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INERTIAL MANIFOLD FOR A REACTION DIFFUSION EQUATION MODEL OF COMPETITION IN A CHEMOSTAT

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Abstract

The existence of an inertial manifold for a reaction-diffusion equation model of the chemostat is established.

1. Introduction

The purpose of this paper is to show that inertial manifolds exist for a system of reaction diffusion equations which was used to model competition in a chemostat (c.f. So and Waltman [8]). The equations are:

$$\begin{cases} S_t = S_{xx} - f(S)u - g(S)v \\ u_t = u_{xx} + f(S)u \\ v_t = v_{xx} + g(S)v \end{cases} \tag{1.1}$$

where $S(t, x)$ (respectively $u(t, x)$, $v(t, x)$) denotes the concentration of the limiting substrate (respectively the competing micro–organisms) at time $t \geq 0$ and position $0 \leq x \leq L$. Here

$$\begin{cases} f(S) := mS/(a + S) \\ g(S) := nS/(b + S) \end{cases} \tag{1.2}$$

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for $S \geq 0$, where m, a, n and $b > 0$. The boundary conditions are

$$\begin{cases} S_x(t, 0) = -S^{(0)} \\ u(t, 0) \ = v_x(t, 0) = 0 \\ S(t, L) + \gamma S(t, L) = u_x(t, L) + \gamma u(t, L) = v_x(t, L) + \gamma v(t, L) = 0 \end{cases} \tag{1.3}$$

where $S^{(0)}$ and $\gamma > 0$.

Let $z = S + u + v$. Then z satisfies

$$z_t = z_{xx} \tag{1.4}$$

with boundary conditions

$$\begin{cases} z_x(t, 0) = -S^{(0)}, \\ z_x(t, L) + \gamma z(t, L) = 0. \end{cases} \tag{1.5}$$

We need the following form of the Poincaré inequality.

Proposition 1.1. (c.f. Theorem 11.11 of Smoller [7]). *Let $w \epsilon W^{1,2}[0, L]$. Then*

$$\|w'\|_2^2 + \gamma w(L)^2 \geq c\|w\|_2^2, \tag{1.6}$$

where $c > 0$ is the smallest eigenvalue of the boundary-value problem

$$-w'' = \lambda w, \quad w'(0) = w'(L) + \gamma w(L) = 0. \tag{1.7}$$

Proposition 1.2. *Let $z(t, x)$ be a solution of (1.4) and (1.5). Then $z(t, x)$ converges to the steady state solution $\hat{z}(x) := S^{(0)}(L + 1/\gamma - x)$ of (1.4), (1.5) in the L^2 norm.*

Proof. Let $w = z - \hat{z}$. Then w satisfies $w_t = w_{xx}$ and $w_x(t, 0) = w_x(t, L) + \gamma w(t, L) = 0$. Now

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\int_0^L w^2\, dx\right) &= \int_0^L w \frac{dw}{dt} dx = \int_0^L w w_{xx} dx \\ &= [w w_x]_0^L - \int_0^L w_x^2 dx = -\gamma w(t, L)^2 - \int_0^L w_x^2 dx. \end{aligned}$$

By Proposition (1.1),

$$\frac{1}{2}\frac{d}{dt}\|w(t, .)\|_2^2 \leq -c\|w(t, .)\|_2^2$$

which in turn implies

$$\|w(t, .)\|_2 \leq e^{-ct}\|w(0, .)\|_2.$$

Since we are only interested in asymptotic behavior, we replace $\hat{z}(x)$ in (1.1) and (1.3) to obtain

$$\begin{cases} u_t = u_{xx} + f(\hat{z}(x) - |u| - |v|)u \\ v_t = v_{xx} + g(\hat{z}(x) - |u| - |v|)v \end{cases} \tag{1.8}$$

with boundary conditions:

$$u_x(t, 0) = v_x(t, 0) = u_x(t, L) + \gamma u(t, L) = v_x(t, L) + \gamma v(t, L) = 0, \tag{1.9}$$

where

$$f(S) := \begin{cases} mS/(a + |S|) & \text{for } S \geq -1 \\ -m/(a+1) & \text{for } S < -1 \end{cases}$$

$$g(S) := \begin{cases} nS/(b + |S|) & \text{for } S \geq -1 \\ -n/(b+1) & \text{for } S < -1 \end{cases}$$

Note that this re-definition of $f(S)$ and $g(S)$ will not affect solutions $(S(t, x), u(t, x), v(t, x))$ of (1.1), (1.3) satisfying $S(t, x), u(t, x), v(t, x) \geq 0$ and $S(t, x) + u(t, x) + v(t, x) = \hat{z}(x)$. It is (1.8), (1.9) for which we shall show that inertial manifolds exist.

We shall need the following simple estimates on f and g.

Proposition 1.3. *For all S, S_1 and S_2, we have*

$$|f(S)| \leq m, \quad |g(S)| \leq n,$$
$$|f(S_1) - f(S_2)| \leq (m/a)|S_1 - S_2|, \quad |g(S_1) - g(S_2)| \leq (n/b)|S_1 - S_2|.$$

2. Inertial manifolds: general theory

There are a number of existence theories for inertial manifolds (e.g. Kamaev [4], Mora [6], Foias, Sell and Teman [2], Mallet-Paret and Sell [5], Chow and Lü [1] and Teman [9]). In this section we recall one that is immediately applicable to (1.8) and (1.9).

Consider an abstract evolution equation of the form

$$\frac{dw}{dt} + Aw = R(w) \tag{2.1}$$

on a Hilbert space H. A is a linear, unbounded, self-adjoint operator on H with dense domain, $D(A)$, in H. Moreover, A is assumed to be positive and that A^{-1} is compact. Under these assumptions on A, there exists an orthonormal basis $\{w_j\}$ of H consisting of eigenvectors of A, $Aw_j = \lambda_j w_j$, where the eigenvalues satisfy $0 < \lambda_1 \le \lambda_2 \le \dots$, $\lambda_j \to \infty$ as $j \to \infty$. The nonlinear term $R : H \to H$ is assumed to be locally Lipschitz continuous.

DEFINITION 2.1. A subset M of H is said to be an *inertial manifold* for (2.1) if it satisfies the following properties:

(i) M is a finite dimensional Lipschitz manifold,
(ii) M is positively invariant, and
(iii) M attracts exponentially all solutions of (2.1).

Assume that (2.1) is *dissipative*, i.e., there is a $\rho_0 > 0$ such that

$$\limsup_{t\to\infty} \|w(t)\|_2 \le \rho_0, \tag{2.2}$$

for all solutions $w(t)$ of (2.1). In this case, one can modify (2.1) to the so-called *prepared* equation

$$\frac{dw}{dt} + Aw = \theta_\rho(|w|)R(w). \tag{2.3}$$

Here, $\theta : [0, \infty) \to [0, 1]$ is a fixed smooth function with $\theta(s) = 1$ for $0 \le s \le 1$, $\theta(s) = 0$ for $s \ge 2$ and $|\theta'(s)| \le 2$ for $s \ge 0$. And $\theta_\rho(s) = \theta(\frac{s}{\rho})$ for $s \ge 0$, where $\rho = 2\rho_0$.

THEOREM 2.2. (Theorem 2.2 of [Foias, Sell and Teman]). *Under the above assumptions, there exist* N_0, K_{12}, $K_{13} > 0$ *such that if one has*

$$N \ge N_0, \quad \lambda_{N+1} \ge K_{12}, \quad \lambda_{N+1} - \lambda_N \ge K_{13}, \tag{2.4}$$

then (2.3) *possesses an inertial manifold of dimension* N.

3. Inertial manifolds: our model

In order to show that (1.8), (1.9) possess an inertial manifold, we will first cast them in the form (2.1) and verify the hypotheses of Theorem 2.2. Let H be the Hilbert space $L^2[0, L] \times L^2[0, L]$. Let A be the linear operator $(-d^2/dx^2, -d^2/dx^2)$ defined on the subspace of H consisting of all pairs (u, v), where $u, v \in C^2[0, L]$ satisfy the boundary conditions (1.9). By Friedrichs' extension theorem, we can extend A to a closed operator, again denoted by A. Then A is an unbounded, self-adjoint, positive operator from its domain $D(A)$ to H with A^{-1} compact. Moreover, if we denote the eigenvalues of A by $0 < \lambda_1 \le \lambda_2 \le \dots$, then $\lambda_{2n-1} = \lambda_{2n} = \mu_n^2$, where μ_n is the n-th positive root of the equation $\tan(\mu L) = \gamma/\mu$. Since $(n-1)\pi L^{-1} < \mu_n < (n - \frac{1}{2})\pi L^{-1}$, (2.4) can be satisfied with a large enough N.

Let $R : H \to H$ denote the Nemitski operator corresponding to the reaction term, i.e.

$$R(u, v)(x) = \Big(f(\hat{z}(x) - |u(x)| - |v(x)|)u(x),\ g(\hat{z}(x) - |u(x)| - |v(x)|)v(x)\Big). \tag{3.1}$$

We first show that R is globally Lipschitz continuous on H. Consider the integral

$$I := \int_0^L \left| f(\hat{z}(x) - |u_1(x)| - |v_1(x)|)u_1(x) - f(\hat{z}(x) - |u_2(x)| - |v_2(x)|)u_2(x)\right|^2 dx$$

Then

$$I = \int_{M_1^+ \cap M_2^+} + \int_{M_1^- \cap M_2^+} + \int_{M_1^+ \cap M_2^-} + \int_{M_1^- \cap M_2^-},$$

where

$$M_i^+ := \{x \in [0, L] : \hat{z}(x) - |u_i(x)| - |v_i(x)| > -1\},$$
$$M_i^- := \{x \in [0, L] : \hat{z}(x) - |u_i(x)| - |v_i(x)| \leq -1\}.$$

Denote these integrals by I_1, I_2, I_3 and I_4, respectively.

For $x \in M_1^- \cap M_2^-$, the absolute value (i.e. without the square) in the integrand of I is (with the x suppressed):

$$\leq \left|(-\frac{m}{a+1}u_1) - (-\frac{m}{a+1}u_2)\right| \leq \frac{m}{a+1}|u_1 - u_2|.$$

Therefore, $I_4 \leq c_4 \|u_1 - u_2\|_2^2$, for some $c_4 > 0$.

For $x \in M_1^+ \cap M_2^+$, by Proposition 1.3, the absolute value is:

$$\begin{aligned}
&\leq \left| f(\hat{z} - |u_1| - |v_1|)(u_1 - u_2)\right| + \left|(f(\hat{z} - |u_1| - |v_1|) - f(\hat{z} - |u_2| - |v_2|))u_2\right| \\
&\leq m|u_1 - u_2| + \frac{m}{a}\Big||u_1| + |v_1| - |u_2| - |v_2|\Big||u_2| \\
&\leq m|u_1 - u_2| + \frac{m(\hat{z}(0)+1)}{a}\Big(|u_1 - u_2| + |v_1 - v_2|\Big).
\end{aligned}$$

Therefore, $I_1 \leq c_1(\|u_1 - u_2\|_2 + \|v_1 - v_2\|_2)^2$, for some $c_1 > 0$.

There are similar estimates on I_2 and I_3 as well as on the second component of R. Hence, R is globally Lipschitz continuous.

Next we will show that the dissipative condition (2.2) is satisfied. Integrating

$$uu_t = uu_{xx} + f(\hat{z} - |u| - |v|)u^2$$

we get

$$\frac{1}{2}\frac{d}{dt}\int_0^L u^2 = -\gamma u(.,L)^2 - \int_0^L u_x^2 + \int_0^L f(\hat{z}-|u|-|v|)u^2$$
$$\leq -c\int_0^L u^2 + \int_0^L f(\hat{z}-|u|-|v|)u^2,$$

by Proposition 1.1. Fix any t and consider the integral

$$I := \int_0^L f(\hat{z}-|u|-|v|)u^2 = \int_{M^+} f(\hat{z}-|u|-|v|)u^2 + \int_{M^-} f(\hat{z}-|u|-|v|)u^2$$

where

$$M^+ := \{x \in [0, L] : \hat{z}(x) - |u(t,x)| - |v(t,x)| > -1\}$$
$$M^- := \{x \in [0, L] : \hat{z}(x) - |u(t,x)| - |v(t,x)| \leq -1\}.$$

Denote these integrals by I_1 and I_2 respectively. The first integral I_1 is bounded above by

$$m\int_{M^+} u^2 \leq m(\hat{z}(0)+1)^2 L := K.$$

Let $\bar{\rho} > 0$ be such that $\bar{\rho}^2 = \max\{\frac{(a+1)K}{m}, \frac{(b+1)K}{n}\} + (\hat{z}(0)+1)^2 L$ and pick any $\rho_0 > \bar{\rho}$. Suppose $\|u(\bar{t},.)\|_2 \geq \rho_0$ for some $\bar{t}$. Then for $t = \bar{t}$, we have

$$\int_{\left\{\substack{x\in[0,L];\\ |u(t,x)|<\hat{z}(x)+1}\right\}} u^2 + \int_{\left\{\substack{x\in[0,L];\\ |u(t,x)|\geq\hat{z}(x)+1}\right\}} u^2 \geq \rho_0^2$$

which implies

$$\int_{\left\{\substack{x\in[0,L];\\ |u(t,x)|\geq\hat{z}(x)+1}\right\}} u^2 \geq \rho_0^2 - \int_{\left\{\substack{x\in[0,L];\\ |u(t,x)|<\hat{z}(x)+1}\right\}} u^2$$
$$\geq \rho_0^2 - (\hat{z}(0)+1)^2 L > \frac{(a+1)K}{m}$$

Therefore, at $t = \bar{t}$,

$$I_2 \leq -\frac{m}{a+1}\int_{M^-} u^2 \leq -\frac{m}{a+1}\int_{\left\{\substack{x\in[0,L];\\ |u(t,x)|\geq\hat{z}(x)+1}\right\}} u^2 < -K.$$

Hence, $I < 0$ and consequently $\frac{d}{dt}\|u(t,.)\|^2 \leq -2c\|u(t,.)\|^2$, whenever $\|u(t,.)\|^2 \geq \rho_0$. Similarly, $\frac{d}{dt}\|v(t,.)\|^2 \leq -2c\|v(t,.)\|^2$, whenever $\|v(t,.)\|^2 \geq \rho_0$.

Thus, by Theorem 2.2, we have proved that the prepared equation for (1.8), (1.9) possesses an inertial manifold M_{ρ_0}.

Actually the above argument shows a little more. If we let

$$B := \{(u, v) \in H : \|u\|_2,\ \|v\|_2 < \rho_1\},$$

where $\rho_1 > \bar{\rho}$ then B is positively invariant and absorbing, i.e., if we denote the solution operator for (1.8), (1.9) by $T(t)$ then $T(t)B \subset B$ and for each bounded set B_1, there exists t_1 such that $T(t)B_1 \subset B$ for all $t \geq t_1$. Moreover, $T(t)$ maps bounded sets to bounded sets. Hence, by Theorem 4.2.4 of Hale [3], (1.8), (1.9) possess a global attractor which lies in B. If we now pick $\rho_0 > \rho_1$ so large that the ball in H with radius ρ_0 and centered at the origin contains the B, then $B \cap M_{\rho_0}$ is an inertial manifold for (1.8), (1.9). Thus, we have proved that

THEOREM 3.1. *Under the above assumptions,* (1.8), (1.9) *possess an inertial manifold.*

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