet another equivalent condition for finite C^1 embeddability would be that TX be a quasibundle and X be bi-Lipschitz embeddable in a finite dimensional Euclidean space. For a counterexample, see [20]; for the convenience of the reader, we briefly recall this example in Section 8 below.

7 Smoothing and Disjunction

In this section we wish to develop a smoothing theorem somewhat in the general spirit of [10]; that is, we show that suitable sets have tangent quasibundles. We will show that if a quasibundle F can play the role, in a suitable Lipschitz sense, of a tangent quasibundle for a subset X of $\mathbb{H}$, then there is an arbitrarily small bi-Lipschitz isotopy moving X to a new set (not necessarily C^1 diffeomorphic to X) which has a tangent quasibundle. (See Section 2.)

We note that if F is a quasibundle over X and Y is a closed subset of X, and G is a quasibundle over Y such that $F_x \subseteq G_x$ for every point $x \in Y$, then we may extend G to a quasibundle over X by defining $G_x = F_x$ when $x \in X \setminus Y$. Furthermore, if $f\colon X \to Y$ is continuous and G is a quasibundle over Y, we may define the *induced quasibundle* f^*G by setting $f^*G_x = G_{f(x)}$. We let π_{F_x} denote the orthogonal projection onto F_x.

If F and G are quasibundles over the same set X, we say that F and G are *equivalent* if there is a homeomorphism from $|F|$ to $|G|$ of the form $(x, y) \mapsto \big(x, \phi_x(y)\big)$ where each ϕ_x is a linear isomorphism from F_x to G_x. The next lemma justifies a rather simple method for deforming a quasibundle into an equivalent one.

Lemma 7.1 *Suppose that $p_0 \in X \subset \mathbb{H}$, that $F\colon X \to \mathbb{G}(\mathbb{H})$ is a quasibundle, and that γ is a continuous function mapping X into $[0,1]$ whose support is contained in a neighborhood U of p_0 for which there is an $\alpha > 0$ with $\|\pi_{F_{p_0}} v\| \geq \alpha \|v\|$ for all $v \in F_z$ and $z \in U \cap X$. For all $y \in X$, let $G_y = \{\big(1-\gamma(y)\big)v + \gamma(y)\pi_{F_{p_0}} v : v \in F_y\}$. Then G is a quasibundle equivalent to F.*

Proof First we must check that G is a quasibundle. By Lemma 2.3, we need only show that $\sigma(G)$ is compact. So suppose that $\{x_n\}_{n\geq 1} \subset X$ and $v_n \in G_{x_n}$ and that the x_n converge to $x \in X$ and the v_n have norm 1. We must show that a subsequence of $\{v_n\}_{n\geq 1}$ converges to a point of G_x.

If x is not in the support of γ, then for all large n, $G_{x_n} = F_{x_n}$ and we are done. If x is in the support of γ, then, for large n, $x_n \in U$. Choose $w_n \in F_{x_n}$ such that $v_n = \big(1-\gamma(x_n)\big)w_n + \gamma(x_n)\pi_{F_{p_0}} w_n$. Then $x_n \in U$ implies that $\|\pi_{F_{p_0}} w_n\| \geq \alpha \|w_n\|$. Also, elementary geometry reveals that $\|v_n\| \geq \|\pi_{F_{p_0}} w_n\|$. Therefore the sequence $\{z_n\}_{n\geq 1}$ is bounded and we may as well assume it converges to $w \in F_x$ because $\sigma(F)$ is compact. Then the sequence $\{v_n\}_{n\geq 1}$ converges to $v = \big(1-\gamma(x)\big)w + \gamma(x)\pi_{F_{p_0}} w$ in G_x.

Secondly, we must show that G is equivalent to F. Define $\phi_x(v) = \big(1-\gamma(x)\big)v + \gamma(x)\pi_{F_{p_0}} v$. Then ϕ_x is clearly linear with image G_x. We must show it to be an isomorphism. If x is not in the support of γ, ϕ_x is the identity. Otherwise, $x \in U$ and the Lipschitz condition guarantees that ϕ_x has an empty kernel.

Finally, it is a routine exercise, left to the reader, to prove that the map $(x, v) \to \big(x, \phi_x(v)\big)$ and its inverse are both continuous. ■

The lemma above supplies us with the facts that we need. We will, however, require some further terminology. A quasibundle G *expands* F (or F is a *sub-bundle* of G) (in symbols $G \geq F$) if for all x, $F_x \subseteq G_x$. A quasibundle F is *pseudo-tangent* if for each point $x \in X$, the orthogonal projection onto F_x is bi-Lipschitz on some $U \cap X$ where U is a neighborhood of x. Lemma 4.3 says that the tangent quasibundle is pseudo-tangent in this sense.

Theorem 7.1 **(The Local Smoothing Theorem)** *Let X be a compact subset of $\mathbb{H}$, F a pseudo-tangent quasibundle over X, and p_0 a point of X. Then there is an arbitrarily small bi-Lipschitz isotopy i_t of the inclusion of X in $\mathbb{H}$ and a quasibundle F' over $i_1(X)$ and an expansion G of F' with the following properties (in what follows, the phrases "near to" and "far from" all refer to the same open set):*

(1) $i_1^* F'$ is equivalent to F.

(2) $\dim G_{i_1(p)} \le \dim F_{p_0}$ *for* p *near* p_0.
(3) $G_{i_1(p)} = F'_p$ *for* p *far from* p_0.
(4) $T_{i_1(p)} i_1(X)$ *is a subspace of* $G_{i_1(p)}$ *for* p *near* p_0.
(5) *The orthogonal projection* $\pi_{G_{i_1(p)}}$ *restricts to a bi-Lipschitz embedding of a neighborhood in* $i_1(X)$ *of* $i_1(p)$ *into* $G_{i_1(p)}$.

Proof Let U be related to p_0 as in Lemma 7.1. Let γ be a continuous function mapping X into $[0,1]$ with support contained in U which is equal to one on a neighborhood, V, of p_0. By taking U small enough, the pseudo-tangent property allows us to assume that $\pi_{F_{p_0}}$ is a bi-Lipschitz homeomorphism on $U \cap X$. Let λ be the inverse of $\pi_{F_{p_0}} \restriction (U \cap X)$. As in the proof of the Weak Projection Theorem 4.2, there is a Lipschitz map $f \colon \pi_{F_{p_0}}(V) \to F_{p_0}^{\perp}$ such that $\lambda(x) = x + f(x)$. Let $x_0 = \pi_{F_{p_0}} p_0$ and let

$$g(x) = \Big(1 - \gamma\big(\lambda(x)\big)\Big) f(x) + \gamma\big(\lambda(x)\big) f(x_0).$$

Let

$$X' = (X \setminus U) \cup \operatorname{graph}(g).$$

We next define a bi-Lipschitz isotopy $i_t \colon X \to \mathbb{H}$ such that i_0 is the identity and $i_1(X) = X'$. Let

$$i_t(p) = p \quad \text{if } p \in X \setminus U$$
$$i_t\big(x + f(x)\big) = x + (1-t) f(x) + t g(x) \quad \text{otherwise.}$$

Clearly this isotopy is bi-Lipschitz and we may make it as small as we like by choosing U sufficiently small.

Define the new quasibundle F' over X' as in Lemma 7.1 by setting

$$F'_{i_1(p)} = \big\{\big(1 - \gamma(p)\big) v + \gamma(p) \pi_{F_{p_0}} v : v \in F_p\big\}.$$

Using Lemma 7.1, we see that $i_1^* F'$ is equivalent to F.

Also note that for $p \in \bar{V}$, $\gamma(p) = 1$ and consequently, $F'_{i_1(p)} = \pi_{F_{p_0}} F_p$. Therefore, for $p \in \bar{V}$, we have $F'_{i_1(p)} \subseteq F_{p_0}$. Accordingly, we may expand the quasibundle F' to a quasibundle G by setting $G_{i_1(p)} = F_{p_0}$ for $p \in \bar{V}$ and $G_{i_1(p)} = F'_{i_1(p)}$ otherwise. Furthermore, if $p = \lambda(x) \in V$, then $g(x) = f(x_0)$ and so $\operatorname{graph}(g) \cap V$ is a translation of a subset of F_{p_0} and therefore $T_{i_1(p)} X' \subseteq F_{p_0}$. Thus far we have verified conclusions (1)–(4) of the theorem.

To verify conclusion (5), it suffices to show that $\pi_{F_{p_0}} \restriction i_1(U)$ is a bi-Lipschitz homeomorphism. Any projection is bounded above, so we need to show it is bounded below. Since g is Lipschitz, $\|g(x) - g(y)\| \le L \|x - y\|$ for some number L. Then

$$\frac{1}{\sqrt{L^2 + 1}} \sqrt{\|x - y\|^2 + \|g(x) - g(y)\|^2} \le \|x - y\|.$$

Since $\pi_{F_{p_0}}\big(x + g(x)\big) = x$, this says

$$\frac{1}{\sqrt{L^2 + 1}} \|x + g(x) - \big(y + g(y)\big)\| \le \|\pi_{F_{p_0}}\big(x + g(x)\big) - \pi_{F_{y_0}}\big(y + g(y)\big)\|,$$

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which is what was to be shown. ■

A global version of Theorem 7.1 can be derived by piecing together the local isotopies as in the proof of Lemma 4.6. We state the following theorem whose proof is left to the reader:

Theorem 7.2 *Let $\mathcal{U}$ be an open cover of a compact subset X of a Hilbert space $\mathbb{H}$ and let F be a pseudo-tangent quasibundle over X. Then there exist an arbitrarily small bi-Lipschitz isotopy $i_t \colon X \to \mathbb{H}$ with i_0 the identity, a quasibundle F' over $i_1(X)$, an expansion G of F', and an open refinement $\mathcal{V}$ of $\mathcal{U}$ with the following properties:*

(1) $i_1^* F'$ is equivalent to F.
(2) For each point $p \in X$ there exists $V \in \mathcal{V}$ such that $p \in V$ and $\max\{\dim G_{i_1(p)} : p \in V\} \le \max\{\dim F_p : p \in V\}$.
(3) $Ti_1(X)$ is equivalent to a sub-bundle of G and therefore is a quasibundle.
(4) G is pseudo-tangent to $i_1(X)$.

We note that $i_1(X)$ has, as promised, tangent quasibundle $Ti_1(X)$ as well as a pseudo-tangent quasibundle G. We do not know, however, whether some such G can be made to coincide with $Ti_1(X)$. For an example, in Euclidean space, illustrating the statement of this theorem, consider the edges of a square embedded in a concentric circle. For each point p on the square, let F_p be the tangent line to the corresponding point on the circle. Then projection onto F_p is locally bi-Lipschitz even at the corners and, of course, the square can be isotopically transformed into the circle so that F_p becomes the actual tangent space. Note that Theorems 7.1 and 7.2 do not require X (in a Hilbert space) to be spherically compact. Unfortunately, even if X is spherically compact, $i_1(X)$ need not be. (See Proposition 8.4 and Example 8.2.)

In Euclidean space, using the same kind of reasoning as in Theorem 7.2, we may obtain a smooth disjunction theorem along the lines of [13]. Let $\dim_H X$ be the Hausdorff dimension of X and recall that $\dim_S X = \max\{\dim T_p X : p \in X\}$.

***Theorem 7.3* (The Smooth Disjunction Theorem)** *Let X and Y be locally compact subsets of $\mathbb{R}^N$ with $\dim_S X + \dim_H Y < N$. There is a small C^1 isotopy $i_t \colon X \to \mathbb{R}^N$ with i_0 the identity and $i_1(X) \cap Y = \varnothing$.*

The main idea of the proof is that we may use the Projection Theorem (Theorem 5.1) near a point p_0 to represent X as the graph of a C^1 map $f \colon T_{p_0} X \to (T_{p_0})^\perp$. Since $\dim(T_{p_0} X)^\perp > \dim_H Y$, the portion of Y near p_0 projects onto a nowhere dense subset of $(T_{p_0} X)^\perp$ and X may thus be smoothly isotoped near p_0 to miss Y.

Analogously, there is a Lipschitz disjunction theorem. Define the notion of the Lipschitz dimension of a locally compact set X in $\mathbb{R}^N$ as

$$\dim_L X = \min\{\max\{\dim F_p : p \in X\} : F \text{ a pseudo-tangent quasibundle over } X\}.$$

***Theorem 7.4* (The Lipschitz Disjunction Theorem)** *Let X and Y be locally compact subsets of $\mathbb{R}^N$ with $\dim_L X + \dim_H Y < N$. There is a small bi-Lipschitz isotopy $i_t \colon X \to \mathbb{R}^N$ with i_0 the identity and $i_1(X) \cap Y = \varnothing$.*

The proof of this theorem is similar to Theorem 7.3, except that X is represented locally as the graph of a bi-Lipschitz map $f\colon F_{p_0} \to F_{p_0}^{\perp}$ where F is a quasi-tangent quasibundle to X such that $\dim F = \dim_L X$. Of course, the above two theorems hold, more easily, in a Hilbert space.

8 Remarks and Examples

1) Let X be a compact and tractable subset of a Hilbert space $\mathbb{H}$ with TX a quasibundle. Referring back to our embedding theorem (Theorem 5.3 or Theorem 6.1), we note that it is almost trivial (as in Lemma 4.3 or Lemma 6.1) to show that there is a projection $\pi\colon \mathbb{H} \to F$, where F is a finite dimensional linear subspace of $\mathbb{H}$, which bi-Lipschitz embeds X into F with non-singular differential at every point. The whole point of the technical Sections 4 and 5 is to prove that $(\pi \restriction X)^{-1}$ is also C^1.

2) In [23], Repovš, Spokenkov, and Ščepin show that a compact C^1-homogeneous subset of $\mathbb{R}^N$ is a C^1-submanifold of $\mathbb{R}^N$. This fact together with our Embedding Theorem (Theorem 5.3) establish the following result.

Theorem 8.1 *A compact subset X of a Hilbert space $\mathbb{H}$ is a finite dimensional C^1-submanifold of $\mathbb{H}$ if and only if X is C^1-homogeneous, spherically compact, and TX is a quasibundle.*

Alternatively, by combining Theorem 6.1 with the results of [23], we arrive at the following result which is equivalent to the above theorem.

Theorem 8.2 *A compact subset X of a Hilbert space $\mathbb{H}$ is a finite dimensional C^1-submanifold of $\mathbb{H}$ if and only if X is C^1-homogeneous, tractable at a point p, with TX a quasibundle.*

One may surmise that the C^1-homogeneity condition is sufficiently strong to allow replacement of the condition that TX be a quasibundle with the condition that some T_pX be finite dimensional. We are unable to determine whether this conjecture is true. However, the condition that X be spherically compact is necessary, as Example 8.1 below shows.

3) We use the same Example 8.1 below to establish three facts.

Fact 8.1 Let X be a subset of a Hilbert space $\mathbb{H}$. Then X compact with TX a quasibundle does not imply that X is spherically compact.

Fact 8.2 Let X be a subset of a Hilbert space $\mathbb{H}$. Then X compact and C^1-homogeneous with TX a quasibundle does not imply that X is spherically compact.

Fact 8.3 Let X be a subset of a Hilbert space $\mathbb{H}$. Then X compact and bi-Lipschitz embeddable into a finite dimensional Euclidean space, with TX a quasibundle does not imply that X is C^1 embeddable in a finite dimensional Euclidean space.

The example is actually constructed in [20]; we recall here the construction of [20] in order to produce some examples. Let (X, d, μ) consist of a compact space X, a metric d yielding the topology of X, and a Borel probability measure μ on X which is positive on nonempty open sets. Then a *canonical map* $\iota\colon X \to L^p(\mu)$, $1 \le p < \infty$, *for the metric* d is defined by setting $\iota(x) = d(x, \ )$. It is easy to check that there exists $s \ge 1$ such that

$$d(x,y)^s \le \|\iota(x) - \iota(y)\|_{L^2} \le d(x,y).$$

If d is an ultrametric (*i.e.* $d(x,y) \le \max\{d(x,z), d(y,z)\}$), then for any $t > 0$ the power distance function d^t is another ultrametric yielding the same topology. Next, we define

$$\mathrm{Dim}(X,d,\mu) = \sup\left\{ \frac{\log \mu\big(B_d(x,r)\big)}{\log r} : x \in X, r > 0 \right\},$$

where $B_d(x,r)$ is the closed ball (in X) of radius r centered at $x \in X$.

Theorem 8.3 (**[20, Lemma 3.1]**) *If* (X,d,μ) *is as above with* d *ultrametric and* $\mathrm{Dim}(X,d,\mu) = D < 2$, *then the canonical map* ι *for the metric* $d^{1-D/2}$ *is a bi-Lipschitz embedding of* (X,d) *into* $L^2(\mu)$.

In fact, for $1 \le p < \infty$ and any $0 < D < p$, the canonical map for the ultrametric $d^{1-D/p}$ is a bi-Lipschitz embedding of (X,d) into $L^p(\mu)$.

Example 8.1 It is shown in [20] that the image $\iota(X) \subset L^2(\mu)$ is smoothly zero-dimensional but not spherically compact; consequently TX is a constant map and so is a quasi-bundle.

We may see that X cannot be spherically compact within our present context, without reference to [20]: If X were spherically compact, then it would be C^1 diffeomorphic to a compact subset Y of some finite dimensional Euclidean space. But Y would have limit points so that $|TY|$ could not be zero dimensional; then $|TX|$ could not be zero dimensional either.

Now Fact 8.1 is established. For Fact 8.2, we use $X = \mathbb{Z}_2^\omega$, the topological group of 2-adic integers with the standard ultrametric and the corresponding Haar measure μ. Then $\mathbb{Z}_2^\omega$ acts on $L^2(\mu)$ in a C^1 way via the regular representation, and $\iota(X)$ is the orbit $\mathbb{Z}_2^\omega \cdot f$, where $f(x) = d(1,x)$. Thus $\iota(X)$ is C^1-homogeneous and Fact 8.2 is established. Finally, for Fact 8.3 we note that $\mathbb{Z}_2^\omega$ is bi-Lipschitz isomorphic to the standard Cantor ternary set.

For our next fact, we build on Example 8.1 to produce Example 8.2. The fact itself is a little complicated, but perhaps still surprising. In the sequel, by the phrase "is C^1-finitely embeddable" we mean "admits a C^1 embedding into a finite dimensional Euclidean space."

Fact 8.4 Let Γ be a compact and C^1-finitely embeddable subset of a Hilbert space $\mathbb{H}$. Let $\pi\colon \mathbb{H} \to \mathbb{H}$ be an orthogonal projection which restricts to a bi-Lipschitz equivalence $\pi\colon \Gamma \to \pi(\Gamma)$. It does not follow that $\pi(\Gamma)$ is C^1-finitely embeddable, not even if the kernel of π is finite dimensional.

Example 8.2 Let (X,d,μ) be one of the ultrametric-measure spaces in Example 8.1 and let $f\colon X \to \mathbb{R}^N$ be a bi-Lipschitz embedding. Let $Y = f(X)$ and define $g\colon Y \to L^2(\mu)$ by setting $g = \iota \circ f^{-1}$, where ι is the canonical map for the metric $d^{1-D/2}$. Finally, let $\Gamma = \{\big(y, g(y)\big) : y \in Y\} \subset \mathbb{R}^N \times L^2(\mu) = \mathbb{H}$.

The following easy variant of the argument in [20] shows that Γ is spherically compact and $T\Gamma$ is a quasibundle: For $n \geq 1$, let

$$u_n = \frac{\big(x_n - y_n, \iota(x_n) - \iota(y_n)\big)}{\sqrt{\|x_n - y_n\|^2_{\mathbb{R}^N} + \|\iota(x_n) - \iota(y_n)\|^2_{L^2(\mu)}}}$$

be a sequence of normalized secants with $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. (This is the hard case.) We may assume (by choosing subsences) that $\lim_{n\to\infty}(x_n - y_n)/\|x_n - y_n\|_{\mathbb{R}^N} = v \in \mathbb{R}^N$ exists. The key observation is that the ultrametric property implies the isoceles property: If $d(x,z) > d(x,y)$, then $d(x,z) = d(y,z)$; thus $\iota(x)(z) - \iota(y)(z) = 0$ and so $\iota(x) - \iota(y)$ has support in $B\big(x, d(x,y)\big) = B\big(y, d(x,y)\big)$. Again, we may assume (by choosing subsequences) that the sequence $\big\{\big(\iota(x_n) - \iota(y_n)\big)/\|\iota(x_n) - \iota(y_n)\|_{L^2(\mu)}\big\}_{n\geq 1}$ has a weak limit w; the observation implies that $w = 0 \in L^2(\mu)$. Then it is easy to check that our sequence $\{u_n\}_{n\geq 1}$ of normalized secants must have norm limit $(v, 0)$. Thus Γ is spherically compact, and for any $p \in \Gamma$ we have $T_p\Gamma \subset \mathbb{R}^N \times 0$. It follows that $|T\Gamma|$ is contained in the constant quasibundle $|\mathbb{R}^N \times 0|$, and so by Theorem 2.1, $T\Gamma$ is a quasibundle. Hence, Γ is C^1-finitely embeddable by the Generalized Whitney Embedding Theorem (Theorem 5.3). Now let $\pi\colon \mathbb{R}^N \times L^2(\mu) \to L^2(\mu)$ be the orthogonal projection $(x, g) \mapsto g$. Then it is easy to check that $\pi\colon \Gamma \to \pi(\Gamma)$ is bi-Lipschitz and that $\pi(\Gamma) = \iota(X)$, which is not C^1-finitely embeddable.

Our next example is self explanatory.

Example 8.3 The element $(1, 1, 1, \dots) \in \mathbb{Z}_2^\omega$ acts on $L^2(\mu)$ to define a C^2 (even linear and unitary!) dynamical system. Then *no* variant of the Takens algorithm [25] can C^1 embed the closed orbit $\iota(\mathbb{Z}_2^\omega)$ in a finite dimensional Euclidean space in spite of the fact that all the fractal dimensions (such as Dim defined above) are finite. (See [11] for a comprehensive treatment of fractal dimensions and the embedding problem. Also, see [19] for another way a Takens-like theorem fails in a Hilbert space.)

4) We turn now to curious and useful examples of curves in Hilbert space. To begin, the most important property of the examples above is that $\iota(X)$ has a tangent quasibundle but is not spherically compact. Thus, the hypothesis that X be spherically compact cannot be dropped from the Generalized Whitney Embedding Theorem 5.3.

Fact 8.5 The Generalized Whitney Embedding Theorem 5.3 becomes false if either the hypothesis that X be spherically compact or the hypothesis that TX be a quasibundle is dropped.

To justify the construction for Example 8.4 (which shows that the hypothesis that TX be a quasibundle cannot be dropped), we need the following easy lemma. (In what follows, l^2 denotes the Hilbert space of square summable sequences of real numbers.)

Lemma 8.1 *Let $M \subset \mathbb{R}^N$ be compact and suppose that there exists a sequence of Lipschitz functions $f_n\colon M \to [0,1]$ with* $\mathrm{Lip}(f_n) \leq 1/n$ *for $n \geq 1$. Let $F\colon M \to \mathbb{R}^N \times l^2$ by setting*

$$F(x) = \big(x, f_1(x), f_2(x), \dots\big),$$

where $x \in M$. Then the map F is bi-Lipschitz and the set $F(M) \subset \mathbb{R}^N \times l^2$ is spherically compact.

Proof Let $\{x_k\}_{k\geq 1}$ and $\{y_k\}_{k\geq 1}$ be sequences in M with $x_k \neq y_k$ for all k. We may write

$$\frac{F(x_k) - F(y_k)}{\|F(x_k) - F(y_k)\|} = \frac{\|x_k - y_k\|}{\|F(x_k) - F(y_k)\|}\left(\frac{x_k - y_k}{\|x_k - y_k\|}, \dots, \frac{f_n(x_k) - f_n(y_k)}{\|x_k - y_k\|}, \dots\right).$$

Applying the Cantor Diagonalization Process, we may assume that the sequences $\{\|x_k - y_k\|/\|F(x_k) - F(y_k)\|\}_{k\geq 1}$, $\{(x_k - y_k)/\|x_k - y_k\|\}_{k\geq 1}$, and $\{u_n(k)\}_{k\geq 1} = \{\big(f_n(x_k) - f_n(y_k)\big)/\|x_k - y_k\|\}_{k\geq 1}$ converge to $\alpha \in [1/\sqrt{1 + \sum_n 1/n^2}, 1]$, $u \in \mathbb{R}^N$, and $u_n \in [-1/n, 1/n]$, respectively. Therefore, the sequence $\{\big(F(x_k) - F(y_k)\big)/\|F(x_k) - F(y_k)\|\}_{k\geq 1}$ converges *weakly* to the point $\alpha(u, u_1, u_2, \dots)$ in the unit ball of $\mathbb{R}^N \times l^2$. Because $|u_n(k)| \leq 1/n$, the series $\sum_n u_n(k)^2$ converges uniformly with respect to k. Therefore,

$$\begin{aligned}\lim_{k\to\infty}\left(\frac{\|F(x_k) - F(y_k)\|}{\|x_k - y_k\|}\right)^2 &= \lim_{k\to\infty}\Big(1 + \sum_n u_n(k)^2\Big)\\ &= 1 + \sum_n u_n^2 = \|(u, u_1, u_2, \dots)\|\end{aligned}$$

implying that $\|\alpha(u, u_1, u_2, \dots)\| = 1$. Hence, the sequence, $\left\{\frac{\big(F(x_k)-F(y_k)\big)}{\|F(x_k)-F(y_k)\|}\right\}_{k\geq 1}$ converges in *norm* to the point $\alpha(u, u_1, u_2, \cdots)$ and the lemma is proved. ∎

With this lemma at hand, we may now give our example.

Example 8.4 Choose first a sequence $b(1,1) > a(1,1) > b(1,2) > a(1,2) > b(1,3) > a(1,3) > \cdots$ in the interval $(0,1)$ converging to 0. Choose a second sequence $b(2,1) > a(2,1) > b(2,2) > a(2,2) > b(2,3) > a(2,3) > \cdots$ so that we have $a(1,j) > b(2,j) > a(2,j) > b(1,j+1) > \cdots$ for $j = 1, 2, 3, \dots$. Finally, inductively on $n \geq 2$, choose a sequence $b(n{+}1,1) > a(n{+}1,1) > b(n{+}1,2) > a(n{+}1,2) > b(n{+}1,3) > a(n{+}1,3) > \cdots$ so that we have $a(1,j) > b(n+1,j) > a(n+1,j) > b(n,j+1) > \cdots$ for $j = 1, 2, \dots$. Let

$$J_n = \bigcup_{j\geq 1}[a(n,j), b(n,j)]$$

and note that $J_n \cap J_m = \varnothing$ for $n \neq m$. We define, for each $n \geq 1$, a function $f_n \colon [0,1] \to [0,1]$ by setting

$$f_n(t) = \frac{1}{n}\int_{[0,t]\cap J_n} ds$$

and note that f_n is Lipschitz with $\mathrm{Lip}(f_n) \leq 1/n$. Then, by the above lemma, the map $F \colon [0,1] \to l^2$ defined by setting $F(t) = \big(t, f_1(t), f_2(t), \dots\big)$ has spherically compact image $X = F([0,1]) \subset l^2$. Let $e_n = (0, \dots, 0, 1, 0, \dots)$ be the sequence with the nonzero entry at the n-th place and, for each $j \geq 1$, let t_{n_j} be the midpoint of $[a(n,j), b(n,j)]$. Then the vector $e_1 + \frac{1}{n}e_{n+1} \in T^0_{F(t_{n_j})}X$, for all $j \geq 1$. Thus, holding n fixed and letting j tend to infinity, we see that $e_1 + \frac{1}{n}e_{n+1} \in T^1_0 X$ for all $n \geq 1$ and so $T^1_0 X$ is infinite dimensional implying that TX cannot be a quasibundle.

As a stepping-stone to Example 8.6, we introduce the next example.

Example 8.5 There exists a topological embedding $\varphi\colon [0,1] \hookrightarrow \mathbb{H}$, which is a C^1 map, with infinite dimensional $T_{\varphi(0)}\varphi([0,1])$. Indeed, let $\mathbb{H}$ be a separable Hilbert space and let $\{e_n\}_{n\geq 1}$ be an orthonormal basis for $\mathbb{H}$. Let $\{n_k\}_{k\geq 1}$ be a sequence of natural numbers such that $n_k \geq 2$, for all k, and such that each natural number ≥ 2 appears infinitely often. Let $\psi\colon (0,1] \to \mathbb{H}$ be C^1 such that

(i) $\psi(1) = e_2$,
(ii) $\psi(1/k) = e_{n_{k+1}}$,
(iii) $\psi\bigl([1/(k+1), 1/k]\bigr) \subset \operatorname{span}\{e_{n_k}, e_{n_{k+1}}\}$ for $k \geq 2$, and
(iv) $\|\psi(t)\| = 1$, for $t \in (0,1]$.

Define $\varphi\colon [0,1] \to \mathbb{H}$ by setting

$$\varphi(t) = \begin{cases} 0, & \text{for } t = 0,\\ e^{-1/t^2}\psi(t), & \text{for } t > 0. \end{cases}$$

Then φ is C^1 with $e_{n_k} \in C^0_0\varphi([0,1])$ for all $k \geq 2$, implying that $T_0\varphi([0,1])$ is infinite dimensional.

The above example leads to a more interesting one, justifying the following fact.

Fact 8.6 It is not true that a finite union of compact C^1 finitely embeddable sets is C^1 finitely embeddable, not even if they meet in a single point.

Example 8.6 There exist compact subsets X_1 and X_2 of a Hilbert space $\mathbb{H}$ with $X_1 \cap X_2$ a single point, X_1 and X_2 each C^1-diffeomorphic to $[0,1]$ but $X_1 \cup X_2$ not C^1 embeddable in any finite dimensional Euclidean space. Let $\varphi\colon [0,1] \hookrightarrow \mathbb{H}_1$ be the C^1 homeomorphism defined in Example 8.5, where $\mathbb{H}_1$ is the orthogonal complement of the basis vector e_1. Define $\sigma_1, \sigma_2\colon [0,1] \hookrightarrow \mathbb{H}$ by setting

$$\sigma_1(t) = \varphi(t) + te_1 \quad \text{and} \quad \sigma_2(t) = te_1$$

and let $X_1 = \sigma_1([0,1])$ and $X_2 = \sigma_2([0,1])$. Of course, it is clear that $X_1 \cap X_2 = \{0\}$ and that X_2 is diffeomorphic to $[0,1]$. To see that X_1 is diffeomorphic to $[0,1]$, let $\pi_1\colon \mathbb{H} \to \mathbb{R}$ be the C^1 projection $\pi_1(x) = \langle x, e_1\rangle$. Then we have

$$\sigma_1 \circ \pi_1 = \mathrm{id}_{X_1} \quad \text{and} \quad \pi_1 \circ \sigma_1 = \mathrm{id}_{[0,1]}\,.$$

But for $X = X_1 \cup X_2$ we have that every basis vector is in $C^0_0 X$ implying that $T_0 X$ is infinite dimensional.

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5) We are grateful to an anonymous reader for constructing Example 8.7 below. It establishes the following surprising fact, showing that Theorem 2.1 is sharp.

Fact 8.7 Let X be a compact subset of a Hilbert space $\mathbb{H}$. Then $\dim T_pX < \infty$ for every $p \in X$ does not imply that TX is a quasibundle.

Example 8.7 Consider a separable Hilbert space $\mathbb{H}$, and let $e_1, e_2, \dots$ be an orthonormal basis of $\mathbb{H}$. For each $n \geq 1$, let $H_n = \operatorname{span}\{e_{n^2}, \dots, e_{(n+1)^2-1}\}$ so that $\mathbb{H} = H_1 \oplus H_2 \oplus \cdots$. Let X_n denote the intersection of H_n with the closed ball of radius $1/(2n)$ centered at $\frac{1}{n}e_{n^2}$ and set

$$X = \{0\} \cup \Big\{ \bigcup_{n \geq 1} X_n \Big\}.$$

Then X is compact and each X_n is open and closed in X. Consequently, $T_p^0 X_n = H_n$ for $p \in X_n$. In addition, one may check that $T_0^0 X = \{0\}$. Furthermore, for each ordinal α we have $T_p^\alpha X_n = H_n$ for $p \in X_n$ and $T_0^\alpha X = \{0\}$ implying that $T_p X_n = H_n$ for $p \in X_n$ while $T_0 X = \{0\}$. Thus $\dim T_pX < \infty$ for each $p \in X$. However, if $p \in X_n$, then $\dim T_pX = \dim H_n = 2n + 1 \to \infty$ as $n \to \infty$.

6) For further application of this material we refer the reader to [21], where the following generalization of the Smale-Hirsch Immersion Theorem is proven.

Theorem 8.4 (**[21, Theorem 6.1]**) *Let X be a locally compact subset of $\mathbb{R}^N$ and suppose that for some n there exists a quasibundle monomorphism*

$$\begin{array}{ccc} TX & \xrightarrow{\varphi} & \mathbb{R}^n \\ \downarrow & & \downarrow \\ X & \longrightarrow & * \end{array}$$

Then there exists an immersion $f\colon X \to \mathbb{R}^n$ with df monotopic to φ.

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