ON THE NUMBER OF VERTICES OF A CONVEX POLYTOPE

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Introduction. As is well known, the theory of linear inequalities is closely related to the study of convex polytopes. If the bounded subset P of euclidean d-space $\mathfrak{R}^d$ has a non-empty interior and is determined by i linear inequalities in d variables, then P is a d-dimensional convex polytope (here called a *d-polytope*) which may have as many as i faces of dimension $d - 1$, and the vertices of this polytope are exactly the basic solutions of the system of inequalities. Thus, to obtain an upper estimate of the size of the computation problem which must be faced in solving a system of linear inequalities, it suffices to find an upper bound for the number $f_0(P)$ of vertices of a d-polytope P which has a given number $f_{d-1}(P)$ of $(d-1)$-faces. A weak bound of this sort was found by Saaty **(14)**, and several authors have posed the problem of finding a sharp estimate. Dantzig **(3)** mentions the closely related problem (arising naturally in connection with the simplex method for linear programming) of determining those convex sets which have the maximum number of extreme points, among all sets which are determined by a system of m linear equations in n non-negative variables.

Our main concern here is with the conjectured inequality

$$(1) \qquad f_0 \leqslant \binom{f_{d-1} \quad -[\frac{1}{2}(d+1)]}{f_{d-1} \quad -d} + \binom{f_{d-1} \quad -[\frac{1}{2}(d+2)]}{f_{d-1} \quad -d}$$

and its dual equivalent

$$(1^*) \qquad f_{d-1} \leqslant \binom{f_0 \quad -[\frac{1}{2}(d+1)]}{f_0 \quad -d} + \binom{f_0 \quad -[\frac{1}{2}(d+2)]}{f_0 \quad -d},$$

where $[k]$ denotes the greatest integer $\leqslant k$ and f_s denotes the number of s-faces of a d-polytope. The validity of these inequalities for all d-polytopes was conjectured by Jacobs and Schell **(10)** and by Gale **(8, 9)**, who observed that the proposed upper bound in (1*) is attained by the neighbourly d-polytopes (studied by Brückner **(1)**, Carathéodory **(2)**, Gale **(7, 8)**, and Motzkin **(13)**) having the remarkable property that for all $m \leqslant [d/2]$, each m vertices determine an $(m-1)$-face. Dually, equality in (1) is attained for d-polytopes such that for all $m \leqslant [d/2]$, each m $(d-1)$-faces intersect in a $(d-m)$-face.

The assertions (1) and (1*) are trivial for $d \leqslant 2$, where equality always holds. For $d = 3$ they become $f_0 \leqslant 2f_2 - 4$ and $f_2 \leqslant 2f_0 - 4$, facts known to Euler **(5)**. Saaty's bound **(14)** was sharp for $d \leqslant 4$. The inequalities (1)

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and (1*) were established by Fieldhouse for all $d \leqslant 6$, and by Gale **(9)** for arbitrary d when $f_{d-1} = d + 2$ or $d + 3$. (I have not actually seen the thesis of Fieldhouse, but have read a review of it **(6)**.) Thus Gale shows that (1) holds whenever f_{d-1} is small enough. We show here that it holds whenever f_{d-1} is large enough, specifically when $f_{d-1} \geqslant (d/2)^2 - 1$. This covers the case $d \leqslant 6$ and thus includes the result of Fieldhouse, but it does not include Gale's theorem when $d > 6$ and does not fully settle the conjecture.

Under the restriction $f_0 \geqslant (d/2)^2 - 1$, the inequality (1*) is established not only for d-polytopes, but also for an arbitrary Eulerian $(d - 1)$-manifold of Euler characteristic $1 - (-1)^d$, where an *Eulerian n-manifold* (as introduced in **(12)**) is a finite simplicial n-complex M^n such that for each s-simplex $\sigma^s \in M^n$, the linked complex $L(\sigma^s, M^n)$ has the same Euler characteristic $1 - (-1)^{n-s}$ as an $(n - s - 1)$-sphere. The principal tool is a formula from **(12)**, applying to all Eulerian $(d - 1)$-manifolds, which expresses f_{d-1} linearly in terms of

$$f_{[d/2]-1}, f_{[d/2]-2}, \ldots, f_1, f_0,$$

and the Euler characteristic χ. With the aid of similar formulae for $f_{d-2}, \ldots, f_{[d/2]}$, we are able to show that whenever f_0 is sufficiently large, then among all of the d-polytopes (or Eulerian $(d - 1)$-manifolds with $\chi = 1 - (-1)^d$) which have f_0 vertices, the neighbourly d-polytopes maximize not only f_{d-1} but also all of the other functions f_s $(1 \leqslant s \leqslant d - 2)$. The results for Eulerian manifolds appear in §1 below, and they apply directly to d-polytopes which are $(d - 1)$-simplicial. (A polytope is *s-simplicial* if each of its s-faces is a simplex.) A construction in §2 reduces the problem for general d-polytopes to those which are $(d - 1)$-simplicial. It is also proved there that if a d-polytope P is not a d-simplex, then

$$f_s(P) \geqslant \binom{d+1}{s+1} + \binom{d-1}{s} \quad \text{for} \quad 0 \leqslant s \leqslant d - 1,$$

where the lower bound is sharp. §3 discusses the inequalities (1) and (1*) for general d-polytopes, and §4 is devoted to Dantzig's problem. §4 also contains a characterization of those convex polyhedra (not necessarily bounded) which are affinely equivalent to the intersection of some flat with $\mathfrak{O}^n$, the positive orthant in $\mathfrak{R}^n$.

1. Eulerian manifolds. Let $\mathbf{K}$ denote the class of all finite simplicial complexes. For $K \in \mathbf{K}$ and $m \geqslant 0$, let $s_m(K)$ denote the m-skeleton of K; that is, $s_m(K)$ is the set of all simplices $\sigma^s \in K$ for which $s \leqslant m$. (Note that $s_m(K) = K$ if and only if $\dim K \leqslant m$, so $\dim s_m(K) = m$ if and only if $\dim K \geqslant m$.) For each subclass $\mathbf{J}$ of $\mathbf{K}$ we define

$$s_m(\mathbf{J}) = \{s_m(J) : J \in \mathbf{J}\},$$

$$J_v = \{J \in \mathbf{J} : J \text{ has exactly } v \text{ vertices}\},$$

and

$$\mathbf{J}[m] = \{J \in \mathbf{J} : s_m(J) \text{ is a complete } m\text{-complex}\}.$$

Thus $J \in \mathbf{J}[m]$ if and only if $J \in \mathbf{J}$, J has at least $m + 1$ vertices, and each $m + 1$ vertices of J determine an m-simplex of J.

Now suppose that $\mathbf{J} \subset \mathbf{K}$ and ϕ is a real-valued function on J. The function ϕ will be called *m-invariant* provided $\phi(J) = \phi(J')$ whenever J and J' are members of $\mathbf{J}$ such that $f_s(J) = f_s(J')$ for all $s \leqslant m$. We shall say that ϕ is *proper for* $(\mathbf{J}, m, v)$ provided ϕ is m-invariant, $\mathbf{J}_v[m] \neq \emptyset$, and $\sup \phi(\mathbf{J}_v) = \phi(\mathbf{J}_v[m])$. If $\phi(J) < \phi(\mathbf{J}_v[m])$ for all $J \in \mathbf{J}_v \sim \mathbf{J}_v[m]$, then ϕ is said to be *strictly proper for* $(\mathbf{J}, m, v)$.

1.1. Proposition. *Suppose that* $\mathbf{J} \subset \mathbf{K}$, $0 \leqslant l \leqslant m$, *and* ψ *and* ϕ *are real-valued functions on* $\mathbf{J}$. *If* ψ *is proper for* $(\mathbf{J}, l, v)$ *and* ϕ *is proper* [*strictly proper*] *for* $(\mathbf{J}, m, v)$, *then* $\psi + \phi$ *is proper* [*strictly proper*] *for* $(\mathbf{J}, m, v)$.

Proof. Let $J_0 \in \mathbf{J}_v[m] \subset \mathbf{J}_v[l]$. Then

$$\begin{aligned}\sup(\psi + \phi)(\mathbf{J}_v) &\leqslant \sup \psi(\mathbf{J}_v) + \sup \phi(\mathbf{J}_v)\\ &= \psi(J_0) + \phi(J_0) = (\psi + \phi)(J_0).\end{aligned}$$

Thus $\psi + \phi$ is proper for $(\mathbf{J}, m, v)$. If ϕ is strictly proper and $J \in \mathbf{J}_v \sim \mathbf{J}_v[m]$, then $\phi(J) < \phi(J_0)$ and consequently $(\psi + \phi)(J) < (\psi + \phi)(J_0)$.

Now let A denote the set of all eventually zero sequences $\alpha = (\alpha_0, \alpha_1, \ldots)$ of real numbers. For $\alpha \in A$ and $K \in \mathbf{K}$ define

$$\alpha(K) = \sum_{s=0}^{\infty} \alpha_s f_s(K),$$

where $f_s(K)$ is the number of s-simplices of K. It is clear that if $\alpha_s = 0$ for all $s > m$, then the function α is m-invariant on K.

We shall denote by $n_{(r)}$ the falling factorial $n(n-1)\ldots(n-r+1)$ and by $n^{(r)}$ the rising factorial $n(n+1)\ldots(n+r-1)$, with the convention that $n_{(0)} = 1 = n^{(0)}$.

1.2. Theorem. *Suppose that* $\alpha \in A$ *with* $\alpha_s = 0$ *whenever* $s > m$ *and whenever* $s < m - k$. *Then the function* α *is proper for* $(\mathbf{K}, m, v)$ *if*

$$(a_j) \qquad \sum_{s=0}^{j} \alpha_{m-s}(m+1)_{(s)}(v-s)^{(j-s)} \geqslant 0 \qquad (0 \leqslant j \leqslant k),$$

and is strictly proper when the conditions (a_j) $(0 \leqslant j \leqslant k)$ *are valid with strict inequality.*

Proof. Suppose that $K \in \mathbf{K}_v$ and $K_0 \in \mathbf{K}_v[m]$. Each s-simplex σ^s of K is determined in $s + 1$ different ways by specification of one of the $(s-1)$-faces (having s vertices) of σ^s together with the remaining vertex of σ^s (which is one of $v - s$ vertices of K). Thus

$$(b_s) \qquad (s+1)f_s(K) \leqslant (v-s)f_{s-1}(K).$$

Writing f_s for $f_s(K)$, we obtain the following inequalities, whose justification is indicated in parentheses to the right:

$$(b_m) \qquad (m+1)f_m \leqslant (v-m)f_{m-1}$$

$$(1) \qquad (m+1)(\alpha_{m-1}f_{m-1} + \alpha_m f_m) \leqslant f_{m-1}[\alpha_{m-1}(m+1) + \alpha_m(v-m)] \qquad (a_0, b_m)$$

$$(b_{m-1}) \qquad mf_{m-1} \leqslant (v-m+1)f_{m-2}$$

$$(2) \qquad (m+1)m(\alpha_{m-2}f_{m-2} + \alpha_{m-1}f_{m-1} + \alpha_m f_m) \leqslant f_{m-2}[\alpha_{m-2}(m+1)m + \alpha_{m-1}(m+1)(v-m+1) + \alpha_m(v-m)(v-m+1)] \qquad (1, a_1, b_{m-1})$$

. . . .

$$(k) \qquad (m+1)_{(k)}(\alpha_{m-k}f_{m-k} + \alpha_{m-k+1}f_{m-k+1} + \ldots + \alpha_{m-1}f_{m-1} + \alpha_m f_m) \leqslant f_{m-k}\left[\sum_{s=0}^{k} \alpha_{m-s}(m+1)_{(s)}(v-s)^{(k-s)}\right] \qquad (k-1, a_{k-1}, b_{m-k+1})$$

But, of course,

$$f_{m-k} \leqslant \binom{v}{m-k+1},$$

and in conjunction with the inequalities (k) and (a_k) this implies that

$$(k^*) \qquad \alpha(K) \leqslant [(m+1)_{(k)}]^{-1}\binom{v}{m-k+1}\sum_{s=0}^{k} \alpha_{m-s}(m+1)_{(s)}(v-m)^{(k-s)} = \sum_{s=0}^{k} \alpha_{m-s}\binom{v}{m-s+1} = \alpha(K_0),$$

whence α is proper for $(\mathbf{K}, m, v)$.

Suppose, finally, that all of the inequalities (a_j) $(0 \leqslant j \leqslant k)$ are strict and that $\alpha(K) = \alpha(K_0)$. Since the inequality in (k^*) is strict unless

$$f_{m-k} = \binom{v}{m-k+1},$$

we conclude that $K \in \mathbf{K}_v[m-k]$. An inequality $((k-1)^*)$ (which is related to the inequality $(k-1)$ as (k^*) is to (k)) then shows that $K \in \mathbf{K}_v[m-k+1]$; continuing the process, we conclude after a number of steps that $K \in \mathbf{K}_v[m]$.

1.3. Corollary. *Suppose that $\alpha = (\alpha_0, \ldots, \alpha_m, 0, \ldots) \in A$ (that is, $\alpha_s = 0$ for all $s > m$). If $\alpha_m > 0$, then the function α is strictly proper for $(\mathbf{K}, m, v)$ whenever v is sufficiently large.*

Proof. Note that condition (a_j) in 1.2 is equivalent to an inequality of the form

$$\alpha_m v^j + p_j(v) \geqslant 0,$$

where p_j is a polynomial of degree $j - 1$ whose coefficients are determined by the values of $\alpha_0, \alpha_1, \ldots, \alpha_m$.

1.4. COROLLARY. *Suppose that* $\alpha = (0, \ldots, 0, \alpha_{m-1}, \alpha_m, 0, \ldots) \in A$. *If the numbers* α_m *and* $\alpha_{m-1}(m + 1) + \alpha_m(v - m)$ *are both* $\geqslant 0$ $[>0]$, *then the function* α *is proper* [*strictly proper*] *for* $(\mathbf{K}, m, v)$.

Proof. This is merely the case $k = 1$ of 1.2.

For each positive integer n, let $\mathbf{E}^n$ denote the class of all Eulerian n-manifolds (as defined in the Introduction and in **(12)**). When n is odd and $M \in \mathbf{E}^n$, the Euler characteristic $\chi(M)$ is necessarily equal to 0 **(12**; 3.2**)**. When n is even and c is an integer, $\mathbf{E}^{n,c}$ will denote the class of all Eulerian n-manifolds M for which $\chi(M) = c$. We recall from **(12**, 3.2**)** the fact that if $M \in \mathbf{E}^n$ with $n = 2u - 1$ or $n = 2u - 2$, then

$$(M) = \begin{cases} \displaystyle\sum_{j=0}^{u-1} (-1)^{u-1-j} \frac{j+1}{u} \binom{n-j-1}{u-1} f_j(M) & \text{for } n = 2u - 1. \\ \displaystyle(-1)^{u+1} \binom{n}{u-1} \chi(M) + \sum_{j=0}^{u-2} (-1)^{u-j} 2 \binom{n-j-1}{u-1} f_j(M) & \\ & \text{for } n = 2u - 2. \end{cases}$$

A d-polytope P will be called *m-neighbourly* provided each m vertices of P determine an $(m - 1)$-face of P. Gale **(7, 8)** has proved that for $d/2 < m \leqslant d + 1$, the only m-neighbourly d-polytopes are the d-simplices, while for $m \leqslant d/2$ there exist m-neighbourly d-polytopes having any specified number of vertices $\geqslant d + 1$. Such polytopes must be $(d - 1)$-simplicial when $m = d/2$ **(7)**, but for $m < d/2$ there exist m-neighbourly d-polytopes which are $(d - 1)$-simplicial and also those which are not $(d - 1)$-simplicial (both having any number of vertices $\geqslant d + 1$). As the term will be used here, a *neighbourly d-polytope* is one which is $(d - 1)$-simplicial and $[d/2]$-neighbourly. If K is the complex formed by all of the proper faces of such a polytope, then of course $K \in \mathbf{K}[[(d - 2)/2]]$; that is, K is a simplicial complex whose $[(d - 2)/2]$-skeleton is a complete complex. By **(12**, 3.3**)**, K is an Eulerian $(d - 1)$-manifold, so we conclude that

$$\text{for } v \geqslant 2u + 1, \text{ the class } \mathbf{E}_v^{2u-1}[u - 1] \text{ is non-empty;}$$

$$\text{for } v \geqslant 2u, \text{ the class } \mathbf{E}_v^{2u-2,2}[u - 2] \text{ is non-empty.}$$

1.5. THEOREM. *Suppose that* $n = 2u - 1$ *and* $v \geqslant n + 2$. *Then the functions* $f_n, f_{n-1}, \ldots, f_{u+1}$ *and* f_u *are* $(u - 1)$*-invariant on* $\mathbf{E}^n$, *and for* v *sufficiently large they are strictly proper for* $(\mathbf{E}^n, u - 1, v)$. *In particular, if* $v \geqslant u^2 - 1$, *then*

$$\sup f_n(\mathbf{E}_v^n) = f_n(\mathbf{E}_v^n[u - 1]) = \frac{v}{v - u} \binom{v - u}{u};$$

for $v > u^2 - 1$ *the maximum of* f_n *on* $\mathbf{E}_v^n$ *is attained only on* $\mathbf{E}_v^n[u - 1]$.

Proof. For the $(u-1)$-invariance of the functions $f_n, \ldots, f_u$ on $\mathbf{E}^n$, it suffices to note that on $\mathbf{E}^n$ each of these functions is a linear combination of the functions $f_{u-1}, \ldots, f_0$; cf. **(12**, 2.4**)**. In fact, for $0 \leqslant i \leqslant u-1$ the function f_{n-i} is seen to have the form

$$f_{n-i} = \binom{u}{i} f_{u-i} + \sum_{j=0}^{u-2} \alpha(n, i, j) f_j \qquad (\text{on } \mathbf{E}^n).$$

Since by 1.3 the function

$$\left(\alpha(n, i, 0), \alpha(n, i, 1), \ldots, \alpha(n, i, u-2), \binom{u}{i}, 0, \ldots\right) \quad (\in A)$$

is strictly proper for $(\mathbf{K}, u-1, v)$ whenever v is sufficiently large, and since the class $\mathbf{E}_v{}^n[u-1]$ is non-empty, it follows that f_{n-i} is strictly proper for $(\mathbf{E}^n, u-1, v)$ whenever v is sufficiently large.

Since

$$\frac{v}{v-u}\binom{v-u}{u}$$

is exactly the number of n-faces of a neighbourly $(n+1)$-polytope which has v vertices **(8)**, we may complete the proof of 1.5 by showing that the function f_n is proper for $(\mathbf{E}^n, u-1, v)$ when $v \geqslant u^2 - 1$ and strictly proper when $v > u^2 - 1$. To this end, we employ the formula for f_n stated above, representing f_n as a linear combination of f_{u-1} and f_{u-2}, plus a linear combination of f_{u-3} and f_{u-4}, plus .... For example,

$$f_{13} = (f_6 - 6f_5) + (20f_4 - 48f_3) + (90f_2 - 132f_1) + 132f_0$$

when $n = 13$. In view of 1.1 and 1.4 we may reach the desired conclusion by verifying that if we write $\alpha_j = \alpha(n, 0, j)$, then

$$-\alpha_{j-1}(j+1) + \alpha_j(v-j) \geqslant 0 \qquad \text{for } j = u-1, u-3, \ldots,$$

with strict inequality when we want strict propriety. This is equivalent to the requirement

$$[j] \qquad |\alpha_{j-1}/\alpha_j| \leqslant (v-j)/(j+1).$$

The requirement $[j]$ is satisfied when $j = u-1$; for the left side of the inequality $[u-1]$ is equal to $u-1$ and the inequality is equivalent to the condition $(v \geqslant u^2 - 1)$ which forms part of our hypotheses. Now, as j decreases through the values $u-1, u-2, \ldots$, the right side of $[j]$ actually increases, so to complete the proof of 1.5 it suffices to show that during this same decrease in j, the left side of the inequality $[j]$ decreases in value. This amounts to the requirement that

$$|\alpha_{i-1}/\alpha_i| \leqslant |\alpha_i/\alpha_{i+1}| \qquad (1 \leqslant i \leqslant u-2)$$

or equivalently that

$$i(i+2)\binom{2u-i-1}{u-1}\binom{2u-i-3}{u-1} \leqslant (i+1)^2\binom{2u-i-2}{u-1}.$$

This is equivalent to the assertion that

$$i(i+2)(i-2u+1)(i-u+1) \leqslant (i+1)^2(i-u)(i-2u-2),$$

which on substituting $x-1$ for i reduces to

$$ux^2 - 3ux + 2u^2 \geqslant 0.$$

The discriminant of the quadratic form is $u^2(9-8u)$, which is negative when $u \geqslant 2$. This establishes the propriety of f_n except when $u = 1$, and that case is trivial; further, the strict propriety of f_n is assured when $v > u^2 - 1$.

1.6. THEOREM. *Suppose that $n = 2u - 2$, $v \geqslant n + 2$, and the integer c is given. Then the functions $f_n, f_{n-1}, \ldots, f_u$, and f_{u-1} are $(u-2)$-invariant on $\mathbf{E}^{n,c}$, and for v sufficiently large they are strictly proper for $(\mathbf{E}^{n,2}, u-2, v)$. In particular, if $v \geqslant u^2 - 2$, then*

$$\sup f_n(\mathbf{E}_v^{n,2}) = f_n(\mathbf{E}_v^{n,2}[u-2]) = 2\binom{v-u}{u-1},$$

while for $v > u^2 - 2$ the maximum of f_n on $\mathbf{E}_v^{n,2}$ is attained only on $\mathbf{E}_v^{n,2}[u-2]$.

Proof. (This is a paraphrase of the proof of 1.5.) By **(12**, 2.4**)**, each of the functions $f_n, \ldots, f_{u-1}$ on $\mathbf{E}^n$ is a linear combination of the functions $f_{u-2}, \ldots, f_0$ and χ (where χ is the Euler characteristic). Since the value of χ is fixed $(=c)$ on $\mathbf{E}^{n,c}$, the functions $f_n, \ldots, f_{u-1}$ must be $(u-2)$-invariant on $\mathbf{E}^{n,c}$. Now for $0 \leqslant i \leqslant u-1$, the linear expression of f_{n-i} in terms of $f_{u-2}, \ldots, f_0$ and χ involves f_{u-2} with a *positive* coefficient, and since the class $\mathbf{E}_v^{n,2}[u-2]$ is non-empty it follows that f_{n-i} is strictly proper for $(\mathbf{E}^{n,2}, u-2, v)$ whenever v is sufficiently large.

For $n = 2u-2$ and $v \geqslant n+2$, the type of construction and reasoning which were employed by Gale **(8)** for odd n leads to neighbourly $(n+1)$-polytopes having v vertices and $2\binom{v-u}{u-1}$ n-faces. The proof of 1.6 is completed by showing that if

$$\alpha_j = (-1)^{u-j}2\binom{n-j-1}{u-1}$$

(the coefficient of f_j in the expression for f_n), then for $v \geqslant u^2 - 2$ and $j = u-2, u-3, \ldots$, $|\alpha_{j-1}/\alpha_j| \leqslant (v-j)/(j+1)$. Verification of this is quite analogous to that in 1.5.

2. Polytopes and pyramids.

Recall that a *convex polytope P* in $\mathfrak{R}^d$ is the convex hull of a finite set or (equivalently) is a bounded set which is the intersection of a finite number of closed half-spaces. A *face* of P is either P

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itself, the intersection of P with a supporting hyperplane, or the empty set $\emptyset$. A *proper face* is one other than P or $\emptyset$.

The following is well known and easily proved.

2.1. PROPOSITION. *Suppose X is the set of all vertices of a convex polytope $P = \text{con}\, X$, and Y is a proper subset of X. Then the following three statements are equivalent:*

(i) *Y is the set of all vertices of some face of P;*

(ii) $\text{aff}\, Y \cap \text{con}(X \sim Y) = \emptyset$;

(iii) *X admits a supporting hyperplane H for which $X \cap H = Y$.*

(Here "con" indicates the *convex hull* and "aff" indicates the *affine hull* (smallest containing flat).)

A d-polytope P will be called *pyramidal at q* provided P is the join of q and a $(d-1)$-polytope; an equivalent requirement is that the vertex q of P should not be an affine combination of the remaining vertices of P.

2.2. PROPOSITION. *Suppose that q is a vertex of a face F of a polytope P. If P is pyramidal at q, then so is F.*

2.3. PROPOSITION. *A d-simplex is pyramidal at each of its $d+1$ vertices. If a d-polytope is pyramidal at $d-1$ or more of its vertices, then it is a d-simplex. For each $d \geqslant 2$ there exists a d-polytope P_d which is pyramidal at exactly $d-2$ of its vertices.*

Proofs. We prove only the second and third assertions of 2.3, leaving the rest to the reader. Clearly the first assertion is true if $d \leqslant 2$. Suppose it is known for $d = k - 1 \geqslant 2$ and consider a k-polytope P which is pyramidal at $k-1$ or more of its vertices. Let q be such a vertex and let Q be the $(k-1)$-polytope such that P is the join of q and Q. It follows from 1.2 that Q is pyramidal at $k-2$ or more of its vertices and then from the inductive hypothesis that Q is a $(k-1)$-simplex. Thus the set $P(=\text{con}\,(Q \cup \{q\}))$ is a k-simplex and the second assertion of 2.3 follows by mathematical induction.

To construct the polytopes P_d we start by taking for P_2 an arbitrary convex quadrilateral (which clearly has the desired property), and having defined P_{k-1} we let P_k be the join of P_{k-1} and an additional independent vertex q. (For example, we may assume that $P_{k-1} \subset \mathfrak{R}^{k-1} \subset \mathfrak{R}^k$; then choose $q \in \mathfrak{R}^k \sim \mathfrak{R}^{k-1}$ and let $P_k = \text{con}\,(P_{k-1} \cup \{q\})$.) For later use, note that the s-faces of P_k are just the s-faces of P_{k-1} and in addition the joins of q with the various $(s-1)$-faces of P_{k-1}; hence

$$f_s(P_k) = f_s(P_{k-1}) + f_{s-1}(P_{k-1}).$$

When X is the set of all vertices of a d-polytope con X and q is one of these vertices, we shall say that *X' is obtained from X by pushing q to q'* provided

$X' = (X \sim \{q\}) \cup \{q'\}$, where q' is a point of con X such that the segment $]q, q']$ does not intersect any $(d-1)$-flat determined by points of X. Clearly such a pushing is always possible. The following result amplifies a remark of Gale **(9**, §2**)**.

2.4. THEOREM. *Suppose X is the set of all vertices of a d-polytope* con X, *and X' is obtained from X by pushing q to q'. Then q' is a vertex of the d-polytope* con X', *and each proper face of* con X' *which includes q' is pyramidal at q'. For all $s \leqslant d-1$,*

$$f_s(\text{con } X') \geqslant f_s(\text{con } X) + g_s(q, X) + g_{s+1}(q, X),$$

where $g_r(q, X)$ denotes the number of proper r-faces of con X *which include q but are not pyramidal at q. If every proper face of* con X *which includes q is pyramidal at q, then $f_s(\text{con } X') = f_s(\text{con } X)$ for all s.*

Remark. The purpose for which 2.4 is employed here can also be served by a simpler result (involving *pulling* rather than *pushing*) appearing in **(16)**.

Proof. From the definition of *pushing* it is clear that $q' \notin \text{con}(X \sim \{q\})$, whence con X' is indeed a d-polytope having X' as its set of vertices. Now suppose that q' is a vertex of some proper face F of con X' which is not pyramidal at q', and let G be a $(d-1)$-face of con X' such that $G \supset F$. Then q' lies in the $(d-1)$-flat aff G which is determined by the subset $G \cap X$ of X. This contradicts the definition of pushing, so we conclude that each proper face of con X' which includes q' is pyramidal at q'.

Now suppose that Y is the set of all vertices of an r-face con Y of con X. We shall prove the following statements:

(a) *If $q \notin Y$, then* con Y *is an r-face of* con X'.

(b) *If $q \in Y$ and* con Y *is pyramidal at q, then the set* $\text{con}((Y \sim \{q\}) \cup \{q'\})$ *is an r-face of* con X'.

(c) *If $r \leqslant d-1$, $q \in Y$, and* con Y *is not pyramidal at q, then* $\text{con}(Y \sim \{q\})$ *is an r-face of* con X', *and there is a proper subset S of $Y \sim \{q\}$ such that* con S *is an $(r-1)$-face of* con X', $\text{con}(S \cup \{q'\})$ *is an r-face of* con X', *and* con S *intersects the relative interior of* con Y.

From (a) and (b) it follows that each s-face of con X which misses q or is pyramidal at q contributes one s-face to con X'. And (c) shows that if $s \leqslant d-1$, then at least two more s-faces are contributed to con X' by each s-face of con X which includes q but is not pyramidal at q; while if $s \leqslant d-2$, an additional s-face of con X' arises from each $(s+1)$-face of con X which includes q but is not pyramidal at q. Since there is no duplication among these contributions, the inequality stated in 2.4 is implied by the conjunction of (a), (b), and (c).

If Y is as in (a) and H is a supporting hyperplane of con X such that

$X \cap H = Y$, then $X' \cap H = Y$ and consequently con Y is an r-face of con X'. This proves (a).

Now with $X \subset \mathfrak{R}^d$, suppose Y is as in (b) and let $Z = Y \sim \{q\}$. We want to show that

$$\operatorname{aff}(Z \cup \{q'\}) \cap A = \emptyset, \qquad \text{where } A = \operatorname{con}(X \sim Y).$$

Suppose the contrary. Then we have

$$\alpha_0 q' + \sum_{z \in Z} \alpha_z Z = \sum_{x \in X \sim Y} \beta_x x \in \operatorname{con} X,$$

with

$$\alpha_0 + \sum_{z \in Z} \alpha_z = 1, \qquad \sum_{x \in X \sim Y} \beta_x = 1, \qquad \text{always } \beta_x \geqslant 0.$$

Since q' is interior to con X while con Z is an $(r - 1)$-face of con X, it follows that $\alpha_0 > 0$. Now we assume without loss of generality that $0 \in \operatorname{aff} Z$ and let ξ denote a linear transformation of $\mathfrak{R}^d$ onto $\mathfrak{R}^{d-r+1}$ such that the kernel $\xi^{-1}(0)$ of ξ is equal to aff Z. Then, of course,

$$\xi(\operatorname{aff} Y) = R(\xi q) \quad \text{and} \quad \xi(\operatorname{aff}(Z \cup \{q'\}) = R(\xi q').$$

Since Y is the set of all vertices of a face of con X, we have

$$(\operatorname{aff} Y) \cap A = \emptyset \quad \text{and} \quad R(\xi q) \cap \xi A = \emptyset,$$

where the second statement follows from the first because aff $Y = \xi^{-1}(R(\xi p))$. Thus, the polytope ξA in $\mathfrak{R}^{d-r+1}$ is not intersected by the line $R(\xi q)$ but (recalling that $\alpha_0 > 0$) it is intersected by the ray $]0, \infty[(\xi q')$. Consequently, the segment $]q, q']$ includes a point w such that the ray $]0, \infty[(\xi w)$ intersects a $(d - r - 1)$-face of ξA. There must be $d - r$ vertices $v_0, \ldots, v_{d-r-1}$ of this face whose affine hull is a $(d - r - 1)$-flat in $\mathfrak{R}^{d-r+1}$, and then with $u_i \in \xi^{-1}(v_i) \cap X$ it can be verified that the subset $Z \cup \{u_i\}_0^{d-r-1}$ of X determines a $(d - 1)$-flat in $\mathfrak{R}^d$ which includes the point w of $]p, p']$. This contradicts the definition of pushing and completes the proof of (b).

Now (preparing for (c)) with $X \subset \mathfrak{R}^d$, let us denote by C the union of all rays which emanate from q' and pass through the various points of $\operatorname{con}(X \sim \{q\})$; C is a polyhedral convex cone with vertex q' and C is pointed (contains no line). We claim that con $X' = C \cap \operatorname{con} X$, where inclusion in one direction is obvious. To establish the reverse inclusion we must show that if $p \in \operatorname{con} X \sim \operatorname{con} X'$, then $p \notin C$. When $p \notin \operatorname{con} X'$, we know that p is separated from con X' by a $(d - 1)$-flat H determined by points of X', and since $p \in \operatorname{con} X$, this flat must pass between p and p'. In view of the definition of pushing, this implies that $q' \in H$, whence H is a supporting hyperplane of C and $p \notin C$. We conclude that con $X' = C \cap \operatorname{con} X$. From this it follows that every face F of con X' which includes q' is contained in a face of C, and hence all of the other vertices of F must lie in the set $(X \sim \{q\}) \cap \operatorname{bdry} C$

Now suppose, finally, that Y is as in (c); that is, Y is the set of all vertices of an r-face con Y of con X (with $r \leqslant d - 1$), and $q \in Y$ but con Y is not pyramidal at q. Let $Z = Y \sim \{q\}$. Then it is clear that aff Z = aff Y and (by 2.1) con Z is an r-face of con X'. This is the first assertion of (c). With the cone C as above, the set B = bdry C is the union of a finite number of $(d - 1)$-dimensional polyhedral cones, each having q' as its vertex. Since con $Z \subset C$ and $p \in Y \cap ((\text{aff } Z) \sim C)$, the polytope con Y cuts across B and the intersection $B \cap$ con Y ($=B \cap$ con Z) is the union of a finite number ($\geqslant 1$) of $(r - 1)$-faces of con Z. It can be verified that each of these faces F is an $(r - 1)$-face of con X' and that con$(F \cup \{q'\})$ is an r-face of con X'. This establishes (c) and hence completes the proof of 2.4 except for the statement about the equality of $f_s(\text{con } X)$ and $f_s(\text{con } X')$. That statement follows from (a), (b), and (c) in conjunction with the fact that each proper face of con X' that includes q' is pyramidal at q'.

A polytope will be called s-simplicial provided all of its s-faces are simplices.

2.5. Corollary. *For $v \geqslant d + 1$ and $2 \leqslant s \leqslant d - 1$, let* $\mathbf{M}(d, v, s)$ *denote the class of all d-polytopes which have v vertices and which, among all d-polytopes with v vertices, have the maximum number of s-faces. Then* $\mathbf{M}(d, v, s)$ *includes d-polytopes which are* $(d - 1)$*-simplicial. All of the members of* $\mathbf{M}(d, v, s)$ *are s-simplicial and* (for $s \leqslant d - 2$) $(s + 1)$*-simplicial. If $d = s + 1$, $d = s + 2$, or $d = 2(s + 1)$, then all of the members of* $\mathbf{M}(d, v, s)$ *are* $(d - 1)$*-simplicial.*

Proof. Suppose $Q \in \mathbf{M}(d, v, s)$ and let $q_1, \ldots, q_v$ be the vertices of Q. Let $X_0 = \{q_i : 1 \leqslant i \leqslant u\}$, and for $1 \leqslant i \leqslant v$ let the set X_i be obtained from X_{i-1} by pushing q_i to a new position q_i'. Let $P = \text{con}\{q_i : 1 \leqslant i \leqslant v\}$. From 2.4 it follows that every proper face of P is pyramidal at each of its vertices, and then from 2.3 that P is $(d - 1)$-simplicial. The inequality in 2.4 implies that $f_s(P) \geqslant f_s(Q)$ (whence $P \in \mathbf{M}(d, v, s)$), with strict inequality if some s-face (or, when $s \leqslant d - 2$, some $(s + 1)$-face) of Q fails to be a simplex and hence is non-pyramidal at some vertex. But, of course, strict inequality is impossible, so all of the desired conclusions follow except for the special case $d = 2(s + 1)$. That case is covered by Gale's observation **(8)** that "the faces of a neighborly polytope are simplexes."

There is an open problem connected with 2.5. For each $d \geqslant 3$ let $M(d)$ denote the set of all integers $s \in [1, d - 1]$ such that for each $v \geqslant d + 1$, *all* of the members of $\mathbf{M}(d, v, s)$ are $(d - 1)$-simplicial. From 2.5 it follows that $\{d - 1, d - 2\} \subset M(d)$, and also $(d - 2)/2 \in M(d)$ when d is even. By considering pyramids based on neighbourly polytopes, it can be verified that $s \notin M(d)$ when $s \leqslant (d - 3)/2$. The problem is to determine $M(d)$ for all d. Note that $M(3) = \{1, 2\}$, $M(4) = \{1, 2, 3\}$,

$$\{3, 4\} \subset M(5) \subset \{2, 3, 4\} \quad \text{and} \quad \{2, 4, 5\} \subset M(6) \subset \{2, 3, 4, 5\},$$

but we do not know whether $2 \in M(5)$ or $3 \in M(6)$.

It may be generally known that each d-polytope has at least as many s-faces as has a d-simplex **(14**, 327**)**, but we have not found a proof in the literature. Accordingly, it seems worth while to establish the following stronger result.

2.6. THEOREM. *For all d and s, each d-simplex has exactly* $\binom{d+1}{s+1}$ *s-faces. For all s and for each d-polytope P which is not a d-simplex,*

$$f_s(P) \geqslant \binom{d+1}{s+1} + \binom{d-1}{s};$$

further, there is a d-polytope P_d *having*

$$f_s(P_d) = \binom{d+1}{s+1} + \binom{d-1}{s}$$

for all s.

Proof. The first assertion is obvious, for a d-simplex has $(d+1)$ vertices and each $s+1$ of these determine an s-face. Now for all d and s, let

$$\xi(d, s) = \binom{d+1}{s+1} + \binom{d-1}{s};$$

note that

$$\xi(d, s) = \xi(d-1, s) + \xi(d-1, s-1).$$

The polytopes P_d are constructed as in 2.3; that they have the stated property follows from the above recursion for $\xi(d, s)$ in conjunction with the equation terminating the proof of 2.3.

We want to show that if $d \geqslant 2$ and P is a d-polytope which is not a d-simplex, then $f_s(P) \geqslant \xi(d, s)$ for all s. This is evident in the two-dimensional case, where $\xi(2, 0) = \xi(2, 1) = 4$. Suppose it is known up through the $(d-1)$-dimensional case, and consider a d-polytope P as described. Since P is not a simplex, 2.3 implies that P is non-pyramidal at some vertex p. Let F be a $(d-1)$-face of P which misses p and let q be a vertex of P which is not in $F \cup \{p\}$. Such a q exists by non-pyramidality. Let G be a $(d-1)$-face of P which includes q but not p. Let H be the intersection with P of a $(d-1)$-hyperplane which strictly separates p from the remaining vertices of P, and let K be the intersection with G of a $(d-2)$-hyperplane which (relative to the $(d-1)$-flat aff G) strictly separates q from the remaining vertices of G. The $(d-1)$-polytopes F and G and the $(d-2)$-polytope K may all be simplices, but in any case the inductive hypothesis implies that

$$\begin{aligned} f_s(F) + f_{s-1}(H) + f_{s-1}(K) &\geqslant \binom{d}{s+1} + \binom{d}{s} + \binom{d-1}{s} \\ &= \binom{d+1}{s+1} + \binom{d-1}{s} = \xi(d, s). \end{aligned}$$

Since the numbers $f_s(F)$, $f_{s-1}(H)$, and $f_{s-1}(K)$ are respectively the numbers of s-faces of P which lie in F, which include p, and which lie in G while including q, it follows that $f_s(P) \geqslant \xi(d, s)$.

3. The number of vertices of a convex polytope. Here the results of §§1 and 2 are applied to establish the conjectured inequality (1) from the Introduction, under the restriction that $f_{d-1} \geqslant (d/2)^2 - 1$.

For $1 \leqslant s \leqslant d - 1$ and $v \geqslant d + 1$, let $N(d, v, s)$ denote the number of s-faces of a neighbourly d-polytope which has v vertices. (It follows from formulae in **(12)** that all such polytopes have the same number of s-faces. According to Gale **(8)**, this has also been established by Fieldhouse **(**perhaps in **6)**.)

3.1. THEOREM. *For each integer $d \geqslant 2$ there is an integer $k(d)$ which has the following property:*

Whenever $v \geqslant k(d)$ and P is a d-polytope having v vertices, then $f_s(P) \leqslant N(d, v, s)$ for $1 \leqslant s \leqslant d - 1$; if $s = d - 1$, $s = d - 2$, or P is $(d - 1)$-simplicial and $s \geqslant [(d - 2)/2]$, then $f_s(P) < N(d, v, s)$ unless P is neighbourly.

Proof. Let $k(d)$ be chosen according to 1.5 and 1.6, so that when $d = 2u$ and $v \geqslant k(d)$ the functions $f_{d-1}, \ldots, f_u$ are strictly proper for $(\mathbf{E}^{d-1}, u - 1, v)$, while when $d = 2u - 1$ and $v \geqslant k(d)$ the functions $f_{d-1}, \ldots, f_{u-1}$ are strictly proper for $(\mathbf{E}^{d-1,2}, u - 2, v)$. Consider a d-polytope P which has v vertices, with $v \geqslant k(d)$. If P is $(d - 1)$-simplicial, let $Q = P$. If P is not $(d - 1)$-simplicial, let Q be a $(d - 1)$-simplicial d-polytope such that $f_0(P) = f_0(Q)$, $f_s(P) \leqslant f_s(Q)$ for $1 \leqslant s \leqslant d - 1$, $f_{d-2}(P) < f_{d-2}(Q)$ and $f_{d-1}(P) < f_{d-1}(Q)$. (The existence of such a Q is guaranteed by 2.5.) By **(12**, 3.3), the complex formed by the proper faces of Q is an Eulerian $(d - 1)$-manifold. Thus, from the choice of $k(d)$ it follows that $f_s(Q) \leqslant N(d, v, s)$ and that the inequality is strict for $s \geqslant [(d - 2)/2]$ unless Q is neighbourly. This completes the proof.

3.2. THEOREM. *Suppose that P is a d-polytope having v vertices and f $(d - 1)$-faces. If $d = 2u$, then*

$$f \leqslant \frac{v}{v - u}\binom{v - u}{u} \qquad \textit{when } v \geqslant u^2 - 1$$

and

$$v \leqslant \frac{f}{f - u}\binom{f - u}{u} \qquad \textit{when } f \geqslant u^2 - 1.$$

If $d = 2u - 1$, then

$$f \leqslant 2\binom{v - u}{u - 1} \qquad \textit{when } v \geqslant u^2 - 2$$

and

$$v \leqslant 2\binom{f - u}{u - 1} \qquad \textit{when } f \geqslant u^2 - 2.$$

Proof. For the inequalities "$f \leqslant \ldots$," use 2.5 in conjunction with 1.5 and 1.6, as was done in the proof of 3.1. The inequalities "$v \leqslant \ldots$" then follow with the aid of the standard polarity theory for convex polytopes **(15)**.

3.3. COROLLARY. *At least for $d \leqslant 6$, the inequalities* (1) *and* (1*) *of the Introduction are satisfied by all d-polytopes.*

It seems probable that the extra conditions on v, f, and d are required in 3.1–3.3 merely because our approach is inadequate. We are interested mainly in those cell-complexes which arise as the system of all proper faces of a d-polytope, but have not made full use of all the structure at our disposal. It was used only to restrict attention to Eulerian $(d-1)$-manifolds of Euler characteristic $1-(-1)^d$, and then a formula valid for all such manifolds was used to express f_{d-1} as a linear combination of

$$f_{[d/2]-1}, f_{[d/2]-2}, \ldots, f_1, \text{ and } f_0.$$

From that point on, the reasoning applied to an arbitrary simplicial $(d-1)$-complex, without using even the information contained in 2.6. Presumably, a fuller use of the available structure would lead to a proof of the inequalities (1) and (1*) without additional restrictions. Thus, *we conjecture that $f_s(P) \leqslant N(d, v, s)$ whenever $1 \leqslant s \leqslant d-1$ and P is a d-polytope having v vertices, while (dually) $f_s(P) \leqslant N(d, f, d-1-s)$ whenever $0 \leqslant s \leqslant d-2$ and P is a d-polytope having f $(d-1)$-faces.*

For $2 \leqslant d \leqslant f-1$, let $V(f, d)$ denote the maximum number of vertices achieved by any d-polytope which has f $(d-1)$-faces. Part of the above conjecture is the same as the JSG-*conjecture* (Jacobs and Schell **(10)**, Gale **(8, 9)**)—namely, that

$$V(f, d) = \frac{f}{f-u}\binom{f-u}{u} \quad \text{when } d = 2u$$

and

$$V(f, d) = 2\binom{f-u}{u-1} \quad \text{when } d = 2u-1;$$

this is proved in 3.2 for $u^2 - 1 \leqslant f$. Now it is also of interest to determine the maximum of $V(f, d)$ for other ranges of values of d (when f is given). Partial results in this direction can be obtained from 3.2, and the same line of reasoning leads to the following observation.

3.4. PROPOSITION. *Suppose the* JSG *conjecture is correct, and f is an integer $\geqslant 2$. Then for the polytopes which have f maximal proper faces, the maximum possible number of vertices is the larger of the two numbers*

$$\frac{f}{f-s}\binom{f-s}{s} \quad \textit{with } s = \left[\frac{5f+6-\surd(5f^2-4)}{10}\right]$$

and

$$2\binom{f-t}{t-1} \quad \textit{with } t = \left[\frac{5f+12-\surd(5f^2+4)}{10}\right].$$

Proof. From the JSG conjecture it follows that if $2(u+1) \leqslant f-1$, then

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$$\frac{V(f, 2u)}{V(f, 2(u+1))} = \frac{(f-u-1)(u+1)}{(f-2u)(f-2u-1)},$$

whence

(1) $\qquad V(f, 2u) \leqslant V(f, 2(u+1)) \Leftrightarrow u \leqslant (5f - 4 - \sqrt{(5f^2 - 4)})/10.$

The JSG conjecture implies also that if $2u + 1 \leqslant f - 1$, then

$$\frac{V(f, 2u-1)}{V(f, 2u+1)} = \frac{(f-u)u}{(f-2u+1)(f-2u)},$$

whence

(2) $\qquad V(f, 2u-1) \leqslant V(f, 2u+1) \Leftrightarrow u \leqslant (5f + 2 - \sqrt{(5f^2 + 4)})/10.$

Now let s be the largest integer such that $2 \leqslant 2s \leqslant f - 1$ and

$$V(f, 2s) = \max\{V(f, 2u) : 2 \leqslant 2u \leqslant f - 1\},$$

and let t be the largest integer such that $2 \leqslant 2t - 1 \leqslant f - 1$ and

$$V(f, 2t-1) = \max\{V(f, 2u-1) : 2 \leqslant 2u - 1 \leqslant f - 1\}.$$

Then

$$V(f, 2s-2) \leqslant V(f, 2s) > V(f, 2s+2)$$

and from (1) it follows that

$$s - 1 \leqslant (5f - 4 - \sqrt{(5f^2 - 4)})/10 < s,$$

whence

$$s = [(5f + 6 - \sqrt{(5f^2 - 4)})/10].$$

Similar reasoning based on (2) shows that

$$t = [(5f + 12 - \sqrt{(5f^2 + 4)})/10].$$

This completes the proof, for the maximum which we seek is either $V(f, 2s)$ or $V(f, 2t-1)$ (or both). (This reasoning assumes that

$$1 \leqslant 2s - 2 \leqslant 2s + 2 \leqslant f - 1 \quad \text{and} \quad 1 \leqslant 2t - 3 \leqslant 2t + 1 \leqslant f - 1.$$

The assumption fails for a few small values of f, but these are easily treated directly.)

When $f \leqslant 7$, the validity of 3.4 follows from 3.3 (without using the JSG conjecture). The first alternative in 3.4 arises for $f \in \{3, 4, 6, 7\}$, the second for $f \in \{2, 4, 5\}$.

4. The problem of Dantzig. Dantzig's problem (**3**, no. 7) is not immediately concerned with linear *inequalities* in real variables, but rather with m linear *equations* in n *non-negative* variables. Accordingly, our attention is directed to the positive orthant $\mathfrak{O}^n$, consisting of all points of $\mathfrak{R}^n$ which have exclusively non-negative co-ordinates. A linear equation in n real variables determines a hyperplane in $\mathfrak{R}^n$, and a system of m linear equations determines a flat of dimension $\geqslant n - m$; if the system is not redundant, the dimension of the flat is equal to $n - m$. Thus Dantzig's problem may be stated more geometrically as follows: Among the intersections of $\mathfrak{O}^n$ with the various k-dimensional flats in $\mathfrak{R}^n$, which ones have the maximum number of vertices and what is this maximum number?

Up to this point we have discussed only bounded sets. However, there is no such restriction in Dantzig's problem, and accordingly we define a *d-polyhedron* to be a d-dimensional set which is the intersection of a finite number of closed half-spaces. As is well known **(15)**, a set is a d-polytope if and only if it is a bounded d-polyhedron.

Considering each finite-dimensional linear space to be self-dual with respect to an inner product $\langle\ ,\ \rangle$, we shall use without specific reference the standard polarity theory for convex bodies. The results employed here can be found in **(15)** or **(11)**. We require also the following remark.

4.1. Proposition. *Suppose that E and F are finite-dimensional linear spaces, ζ is a linear transformation of E into F, and ζ^a is the adjoint of ζ. Then for each set $X \subset E$ it is true that*

$$\zeta^a((\zeta X)^0) = X^0 \cap (\zeta^a F).$$

Proof. Here ζ^a is the linear transformation of F into E which is defined by the condition that $\langle x, \zeta^a y\rangle = \langle \zeta x, y\rangle$ for all $x \in E$ and $y \in F$. To establish 4.1 it suffices to note that if $\bar{x} \in E$, $\bar{y} \in F$, and $\bar{x} = \zeta^a \bar{y}$, then the following five statements are equivalent: $\bar{x} \in X^0$; $\langle x, \bar{x}\rangle \leqslant 1$ for all $x \in X$; $\langle x, \zeta^a \bar{y}\rangle \leqslant 1$ for all $x \in X$; $\langle \zeta x, \bar{y}\rangle \leqslant 1$ for all $x \in X$; $\bar{y} \in (\zeta X)^0$.

In applying 4.1 we shall use the fact that the linear transformation ζ^a is non-singular provided that ζ maps E *onto* F.

The next theorem extends an observation of Davis **(4)**.

4.2. Theorem. *Suppose that P is a k-polyhedron in $\mathfrak{R}^k$, with $O \in \operatorname{int} P$. Then the following three statements are equivalent:*

(*a*) *P is affinely equivalent to the intersection of $\mathfrak{O}^n$ with some k-flat in $\mathfrak{R}^n$.*

(*b*) *P contains no line and P has at most n $(k-1)$-faces.*

(*c*) *the polar body P^0 is a k-polytope in $\mathfrak{R}^k$ with at most n vertices other than the origin O (which may be a vertex of P^0 but is not required to be).*

Proof. $(a) \Rightarrow (b)$. Suppose (a) holds. Then there is a non-singular affine transformation ξ of $\mathfrak{R}^k$ into $\mathfrak{R}^n$ such that $\xi P = (\xi \mathfrak{R}^k) \cap \mathfrak{O}^n$. For $1 \leqslant i \leqslant n$ let η_i be the composition of ξ with the ith co-ordinate function on $\mathfrak{R}^n$. Then the k-polyhedron P is the intersection of the n sets $\{x \in \mathfrak{R}^k : \eta_i(x) \geqslant 0\}$ $(1 \leqslant i \leqslant n)$, and since each of these sets is either all of $\mathfrak{R}^k$ or is a closed half-space in $\mathfrak{R}^k$, it follows that P has at most n $(k-1)$-faces. Since $\mathfrak{O}^n$ contains no line, the same is true of P.

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$(b) \Rightarrow (c)$. Suppose (b) holds and $O \in \operatorname{int} P$, whence of course P^0 is bounded. Since P contains no line, P^0 is not contained in a hyperplane in $\mathfrak{R}^k$ and consequently P^0 is k-dimensional. Each vertex of P^0 other than O corresponds to a $(k-1)$-face of P, so P^0 is a k-polytope with at most n vertices other than O.

$(c) \Rightarrow (a)$. Suppose (c) holds and consider the n-simplex

$$S = \{x = (x^1, \ldots, x^n) \in R^n : \textstyle\sum_1^n x^i \leqslant 1;\ x^i \geqslant 0 \text{ for all } i\} \subset \mathfrak{O}^n.$$

Since $O \in P^0$ and P^0 has at most n vertices other than O, there exists a linear transformation ζ of $\mathfrak{R}^n$ onto $\mathfrak{R}^k$ such that $\zeta S = P^0$. From 4.1 it follows that the set P is affinely equivalent to a k-section of the set

$$S^0 = \{x \in R^n : x^i \leqslant 1 \text{ for all } i\},$$

and of course S^0 is equivalent to $\mathfrak{O}^n$. Thus, (c) implies (a) and the proof is complete.

4.3. Corollary. *If P is a k-polyhedron and j is an integer $\geqslant 1$, then the following two statements are equivalent:*

(a) P is affinely equivalent to the intersection of $\mathfrak{O}^n$ with some $(k+j)$-flat in $\mathfrak{R}^n$.

(b) P contains no line and P has at most $n-j-1$ $(k-1)$-faces.

Proof. $(a) \Rightarrow (b)$. Let G be a $(k+j)$-flat in $\mathfrak{R}^n$ such that the intersection $G \cap \mathfrak{O}^n$ is affinely equivalent to the k-polyhedron P. Let $\mathfrak{O}^l$ (an l-dimensional orthant in $\mathfrak{O}^n$) be the smallest face of $\mathfrak{O}^n$ which contains the set $G \cap \mathfrak{O}^n$. If $l = n$, then G intersects the interior of $\mathfrak{O}^n$ and it is clear that

$$\dim(G \cap \mathfrak{O}^n) = \dim G = k + j > k,$$

an impossibility. Thus, $l \leqslant n-1$ and G misses the interior of $\mathfrak{O}^n$. Since $\mathfrak{O}^n$ is polyhedral, the supporting flat G must lie in a supporting hyperplane H of $\mathfrak{O}^n$. By the minimality of l, G includes a point of the relative interior of $\mathfrak{O}^l$, and this implies that $\mathfrak{O}^l \subset H$, whence H contains the linear hull $\mathfrak{R}^l$ of $\mathfrak{O}^l$. It can be verified that

$$\dim(G \cap \mathfrak{R}^l) = \dim(G \cap \mathfrak{O}^l) = k.$$

Since the $(k+j)$-flat G and the l-flat $\mathfrak{R}^l$ both lie in the $(n-1)$-flat H, we conclude from a well-known inequality that

$$(k+j) + l - k \leqslant n - 1,$$

whence $l \leqslant n-j-1$. Since the k-polyhedron P is affinely equivalent to a k-section of $\mathfrak{O}^l$, we conclude from 4.2 that condition (b) is satisfied.

$(b) \Rightarrow (a)$. Suppose P is as in (b), whence by 4.2 P is affinely equivalent to the intersection of $\mathfrak{O}^{n-j-1}$ by a k-flat F in $\mathfrak{R}^{n-j-1}$. We may regard $\mathfrak{O}^{n-j-1}$ as a face of $\mathfrak{O}^n$ and then $\mathfrak{R}^n$ contains a hyperplane H such that

$$H \cap \mathfrak{O}^n = \mathfrak{O}^{n-j-1} \subset \mathfrak{R}^{n-j-1}.$$

In the $(n-1)$-flat H there is a j-flat F' whose intersection with $\mathfrak{R}^{n-j-1}$ consists of a single point of F, and then the affine hull G of $F \cup F'$ is a $(j+k)$-flat in $\mathfrak{R}^n$ such that $G \cap \mathfrak{O}^n = F \cap \mathfrak{O}^{n-j-1}$, a set affinely equivalent to P.

The following result is useful for its corollary, which justifies a restriction to bounded sets in the problem of Dantzig.

4.4. Proposition. *For positive integers d, m, and n the following two statements are equivalent:*

(*a*) *There exists an unbounded d-polyhedron P which contains no line and which has exactly m $(d-1)$-faces and exactly n vertices.*

(*b*) *There exist a d-polytope Q and a boundary point z (not necessarily a vertex) of Q such that Q has exactly m vertices $\neq z$ and exactly n $(d-1)$-faces disjoint from z.*

Proof. To see that (*a*) implies (*b*), suppose that $O \in \operatorname{int} P \subset \mathfrak{R}^d$ and let Q be the polar body of P, $Q = P^0 \subset \mathfrak{R}^d$. With $z = O$, the desired conclusion follows from the standard polarity theory. To see that (*b*) implies (*a*), take $O = z \in \operatorname{bdry} Q \subset \mathfrak{R}^d$ and let $P = Q^0$. Again the polarity theory is applicable.

4.5. Corollary. *Suppose that P is an unbounded d-polyhedron which contains no line and has f $(d-1)$-faces. Then $f \geqslant d$, and if $f \geqslant d+1$, there exists a d-polytope which has f $(d-1)$-faces and has more vertices than P.*

Proof. Recall that P is the intersection of the supporting half-spaces determined by its $(d-1)$-faces. If $f = k < d$, then P contains a flat of deficiency $d-k > 0$, contrary to our assumption. Hence, $f \geqslant d$, and when $f = d$ it is easily verified that P is a convex cone which is affinely equivalent to an orthant in $\mathfrak{R}^d$.

Now suppose that $f = m \geqslant d+1$. Let n denote the number of vertices of P and let Q and z be as in 4.4 (*b*). If z is not a vertex of Q, then Q is a d-polytope having m vertices and more than n $(d-1)$-faces. Translating Q so as to contain the origin in its interior and then forming the polar body, we obtain a d-polytope which has f $(d-1)$-faces and has more vertices than P. Now suppose z is a vertex of Q. If $n = 1$ the assertion of 4.5 is obvious, so we suppose that $n \geqslant 2$ and denote by S the polytope which is generated by the vertices of Q other than z. With $n \geqslant 2$ it is easy to see that S is a d-polytope which has more than n $(d-1)$-faces, and then we proceed as we did earlier with Q.

The next result is a partial solution of Dantzig's problem.

4.6. Theorem. *Suppose the set P in $\mathfrak{R}^n$ is the intersection of the positive orthant $\mathfrak{O}^n$ with a flat of deficiency m in $\mathfrak{R}^n$, where $n - 2\sqrt{(n+1)} < m \leqslant n$ (a restriction that is unnecessary if the* JSG *conjecture is correct). Then the number of extreme points of the set P is at most*

$$\frac{2n}{m+n}\binom{\frac{1}{2}(m+n)}{m} \qquad \textit{when } n-m \textit{ is even}$$

and at most

$$2\binom{\frac{1}{2}(m+n-1)}{m} \qquad \textit{when } n-m \textit{ is odd.}$$

The upper bounds are attained if and only if P is an $(n-m)$-polytope such that each vertex of P is on exactly $n-m$ edges and such that for all $k \leqslant [(n-m)/2]$, each k $(n-m-1)$-faces of P intersect in an $(n-m-k)$-face of P.

Proof. Let $V(f, d)$ denote, as in §3, the maximum number of vertices achieved by any d-polytope which has f $(d-1)$-faces. By 4.5, this is greater than the maximum number of vertices achieved by any unbounded d-polyhedron which has f $(d-1)$-faces. Let $k = \dim P$ and $j = n-m-k \geqslant 0$. If $j = 0$ it follows from 4.2 that $f_0(P) \leqslant V(n, k)$, where equality implies boundedness of P. If $j > 0$ it follows from 4.3 that $f_0(P) \leqslant V(n-j-1, k) < V(n, k)$. We conclude that $f_0(P) \leqslant V(n, n-m)$, where equality cannot obtain unless P is an $(n-m)$-polytope which has n faces of dimension $n-m-1$. Now if $n - 2\surd(n+1) < m \leqslant n$, then $n > \{\frac{1}{2}(n-m)\}^2 - 1$, so from 3.2 it follows that $V(n, n-m)$ is equal to the upper bounds listed in 4.6. By 4.2, $V(n, n-m)$ can really be attained as the number of vertices of some set P of the sort described in 4.6. To characterize those sets P for which the upper bound is actually attained, one applies certain results from §§1 and 2, the reasoning being similar to that of 3.1.

If a flat in $\mathfrak{R}^n$ is determined by a system of m linear equations, then without checking the redundancy of the system we know only that the flat is of deficiency $\leqslant m$. Thus, the following remark is also of interest in connection with Dantzig's problem. It can be proved by the reasoning of 3.4.

4.7. Proposition. *Suppose the* JSG *conjecture is correct. Let m and n be integers with $0 \leqslant m \leqslant n \geqslant 2$ and let v be the maximum number of vertices which is realized by the intersection of $\mathfrak{O}^n$ with a flat of deficiency $\leqslant m$ in $\mathfrak{R}^n$. Let*

$$s = [(5n + 6 - \surd(5n^2 - 4))/10] \textit{ and } t = [(5n + 12 - \surd(5n^2 + 4))/10].$$

Then at least one of the following statements is true:

(*a*) $2s \geqslant n - m$ *and* $v = \dfrac{n}{n-s}\dbinom{n-s}{s}$;

(*b*) $2t - 1 \geqslant n - m$ *and* $v = 2\dbinom{n-t}{t-1}$;

(*c*) $2s < n - m$, $n - m$ *is even, and* $v = \dfrac{2n}{m+n}\dbinom{\frac{1}{2}(m+n)}{m}$;

(*d*) $2t - 1 < n - m$, $n - m$ *is odd, and* $v = 2\dbinom{\frac{1}{2}(m+n-1)}{m}$.

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