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FINITELY GENERATED SOLUBLE GROUPS WITH A CONDITION ON INFINITE SUBSETS

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Abstract

Let G be a group. We say that $G \in \mathcal{T}(\infty)$ provided that every infinite set of elements of G contains three distinct elements x, y, z such that $x \neq y$, $[x, y, z] = 1 = [y, z, x] = [z, x, y]$. We use this to show that for a finitely generated soluble group G, $G/Z_2(G)$ is finite if and only if $G \in \mathcal{T}(\infty)$.

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1. Introduction

Paul Erdős [10] posed the following question. Suppose that every infinite set of elements of a group G contains a pair of elements which commute. Does there exist an upper bound for the order of (finite) subsets of G consisting of pairwise noncommuting elements?

An affirmative answer to this question was given by Neumann [10] who proved that an infinite group G is centre-by-finite if and only if every infinite subset of G contains two distinct commuting elements. Since this paper, problems of a similar nature have been the object of several articles (for example, [1–10]).

Let G be a group and χ a class of groups. We say that G satisfies the condition (χ, ∞) if every infinite subset of G contains a pair of elements which generate a subgroup in the class χ. We also say that G satisfies condition $\mathcal{T}(\infty)$ (or G is in $\mathcal{T}(\infty)$) if every infinite set of elements of G contains three elements x, y, z such that

$$x \neq y, \quad [x, y, z] = 1 = [y, z, x] = [z, x, y].$$

Our terminology and notation are standard and follow [4]. In this paper $Z_n(G)$ denotes the $(n + 1)$th term of the upper central series of G, and $\Gamma_n(G)$ denotes the nth term of the lower central series of G. Let $\mathcal{N}_2$ and $\mathcal{E}_2$ be the classes of nilpotent groups of class at most 2 and 2-Engel, respectively. Obviously

$$(\mathcal{N}_2, \infty) \subseteq (\mathcal{E}_2, \infty) \subseteq \mathcal{T}(\infty).$$

In [3], Delizia proved that a finitely generated soluble group G is in $(\mathcal{N}_2, \infty)$ if and only if $G/Z_2(G)$ is finite. In [2], Abdollahi proved that a finitely generated soluble group G is in $(\mathcal{E}_2, \infty)$ if and only if $G/Z_2(G)$ is finite.

In this paper, we prove the following theorem.

Main Theorem. Let G be a finitely generated soluble group. Then $G \in \mathcal{T}(\infty)$ if and only if $G/Z_2(G)$ is finite.

The main theorem implies that, for a finitely generated soluble group G, the following conditions are equivalent:

$$G \in (\mathcal{N}_2, \infty), \quad G \in (\mathcal{E}_2, \infty), \quad G \in \mathcal{T}(\infty), \quad G/Z_2(G) \text{ is finite.}$$

2. Results

In the first result we prove the sufficiency.

Lemma 2.1. *Let G be a group and suppose that $G/Z_2(G)$ is finite. Then $G \in \mathcal{T}(\infty)$.*

Proof. Let X be an infinite subset of G. There exists an infinite subset X_0 of X such that $xZ_2(G) = yZ_2(G) = zZ_2(G)$, for $x, y, z \in X_0$. So $[x, y, z] = [y, z, x] = [z, x, y] = 1$. □

Recall that a group G is called a restrained group if $\langle x \rangle^{\langle y \rangle}$ is finitely generated for all $x, y \in G$.

Proposition 2.2. *Every group in $\mathcal{T}(\infty)$ is a restrained group.*

Proof. Let G be a group in $\mathcal{T}(\infty)$ and $x, y \in G$ such that y has infinite order. Since $X = \{xy^i \mid i > 1\}$ is an infinite subset of G, there exist three integers $i < j \leq k$ such that

$$[xy^i, xy^j, xy^k] = [xy^j, xy^k, xy^i] = [xy^k, xy^i, xy^j] = 1. \tag{2.1}$$

It follows from the equations $[xy^t, xy^s] = x^{-y^t} x^{y^s}$ and (2.1) that

$$\begin{aligned} xy^{k-j}x^{-1}y^{j-i}xy^{i-k}xx^{-y^i}x^{y^j} &= 1, \\ xy^{i-k}x^{-1}y^{k-j}xy^{j-i}xx^{-y^j}x^{y^k} &= 1, \\ xy^{j-i}x^{-1}y^{i-k}xy^{k-j}xx^{-y^k}x^{y^i} &= 1, \end{aligned}$$

and so $x^{y^k} = x^{y^i}x^{-1}x^{-y^{j-i}}x^{y^{k-i}}x^{-1}$. In this case we conclude that

$$\langle x^{y^i} : i \geq 0 \rangle \leq \langle x^{y^n} : |n| < k \rangle.$$

Now starting from the infinite set $X = \{xy^i \mid i < 1\}$ and repeating the previous argument, we can prove that

$$\langle x^{y^i} : i \leq 0 \rangle \leq \langle x^{y^n} : |n| < k' \rangle,$$

for a suitable integer $k' > 1$. Therefore there exists a positive integer m such that $\langle x \rangle^{\langle y \rangle} = \langle x^{y^n} : |n| < m \rangle$. □

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Lemma 2.3. *Let G be a finitely generated group in $\mathcal{T}(\infty)$. If $G/Z_3(G)$ is finite, then so is $Z_2(Z_3(G))/Z_2(G)$.*

Proof. It is clear that $Z_2(G) \leq Z_2(Z_3(G))$. Let $x \in Z_2(Z_3(G)) \leq Z_3(G)$. Then, for any $y, z, t \in G$,

$$[x, y, z, t] = 1, \quad [x, y, z]^t = [x, y, z^t] = [x, y, z]. \tag{2.2}$$

Let $|G/Z_3(G)| = n$. It follows that, for any $y, z \in G$, $[x, y^n, z^n] = 1$ and so, by (2.2), $[x, y, z]^{n^2} = 1 = [x^{n^2}, y, z]$. Thus $x^{n^2} \in Z_2(G)$ and $Z_2(Z_3(G))/Z_2(G)$ has finite exponent dividing n^2. Now $Z_2(Z_3(G))/Z_2(G)$ is a finitely generated nilpotent torsion group and thus finite as required. □

Lemma 2.4. *Let G be a finitely generated nilpotent group of class at most 3 which satisfies $\mathcal{T}(\infty)$. Then $G/Z_2(G)$ is finite.*

Proof. We consider the following cases.

Case I. Let G be a torsion group. Then G is a finitely generated nilpotent torsion group and thus finite.

Case II. Let G be a torsion-free group. We claim that $G = Z_2(G)$. Since G is nilpotent of class at most 3,

$$[x^n, y^m, z^k] = [x, y, z]^{nmk}, \quad [x, y, z]^g = [x, y, z], \tag{2.3}$$

for all g, x, y, z in G and all integers m, n, k. Now consider the infinite subset $X = \{xy^1, xy^2, xy^3, \ldots\}$ of G. Since G is in $\mathcal{T}(\infty)$, there exist three positive integers $i \neq j, k$ such that

$$[xy^i, xy^j, xy^k] = 1 = [xy^j, xy^k, xy^i] = [xy^k, xy^i, xy^j].$$

Repeated application of (2.3) yields

$$1 = [xy^i, xy^j, xy^k] = ([y, x, x][x, y, y]^{-k})^{i-j}.$$

Since G is torsion-free, $[x, y, y]^k = [y, x, x]$ and also $[x, y, y]^j = [y, x, x]$. Therefore $[x, y, y]^{k-j} = 1$, and hence $[x, y, y] = 1$. Thus G is a 2-Engel group. Now since G is metabelian [11, Theorem 7.36] implies that $\Gamma_3(G) = 1$, and so $G = Z_2(G)$.

Case III. Let G be neither a torsion nor a torsion-free group. Then G/G_t is a torsion-free group, where G_t is a torsion subgroup of the nilpotent group G. Since $\mathcal{T}(\infty)$ is closed under taking subgroups and homomorphic images, we have by Case II that G/G_t is nilpotent of class 2 and thus $\Gamma_3(G) \leq G_t$ is finite. Therefore $G/Z_2(G)$ is finite. □

Proposition 2.5. *Let G be a finitely generated nilpotent group of class c in $\mathcal{T}(\infty)$. Then $G/Z_2(G)$ is finite.*

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Proof. We argue by induction on c. Since $G/Z(G)$ is nilpotent of class $c-1$, we have that $G/Z_3(G)$ is finite. Now $Z_3(G)/Z_2(Z_3(G))$ is also finite by Lemma 2.4. The result follows from Lemma 2.3. □

The following result is analogous to [4, Lemma 1].

Lemma 2.6. *Let G be an infinite residually finite group satisfying the condition $\mathcal{T}(\infty)$. Then the centraliser $C_G(x)$ is infinite, for all x in G.*

Proof. Suppose, for a contradiction, that G has an element x with finite centraliser $C_G(x)$. Since G is residually finite, there exists a normal subgroup N of G such that $N \cap C_G(x) = 1$ and G/N is finite. In particular, N is infinite. Consider the infinite set $\{x^n : n \in N\}$. Then, by the property $\mathcal{T}(\infty)$, there exist three elements $r, s, t \in N$ such that $r \neq s$, $[x^r, x^s, x^t] = [x^t, x^r, x^s] = [x^s, x^t, x^r] = 1$. Now the equation $[x^r, x^s, x^t] = 1$ implies that $[x^p, x^q, x] = 1$ with $p = rt^{-1} \in N$ and $q = st^{-1} \in N$. It follows that $[x^p, x^q] \in N \cap C_G(x) = 1$, since $[x^{pq^{-1}}, x] = [qp^{-1}, x]^{xpq^{-1}}[pq^{-1}, x] \in N$. Hence $[x^p, x^q] = 1$ and $x^{pq^{-1}} \in C_G(x)$. Since $x^{pq^{-1}} = [pq^{-1}, x^{-1}]x$, we get $[pq^{-1}, x^{-1}] \in N \cap C_G(x) = 1$ and $pq^{-1} \in N \cap C_G(x) = 1$, so $p = q$. We thus obtain the contradiction that $r = s$. □

Corollary 2.7. *Let G be an infinite residually finite group satisfying the condition $\mathcal{T}(\infty)$. Then every element x of G is contained in an infinite abelian subgroup of G.*

Lemma 2.8. *Let G be an infinite residually finite group satisfying the condition $\mathcal{T}(\infty)$. Then the centraliser $C_G(X)$ is infinite, for any finite subset X of G.*

Proof. The proof is by induction on $m = |X|$. If $m = 1$, the result is true by Lemma 2.6. Suppose that $m > 1$, $X = \{x_1, \ldots, x_m\}$ and $C_G(x_1, \ldots, x_{m-1})$ is infinite. Then, by Corollary 2.7, there exists an infinite abelian subgroup A of G such that $A \leq C_G(x_1, \ldots, x_{m-1})$. Put $x_m = x$. Since G is residually finite, there exists an infinite descending sequence $(N_i)_{i\in I}$ of normal subgroups of G with G/N_i finite for any $i \in I$ and $\cap N_i = 1$. Therefore, $A \cap N_i$ is infinite for any $i \in I$.

Now, as in the proof of [4, Lemma 3], we can prove that there exist a sequence $(a_n)_{n\in\mathbb{N}}$ of elements of A that are pairwise distinct and a subsequence $(M_n)_{n\in\mathbb{N}}$ of $(N_i)_{i\in I}$ such that for every $n \in \mathbb{N}$ we get $a_{n+1} \in M_n$ and either $[a_n, x, x] = 1$ or $[a_n, x, x] \notin M_n$. Moreover, if $[a_n, x, x] = 1$ and $[a_n, x, x_s] \neq 1$ for some $s \in \{1, \ldots, m-1\}$ then $[a_n, x, x_s]^x \notin M_n$. Now we consider the infinite set $\{a_1x, \ldots, a_nx, \ldots\}$. Since G satisfies the condition $\mathcal{T}(\infty)$, there exist $i, j, k \in \mathbb{N}$ with $i < j \leq k$ such that $[a_ix, a_jx, a_kx] = 1$ and $[a_ix, a_jx, x][a_ix, a_jx, a_k]^x = 1$; then $[a_ix, a_jx, x] \in \langle a_k\rangle^G \leq M_{k-1} \leq M_i$. Since $[a_ix, a_j, x] \in \langle a_j\rangle^G \leq M_{j-1} \leq M_i$, we have $[a_ix, x, x] \in M_i$ and then $[a_i, x, x] \in M_i$ which implies that $[a_i, x, x] = 1$. So $B = \{a \in (a_n)_{n\in\mathbb{N}} : [a, x, x] = 1\}$ is an infinite set. Suppose that $B = \{b_1, \ldots, b_n, \ldots\}$. Let $x_l \in \{x_1, \ldots, x_{m-1}\}$, and consider the infinite set $\{b_ix_lx : b_i \in B\}$. Then there exist $r, s, t \in \mathbb{N}$ with $r < s \leq t$ such that $[b_rx_lx, b_sx_lx, b_tx_lx] = 1$. Since $[b_rx_lx, b_sx_lx, b_t] \in \langle b_t\rangle^G \leq M_{t-1} \leq M_r$, we have $[b_rx_lx, b_sx_lx, x_lx] \in M_s \leq M_r$. Since $[b_rx_lx, b_s, x_lx] \in \langle b_s\rangle^G \leq M_{s-1} \leq M_r$, we have $[b_r, x_lx, x_lx] \in M_r$. It follows that $[b_r, x, x_l] \in M_r$ and then $[b_r, x, x_l] = 1$, as $b_r \in C_G(x_1, \ldots, x_{m-1})$. Thus there exists an infinite subset B^* of B such that

$[b, x, x_l] = 1$ for any $b \in B^*$. We can now easily prove that there exists an infinite subset V of B such that $[c, x] \in C_G(x_1, \ldots, x_{m-1})$ for every $c \in V$. If the set $\{[c, x] : c \in V\}$ is infinite the result follows. Otherwise, there exist $c \in B$ and an infinite subset $\{d_j : j \in J\} \subseteq B$ such that $[c, x] = [d_j, x]$ for any $j \in J$. Then the infinite set $\{cd_j^{-1} : j \in J\}$ is contained in $C_G(x_1, \ldots, x_{m-1})$, and the result follows. □

The following is an immediate corollary of Lemma 2.8.

Corollary 2.9. *Let G be a finitely generated infinite residually finite group in $\mathcal{T}(\infty)$. Then $Z(G)$ is infinite.*

Denote by $hl(G)$ the Hirsch length of G.

Lemma 2.10. *Let G be a finitely generated infinite polycyclic group in $\mathcal{T}(\infty)$. Then $G/Z_2(G)$ is finite.*

Proof. If $hl(G) = 1$ then, by Corollary 2.9, $G/Z(G)$ is finite. Suppose then that $hl(G) > 1$. It follows that $hl(G) > hl(G/Z(G))$. Now, by the induction hypothesis,

$$\frac{G/Z(G)}{Z_2(G/Z(G))} \cong \frac{G}{Z_3(G)}$$

is finite. Therefore, the result follows from Lemma 2.3 and Proposition 2.5. □

Proof of the main theorem. To show that a finitely generated soluble group G in $\mathcal{T}(\infty)$ has $G/Z_2(G)$ finite, it is enough to show that G is polycyclic by Lemma 2.10. It follows from Proposition 2.2 that G' is finitely generated. Since a finitely generated abelian group is polycyclic and the class of a polycyclic group is closed under extensions, induction on the derived length then gives us G polycyclic. The other direction follows immediately from Lemma 2.1.

Corollary 2.11. *Let G be a finitely generated soluble group. Then the following conditions are equivalent.*

(i) $G \in (\mathcal{N}_2, \infty)$.
(ii) $G \in (\mathcal{E}_2, \infty)$.
(iii) $G \in \mathcal{T}(\infty)$.
(iv) $G/Z_2(G)$.

Proof. This follows using also the main theorems of [2, 3]. □

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References

[1] A. Abdollahi, 'Some Engel conditions on infinite subsets of certain groups', *Bull. Aust. Math. Soc.* **62** (2000), 141–148.

[2] A. Abdollahi, 'Finitely generated soluble groups with an Engel condition on infinite subsets', *Rend. Semin. Mat. Univ. Padova* **103** (2000), 47–49.

[3] C. Delizia, 'Finitely generated soluble groups with a condition on infinite subsets', *Ist. Lombardo Accad. Sci. Lett. Rend. A* **128** (1994), 201–208.

[4] C. Delizia, 'On certain residually finite groups', *Comm. Algebra* **24** (1996), 3531–3535.

[5] G. Endimioni, 'Groups covered by finitely many nilpotent subgroups', *Bull. Aust. Math. Soc.* **50** (1994), 459–464.

[6] G. Endimioni, 'Groupes finis satisfaisant la condition $(\mathcal{N}, n)$', *C. R. Acad. Sci. Paris Sér. I Math.* **319** (1994), 1245–1247.

[7] J. C. Lennox and J. Wiegold, 'Extensions of a problem of Paul Erdős on groups', *J. Aust. Math. Soc. Ser. A* **31** (1981), 459–463.

[8] P. Longobardi, 'On locally graded groups with an Engel condition on infinite subsets', *Arch. Math. (Basel)* **76** (2001), 88–90.

[9] P. Longobardi and M. Maj, 'Finitely generated soluble groups with an Engel condition on infinite subsets', *Rend. Semin. Mat. Univ. Padova* **89** (1993), 97–102.

[10] B. H. Neumann, 'A problem of Paul Erdős on groups', *J. Aust. Math. Soc. Ser. A* **21** (1976), 467–472.

[11] D. J. Robinson, *Finiteness Condition and Generalized Soluble Groups, Part 1, 2* (Springer, Berlin, 1972).

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