σ-STABILITY ANALYSIS OF HYDROMAGNETIC INSTABILITIES IN STARS WITH TOROIDAL MAGNETIC FIELDS

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ABSTRACT

σ-stability analysis is used to investigate the adiabatic stability of a star containing an axisymmetric toroidal magnetic field. Necessary and sufficient conditions for σ-stability are derived. Special attention is devoted to the typical hydromagnetic instabilities that can be introduced by a weak toroidal magnetic field in a star that is stably stratified in the absence of any magnetic field. An expression for the maximum growth rate of instability is derived and the basic properties of the displacement fields associated with the instabilities are indicated.

1. INTRODUCTION

According to the usual stability concept a system is stable if no perturbations occur that grow indefinitely in time. Stability can be investigated by means of an energy principle that says that a system is stable if δW is positive for all admissible perturbations $\vec{\xi}$, where δW is the change in potential energy caused by the linear displacement $\vec{\xi}$; otherwise the system is unstable. The usual stability analysis tells us whether the system is stable or not, and where instability sets in in the star. However it does not tell us whether the instability is relevant (the instability may take a very long time to grow) or where the most violent instabilities are located, and it says nothing about the characteristics of the instabilities.

Therefore the concept of σ-stability has been introduced (Goedbloed and Sakana, 1974; Spies, 1974), and a system is said to be σ-stable if no growth faster than $\exp(\sigma t)$ occurs. σ-stability can be investigated by means of a σ-energy principle that says that a system is σ-stable if δW^{σ} is positive for all admissible displacements $\vec{\xi}$, where

$$\delta W^{\sigma} = \delta W + \sigma^2 K \ , \tag{1}$$

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$$K = \frac{1}{2} \int_V \rho |\vec{\xi}|^2 \, dV \ , \tag{2}$$

and σ^2 is a previously fixed positive quantity.

The σ-stability analysis with a σ-energy principle is of the same nature as an analysis of marginal stability by means of the usual energy principle, but the necessary and sufficient conditions for δW^σ to be positive allow one to obtain the maximum growth rate of instability and to define the region of σ-instability, which is the region where the perturbations that grow at least as fast as does $\exp(\sigma t)$ are trapped. Furthermore, the minimization conditions used to arrive at the minimal expression for δW^σ indicate the basic characteristics of the instabilities. Such objectives can, in principle, also be achieved by spectral analysis of the unstable side of the spectrum of normal modes, and, as a matter of fact, such an analysis should, in principle, give complete information about the instabilities. An important difference between σ-stability analysis and normal mode analysis is that, in the latter, the eigenvalue has to be determined, whereas in the former, the value of σ^2 is set beforehand. Moreover the properties of the hydromagnetic instabilities considered here (see § 4) make a normal mode analysis of these instabilities extremely difficult. The principle advantage of σ-stability analysis is that it provides insight into the basic characteristics of the hydromagnetic instabilities in an elegant, analytical way.

In Section 2 we discuss the σ-stability analysis of a star with an axisymmetric toroidal magnetic field. In Section 3 we consider the hydromagnetic instabilities introduced by a weak toroidal magnetic field in a star that is stably stratified in the absence of any magnetic field. In Section 4 we derive the basic characteristics of the instabilities. A more complete description can be found in Goossens (1980), Goossens and Tayler (1980), Goossens et al. (1980).

2. σ-STABILITY ANALYSIS OF A STAR WITH AN AXISYMMETRIC TOROIDAL MAGNETIC FIELD $H_\varphi(r, \theta)$

To obtain a minimal expression for δW^σ, first Fourier analyze the components of the displacement field in a system of spherical polar coordinates (r, θ, φ) as

$$\xi_r = R(r, \theta) \exp(im\varphi) \ , \qquad \xi_\theta = S(r, \theta) \exp(im\varphi) \ ,$$

$$\xi_\varphi = iT(r, \theta) \exp(im\varphi) \ , \tag{3}$$

and then minimize δW^σ with respect to T, and D* to obtain

$$\delta W^\sigma = \pi \int dr \, d\theta \, r^2 \sin\theta \, (a_m^\sigma R^2 + b_m^\sigma RS + c_m^\sigma S^2) \ , \tag{4}$$

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m is the azimuthal wave number and D^* is defined as

$$\operatorname{div} \vec{\xi} = (D^* - mT/r \sin\theta)\exp(im\varphi) \quad . \tag{5}$$

As the integrand in (4) is a quadratic expression in R and S, sufficient conditions for σ-stability are

$$a_m^\sigma > 0 \ , \quad c_m^\sigma > 0 \ , \quad d_m^\sigma = (b_m^\sigma)^2 - 4\, a_m^\sigma c_m^\sigma > 0 \ . \tag{6}$$

These conditions are also necessary (see Goossens and Tayler, 1980). Conditions for stability are

$$a_m > 0 \ , \quad c_m > 0 \ , \quad d_m = b_m^2 - 4\, a_m c_m > 0 \ . \tag{7}$$

Consider now an equilibrium that is dynamically unstable against perturbations with a given wave number m, i.e. if at least one of conditions (7) is violated. The maximum growth rate (M.G.R.) of instability $\sigma^2_{max}(m)$ is the value of σ^2 so that for all $\sigma^2 \geqslant \sigma^2_{max}(m)$, the three conditions (6) are satisfied. The point where $\sigma^2_{max}(m)$ is reached can be considered as the center of the region of instability against perturbations with wave number m, since instabilities with growth rates σ^2 comparable to $\sigma^2_{max}(m)$ are confined to the vicinity of this point. We refer to this point as the point of MGR. The region of instability and of σ-instability are bounded by the stability line and the σ-stability line with equations

$$\min(a_m, c_m, d_m) = 0 \ ; \qquad \min(a_m^\sigma, c_m^\sigma, d_m^\sigma) = 0 \ . \tag{8}$$

3. HYDROMAGNETIC INSTABILITIES INTRODUCED BY A WEAK TOROIDAL MAGNETIC FIELD

When there is no magnetic field, δW^σ and the conditions for σ-stability take the form

$$\delta W^\sigma = \pi \iint dr\, d\theta\, r^2 \sin\theta\, \rho_0\, (N_0^2 + \sigma^2) R^2 \ , \qquad N_0^2 + \sigma^2 \geqslant 0 \ , \tag{9}$$

where the subscript "0" denotes quantities of a spherically symmetric configuration without any magnetic field, and N_0^2 is the square of the Brunt-Väisälä frequency. $N_0^2 > 0$ is the Schwarzschild condition for stability with respect to non-radial oscillations. For a dynamically unstable star ($N_0^2 < 0$ in some region $\alpha \leqslant r \leqslant \beta$), the MGR is

$$\sigma^2_{max} = \max|N_0^2| \qquad \text{for} \quad N_0^2 < 0 \ . \tag{10}$$

and the region of σ-instability is defined by the inequality

$$\sigma^2 \leqslant |N_0^2| \qquad \text{for} \qquad N_0^2 < 0 \ . \tag{11}$$

Hence in a spherically symmetric star, instabilities that grow at least as fast as does $\exp(\sigma t)$ can only have large amplitudes and are trapped in the region defined by Equation (11).

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Consider now a star that is stably stratified in the absence of any magnetic field ($N_0^2 > 0$) and that contains a weak toroidal magnetic field, and let h be a quantity of the order $M/|W| \ll 1$, where M is the magnetic energy and W the gravitational potential energy. We now have

$$a_m^\sigma \approx a_0^\sigma \approx \rho_0 (N_0^2 + \sigma^2) > 0 \ , \tag{12}$$

$$c_0^\sigma \approx 2\rho_0 \tau_H^2 (\cos^2\theta - \frac{1}{H_\varphi} \frac{\partial H_\varphi}{\partial \theta} \sin\theta \ \cos\theta) + \sigma^2 \rho_0 \ . \tag{13}$$

$$c_m^\sigma \approx \rho_0 \ \tau_H^2 (m^2 - 2\cos^2\theta - \frac{2}{H_\varphi} \frac{\partial H_\varphi}{\partial \theta} \sin\theta \ \cos\theta) + \sigma^2\rho_0 + \frac{4\sigma^2 \rho_0 \tau_H^2}{\sigma^2 + m^2 \tau_H^2} \ , \tag{14}$$

$$|b_m^\sigma / a_m^\sigma| = O(h) \ , \qquad |c_m^\sigma / a_m^\sigma| = O(h) \ , \tag{15}$$

and $\tau_H^2 = H_\varphi^2 / (4\pi r^2 \rho_0 \sin^2\theta)$.

Relevant instabilities are now associated with the violation of conditions $c_0 > 0$, $c_m > 0$. The hydromagnetic stability of a star with a weak toroidal magnetic field is not influenced by the changes caused by H_φ in p, ρ, and ϕ; but depends on the angular variation of H_φ. Hydromagnetic instabilities are associated with an unstable stratification of H_φ in the θ-direction. For instance, when $|m| = 1$, instability occurs for $\theta < 45°$ when H_φ^2 increases with $\cos\theta$.

Expressions for the MGR of axisymmetric and non-axisymmetric perturbations are

$$\sigma_{max}^2(0) = \max|c_0|/\rho_0 \qquad \text{for} \quad c_0 < 0 \ , \tag{16}$$

$$\sigma_{max}^2(m) = \frac{1}{2} \max \tau_H^2 \{ (c_m^* - m^2 - 4\cos^2\theta) + \\ + [(c_m^* - m^2 - 4\cos^2\theta)^2 + 4 m^2 c_m^*]^{1/2} \} \ \text{for} \ c_m < 0 \ , \tag{17}$$

where $c_m^* = |m^2 - 2\cos^2\theta - \frac{2}{H_\varphi} \frac{\partial H_\varphi}{\partial \theta}|$.

The σ-stability lines correspond to the level lines of the functions that appear in the right-hand member of Equations (16) and (17). Note also that H_φ and $\rho_0(r)$ suffice to compute MGR and σ-stability lines. Equations (13) and (14) are simple tests for stability and have been used to show that fields such as

$$H_\varphi \sim \rho_0^\beta (r \sin\theta)^{2\beta - 1} \ , \qquad H_\varphi \sim P_\ell^1(\mu) \ , \qquad H_\varphi \sim |P_\ell|^{1/2}$$

are unstable.

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4. CHARACTERISTICS OF THE DISPLACEMENT FIELDS ASSOCIATED WITH THE INSTABILITIES OF THE WEAK FIELD CASE

Take a star that is stably stratified in the absence of any magnetic field and that contains an unstably stratified and weak toroidal magnetic field. Let $c_m^\sigma < 0$ in the region $\mathcal{R}$. Since δW^σ has to be negative the perturbations have to be confined to $\mathcal{R}$ and decay rapidly outside $\mathcal{R}$, and $c_m^\sigma S^2$ has to be the dominant term in the integrand of Equation (4). The latter can be achieved only if

$$|\xi_r/\xi_\theta| = O(\varepsilon) \tag{18}$$

where ε is a small quantity such that $\varepsilon^2 \ll h$, $\varepsilon \ll 1$. The radial component is non-zero but small compared to the θ-component, so that the motion is mainly horizontal and the thermal-buoyancy effect is negligibly small. The hydromagnetic instabilities have only a very minor effect on the radial structure of the star, which is already stably stratified. Further basic characteristics of the instabilities can be derived from the minimization conditions, which, in the weak field approximation, read as

$$\sigma^2 T = 0 \;, \qquad \frac{\partial R}{\partial r} \cong -\frac{1}{r}\frac{\partial S}{\partial \theta} - \frac{S}{r}\cot\theta \quad \text{for } m = 0 \;, \tag{19 ab}$$

$$T \cong \frac{2\, m\, \tau_H^2}{m^2\tau_H^2+\sigma^2}\, S\cos\theta \;, \qquad \frac{\partial R}{\partial r} \cong -\frac{1}{r}\frac{\partial S}{\partial \theta} + \frac{m^2\tau_H^2 - \sigma^2}{m^2\tau_H^2 + \sigma^2}\,\frac{S}{r}\cot\theta \quad \text{for } m \neq 0 \,. \tag{20 ab}$$

From Equations (19 b) and (20 b) it follows that

$$\frac{\lambda_r}{\lambda_\theta} = O(\varepsilon) \tag{21}$$

where λ_r and λ_θ are the wavelengths of the perturbation in the radial direction and in the θ-direction. Equation (21) implies that the unstable perturbations are very narrow in the radial direction. Consider now the φ-component. Equation (19 a) indicates that axisymmetric instabilities have no toroidal component, or that axisymmetric perturbations that have a toroidal component are stable. For non-axisymmetric instabilities, Equation (20 a) shows that there is a simple relation between ξ_θ and ξ_φ. Note that toroidal displacement fields do not satisfy this relation. Furthermore, it can be shown that div $\vec{\xi}$ is non-zero but small both for axisymmetric and non-axisymmetric instabilities.

σ-stability analysis has allowed us to gain insight into the basic properties of hydromagnetic instabilities in stars with weak toroidal magnetic fields. These properties will make normal mode analysis of hydrodynamic instabilities extremely difficult.

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DISCUSSION

A. COX: You made a remark that perhaps went by too fast. You said that in all cases, you get instability?

GOOSSENS: In all cases we considered. The purely toroidal field will probably in almost all cases introduce hydromagnetic instabilities with short e-folding times.

A. COX: And not very much in the radial direction?

GOOSSENS: The star is stably stratified in the radial direction and so the motion must be perpendicular to that direction.

J. COX: Is there any limit to the strength of the magnetic field?

GOOSSENS: Even very small magnetic fields will lead to instabilities with short e-folding times. If the magnetic field gets stronger, then you can do the problem analytically and you get still stronger instability.

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