Bull. Aust. Math. Soc. **110** (2024), 338–345
doi:10.1017/S0004972723001478

ON THE LIMIT SET OF A COMPLEX HYPERBOLIC TRIANGLE GROUP

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(Received 23 October 2023; accepted 13 December 2023; first published online 2 February 2024)

Abstract

Let $\Gamma = \langle I_1, I_2, I_3\rangle$ be the complex hyperbolic $(4, 4, \infty)$ triangle group with $I_1I_3I_2I_3$ being unipotent. We show that the limit set of Γ is connected and the closure of a countable union of $\mathbb{R}$-circles.

2020 *Mathematics subject classification*: primary 22E40; secondary 20H10.

Keywords and phrases: complex hyperbolic triangle groups, limit sets.

1. Introduction

The purpose of this paper is to study the limit set of a discrete complex hyperbolic triangle group.

Recall that a complex hyperbolic (p, q, r) triangle group is a representation ρ of the abstract (p, q, r) reflection triangle group

$$\langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = (\sigma_1\sigma_2)^p = (\sigma_2\sigma_3)^q = (\sigma_3\sigma_1)^r = \mathrm{Id}\rangle$$

into PU(2, 1) such that $I_j = \rho(\sigma_j)$ are complex involutions, where $2 \le p \le q \le r \le \infty$ and $1/p + 1/q + 1/r < 1$.

For a given triple (p, q, r) with $p, q, r \ge 3$, it is a classical fact that there is a 1-parameter family $\{\rho_t : t \in [0, \infty)\}$ of nonconjugate complex hyperbolic (p, q, r) triangle groups (see for example [8]). Here ρ_0 is the embedding of the hyperbolic reflection triangle group, that is, an $\mathbb{R}$-Fuchsian representation (preserving a Lagrangian plane of $\mathbf{H}^2_{\mathbb{C}}$) and so the limit set is an $\mathbb{R}$-circle. In [9], Schwartz conjectured that ρ_t is discrete and faithful if and only if neither $w_A = I_1I_3I_2I_3$ nor $w_B = I_1I_2I_3$ is elliptic. Moreover, ρ_t is discrete and faithful if and only if w_A is nonelliptic when $p < 10$, or w_B is nonelliptic when $p > 13$. For a discrete complex hyperbolic triangle group, it would be interesting to know its limit set.

In [10], Schwartz studied the limit set of the complex hyperbolic $(4, 4, 4)$ triangle group with $(I_1I_2I_1I_3)^7 = \mathrm{Id}$.

This work was partially supported by the NSFC (Grant No. 12271148).

THEOREM 1.1 [10]. *Let $\langle I_1, I_2, I_3\rangle$ be the complex hyperbolic $(4,4,4)$ triangle group with $I_1I_2I_1I_3$ being elliptic of order 7. Let Λ be its limit set and Ω its complement. Then Λ is connected and the closure of a countable union of $\mathbb{R}$-circles in $\partial\mathbf{H}^2_{\mathbb{C}}$. The quotient $\Omega/\langle I_1I_2, I_2I_3\rangle$ is a closed hyperbolic 3-manifold.*

Recently, in [1], Acosta studied the limit set of the complex hyperbolic $(3,3,6)$ triangle group with $I_1I_3I_2I_3$ being unipotent.

THEOREM 1.2 [1]. *Let $\langle I_1, I_2, I_3\rangle$ be the complex hyperbolic $(3,3,6)$ triangle group with $I_1I_3I_2I_3$ being unipotent. Let Λ be its limit set and Ω its complement. Then Λ is connected and the closure of a countable union of $\mathbb{R}$-circles in $\partial\mathbf{H}^2_{\mathbb{C}}$, and contains a Hopf link with three components. The quotient $\Omega/\langle I_1I_2, I_2I_3\rangle$ is the one-cusped hyperbolic 3-manifold m023 in the SnapPy census.*

In this paper, we are interested in describing the limit set of the complex hyperbolic $(4,4,\infty)$ triangle group with $I_1I_3I_2I_3$ being unipotent. The main result is the following theorem.

THEOREM 1.3. *Let Λ be the limit set of the complex hyperbolic $(4,4,\infty)$ triangle group $\langle I_1, I_2, I_3\rangle$ with $I_1I_3I_2I_3$ being unipotent. Then:*

(1) *Λ contains two linked $\mathbb{R}$-circles;*
(2) *Λ is the closure of a countable union of $\mathbb{R}$-circles;*
(3) *Λ is connected.*

However, the quotient of the complement of the limit set has been described as follows.

THEOREM 1.4 [6]. *Let Ω be the discontinuity set of the complex hyperbolic $(4,4,\infty)$ triangle group with $I_1I_3I_2I_3$ being unipotent. Then the quotient $\Omega/\langle I_1I_2, I_2I_3\rangle$ is the two-cusped hyperbolic 3-manifold s782 in the SnapPy census.*

2. Preliminaries

In this section, we briefly recall some basic facts and notation about the complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$. We refer to Goldman's book [5] and Parker's notes [7] for more details.

2.1. The space $\mathbf{H}^2_{\mathbb{C}}$ and its isometries. Let $\mathbb{C}^{2,1}$ denote the three-dimensional complex vector space endowed with a Hermitian form H of signature $(2,1)$. We take H to be the matrix

$$H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The corresponding Hermitian form is given by

$$\langle \mathbf{z}, \mathbf{w}\rangle = z_1\bar{w}_3 + z_2\bar{w}_2 + z_3\bar{w}_1.$$

https://doi.org/10.1017/S0004972723001478 Published online by Cambridge University Press

Here $\mathbf{z} = [z_1, z_2, z_3]^t$ and $\mathbf{w} = [w_1, w_2, w_3]^t$ are column vectors in $\mathbb{C}^{2,1} \setminus \{0\}$. Let $\mathbb{P} : \mathbb{C}^{2,1} \setminus \{0\} \to \mathbb{CP}^2$ be the natural projection map onto complex projective space. Define

$$V_0 = \{\mathbf{z} \in \mathbb{C}^{2,1} \setminus \{0\} : \langle \mathbf{z}, \mathbf{z} \rangle = 0\},$$
$$V_- = \{\mathbf{z} \in \mathbb{C}^{2,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0\},$$
$$V_+ = \{\mathbf{z} \in \mathbb{C}^{2,1} : \langle \mathbf{z}, \mathbf{z} \rangle > 0\}.$$

The complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$ is defined as $\mathbb{P}V_-$ and its boundary $\partial\mathbf{H}^2_{\mathbb{C}}$ is defined as $\mathbb{P}V_0$. We will denote the point at infinity by q_∞. Note that a standard lift of q_∞ is $[1, 0, 0]^t$.

Topologically, the complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$ is homeomorphic to the unit ball of $\mathbb{C}^2$ and its boundary $\partial\mathbf{H}^2_{\mathbb{C}}$ is homeomorphic to the unit 3-sphere S^3. Note that any point $q \neq q_\infty$ of $\mathbf{H}^2_{\mathbb{C}}$ admits a standard lift $\mathbf{q}$ given by

$$\mathbf{q} = \begin{bmatrix} (-|z|^2 - u + it)/2 \\ z \\ 1 \end{bmatrix},$$

where $z \in \mathbb{C}, t \in \mathbb{R}$ and $u > 0$. Let $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$. Then the triple $(z, t, u) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_{\geq 0}$ is called the *horospherical coordinates* of q. Let $\mathcal{N} = \mathbb{C} \times \mathbb{R}$ be the Heisenberg group with group law given by

$$[z_1, t_1] \cdot [z_2, t_2] = [z_1 + z_2, t_1 + t_2 - 2\,\mathrm{Im}(\overline{z_1} z_2)].$$

Then $\partial\mathbf{H}^2_{\mathbb{C}} = \mathcal{N} \cup \{q_\infty\}$.

Let $\mathrm{U}(2, 1)$ be the subgroup of $\mathrm{GL}(3, \mathbb{C})$ preserving the Hermitian form H. Let $\mathrm{SU}(2, 1)$ be the subgroup of $\mathrm{U}(2, 1)$ consisting of unimodular matrices. The full group of holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$ is $\mathrm{PU}(2, 1) = \mathrm{SU}(2, 1)/\{\omega I : \omega^3 = 1\}$, which acts transitively on points of $\mathbf{H}^2_{\mathbb{C}}$ and pairs of distinct points of $\partial\mathbf{H}^2_{\mathbb{C}}$.

An element of $\mathrm{PU}(2, 1)$ is called *elliptic* if it has a fixed point in $\mathbf{H}^2_{\mathbb{C}}$. If an element is not elliptic, then it is called *parabolic* or *loxodromic* if it has exactly one fixed point in $\partial\mathbf{H}^2_{\mathbb{C}}$ or exactly two fixed points in $\partial\mathbf{H}^2_{\mathbb{C}}$, respectively. A parabolic element of $\mathrm{PU}(2, 1)$ is called *unipotent* if it admits a lift to $\mathrm{SU}(2, 1)$ that is unipotent. These terms will also be used for elements of $\mathrm{SU}(2, 1)$.

2.2. Totally geodesic subspaces and related isometries. There is no totally geodesic subspace of real dimension three of $\mathbf{H}^2_{\mathbb{C}}$. Except for the points, geodesics and $\mathbf{H}^2_{\mathbb{C}}$ (they are obviously totally geodesic), there are two kinds of totally geodesic subspaces of real dimension two: complex lines and Lagrangian planes. A *complex line* is the intersection of a projective line in $\mathbb{CP}^2$ with $\mathbf{H}^2_{\mathbb{C}}$. The boundary of a complex line is called a $\mathbb{C}$-*circle*. A *Lagrangian plane* is the intersection of a totally real subspace in $\mathbb{CP}^2$ with $\mathbf{H}^2_{\mathbb{C}}$. The boundary of a Lagrangian plane is called an $\mathbb{R}$-*circle*. In particular, if an $\mathbb{R}$-circle contains $q_\infty = [1, 0, 0]^t$, it is called an *infinite* $\mathbb{R}$-*circle*.

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An elliptic isometry whose fixed point set is a complex line is called a *complex reflection*. The complex reflections we will use in this paper have order 2 and we call them *complex involutions*.

Similarly, every Lagrangian plane is the set of fixed points of an antiholomorphic isometry of order 2, which is called a *real reflection* on the Lagrangian plane.

We will need the following lemma, which is [4, Proposition 3.1].

PROPOSITION 2.1 [4]. *If I_1 and I_2 are reflections on the $\mathbb{R}$-circles $\mathcal{R}_1$ and $\mathcal{R}_2$:*

(i) *$I_1 \circ I_2$ is parabolic if and only if $\mathcal{R}_1$ and $\mathcal{R}_2$ intersect at one point;*
(ii) *$I_1 \circ I_2$ is loxodromic if and only if $\mathcal{R}_1$ and $\mathcal{R}_2$ do not intersect and are not linked;*
(iii) *$I_1 \circ I_2$ is elliptic if and only if $\mathcal{R}_1$ and $\mathcal{R}_2$ are linked or intersect at two points.*

2.3. Limit set. Let Γ be a discrete subgroup of $PU(2,1)$. The *limit set* of Γ is defined as the set of accumulation points of any orbit in $\mathbf{H}^2_{\mathbb{C}}$ under the action of Γ. It is the smallest closed nonempty Γ-invariant subset of $\partial\mathbf{H}^2_{\mathbb{C}}$. The complement of the limit set of Γ in $\partial\mathbf{H}^2_{\mathbb{C}}$ is called the *discontinuity set* of Γ.

3. The group

Let $\omega = -1/2 + i\sqrt{3}/2$ be the primitive cube root of unity. The complex involutions I_1, I_2 and I_3 are given by

$$I_1 = \begin{bmatrix} -1 & 2(1+\omega) & 2 \\ 0 & 1 & -2\omega \\ 0 & 0 & -1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} -1 & 2\omega & 2 \\ 0 & 1 & -2(1+\omega) \\ 0 & 0 & -1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The products I_2I_3 and I_3I_1 are elliptic elements of order 4 and I_2I_1 is unipotent. In fact, $\langle I_1, I_2, I_3\rangle$ is a discrete complex hyperbolic $(4,4,\infty)$ triangle group. Moreover, the element $I_1I_3I_2I_3$ is unipotent.

From Theorem 1.4, one can see that the group $\langle I_1, I_2, I_3\rangle$ is a subgroup of the Eisenstein–Picard modular group $\mathrm{PU}(2,1;\mathbb{Z}[\omega])$ of infinite index and has no fixed point. In [3], Falbel and Parker studied the geometry of the Eisenstein–Picard modular group $\mathrm{PU}(2,1;\mathbb{Z}[\omega])$. Moreover, they obtained a presentation of $\mathrm{PU}(2,1;\mathbb{Z}[\omega])$.

THEOREM 3.1 [3]. *The Eisenstein–Picard modular group* $\mathrm{PU}(2,1;\mathbb{Z}[\omega])$ *has a presentation*

$$\langle P, Q, R \mid R^2 = (QP^{-1})^6 = PQ^{-1}RQP^{-1}R = P^3Q^{-2} = (RP)^3 = 1\rangle,$$

where

$$P = \begin{bmatrix} 1 & 1 & \omega \\ 0 & \omega & -\omega \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 & \omega \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

By using this presentation, the complex involutions I_1, I_2 and I_3 can be expressed as follows.

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PROPOSITION 3.2. *Let $M = PQ^{-1}$ and $T = QM^3$, then:*

- $I_2 = -TQ^4T(PM^2)^{-2}M^3$;
- $I_1 = I_2T^2Q^2$;
- $I_3 = R$.

4. The limit set

LEMMA 4.1. *Let $G_0 = \langle I_1, I_2, I_3I_2I_3\rangle$, as a subgroup of $\langle I_1, I_2, I_3\rangle$. Let $\mathcal{L}_0$ be the Lagrangian plane, whose boundary at infinity is the infinite $\mathbb{R}$-circle given by $\mathcal{R}_0 = \{[x + i\sqrt{3}/2, \sqrt{3}x - \sqrt{3}/2] \in \mathcal{N} : x \in \mathbb{R}\} \cup \{q_\infty\}$. Then G_0 preserves $\mathcal{L}_0$.*

PROOF. Using horospherical coordinates,

$$\mathcal{L}_0 = \{(x + i\sqrt{3}/2, \sqrt{3}x - \sqrt{3}/2, u) \in \mathbf{H}^2_{\mathbb{C}} : x \in \mathbb{R}, u > 0\},$$

and one can compute

$$I_1(x + i\sqrt{3}/2, \sqrt{3}x - \sqrt{3}/2, u) = (-x - 1 + i\sqrt{3}/2, \sqrt{3}(-x - 1) - \sqrt{3}/2, u),$$
$$I_2(x + i\sqrt{3}/2, \sqrt{3}x - \sqrt{3}/2, u) = (-x + 1 + i\sqrt{3}/2, \sqrt{3}(-x + 1) - \sqrt{3}/2, u),$$

$$I_3I_2I_3(x + i\sqrt{3}/2, \sqrt{3}x - \sqrt{3}/2, u)$$
$$= \Big(\frac{(x + 1/2)^2 - 1 + u}{2(x - 1/2)^2 + 2u} + i\frac{\sqrt{3}}{2}, \sqrt{3}\frac{(x + 1/2)^2 - 1 + u}{2(x - 1/2)^2 + 2u} - \frac{\sqrt{3}}{2}, \frac{u}{((x - 1/2)^2 + u)^2}\Big).$$

Thus, $I_1\mathcal{L}_0 = I_2\mathcal{L}_0 = I_3I_2I_3\mathcal{L}_0 = \mathcal{L}_0$. Therefore, the group G_0 preserves the Lagrangian plane $\mathcal{L}_0$ with boundary $\mathcal{R}_0$ at infinity. □

In the same way, we can prove the following result.

LEMMA 4.2. *Let $G_1 = \langle I_1, I_2, I_3I_1I_3\rangle$, as a subgroup of $\langle I_1, I_2, I_3\rangle$. Let $\mathcal{L}_1$ be the Lagrangian plane, whose boundary at infinity is the infinite $\mathbb{R}$-circle given by $\mathcal{R}_1 = \{[x + i\sqrt{3}/2, \sqrt{3}x + \sqrt{3}/2] \in \mathcal{N} : x \in \mathbb{R}\} \cup \{q_\infty\}$. Then G_1 preserves $\mathcal{L}_1$.*

PROPOSITION 4.3. *The limit set of $\langle I_1, I_2, I_3\rangle$ contains an $\mathbb{R}$-circle.*

PROOF. By Lemma 4.1, the subgroup $G_0 = \langle I_1, I_2, I_3I_2I_3\rangle$ is an $\mathbb{R}$-Fuchsian subgroup of $\langle I_1, I_2, I_3\rangle$. Since I_1I_2 and $I_1I_3I_2I_3$ are unipotent and $I_2I_3I_2I_3$ is elliptic of order 2, the restriction $G_0|_{\mathcal{L}_0}$ is a $(2, \infty, \infty)$-reflection triangle group. Thus, the limit set of G_0 is $\partial\mathcal{L}_0 = \mathcal{R}_0$. Therefore, the limit set of $\langle I_1, I_2, I_3\rangle$ contains the $\mathbb{R}$-circle $\mathcal{R}_0$. □

REMARK 4.4. The $(2, \infty, \infty)$-reflection triangle group is a noncompact arithmetic triangle group [11].

Now, let us consider the images of $\mathcal{R}_0$ and $\mathcal{R}_1$ by the group $\langle I_1, I_2, I_3\rangle$. Since $\mathcal{R}_0$ is the limit set of G_0, the image $I_j\mathcal{R}_0$, with $j = 1, 2$, is the limit set of the group $I_jG_0I_j$. One can see that $I_jG_0I_j = G_0$. Thus, $\mathcal{R}_0$ is stabilised by both I_1 and I_2. Similarly, $\mathcal{R}_1$ is stabilised by both I_1 and I_2.

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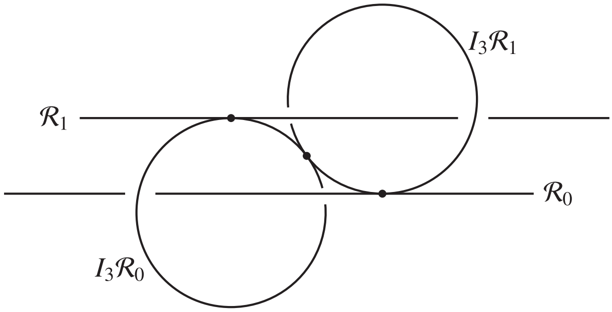

FIGURE 1. A schematic view of the four $\mathbb{R}$-circles. Here $\mathcal{R}_0$ and $\mathcal{R}_1$ are two lines intersecting at infinity.

LEMMA 4.5. *The limit sets $I_3\mathcal{R}_0$ and $\mathcal{R}_0$ are linked and the limit sets $I_3\mathcal{R}_0$ and $\mathcal{R}_1$ intersect at one point.*

PROOF. Since $I_3\mathcal{R}_0$ is the limit set of $I_3G_0I_3 = \langle I_3I_1I_3, I_3I_2I_3, I_2\rangle$, it contains the parabolic fixed point $P_{I_2I_3I_1I_3}$. Therefore, $I_3\mathcal{R}_0 \cap \mathcal{R}_1 = \{P_{I_2I_3I_1I_3}\}$.

Since both $I_3\mathcal{L}_0$ and $\mathcal{L}_0$ contain the elliptic fixed point $P_{I_2I_3I_2I_3} \in I_3\mathcal{L}_0 \cap \mathcal{L}_0$, the product of reflections on the Lagrangian planes $I_3\mathcal{L}_0$ and $\mathcal{L}_0$ is elliptic. Therefore, by Proposition 2.1, the two $\mathbb{R}$-circles $I_3\mathcal{R}_0$ and $\mathcal{R}_0$ must be linked or intersect at two points.

We claim that $I_3\mathcal{R}_0$ and $\mathcal{R}_0$ do not intersect. One can compute that the points of $I_3\mathcal{R}_0$ are given by

$$\left[\frac{8(4x^3-3x+3)}{16x^4+72x^2-48x+21} + i\frac{4\sqrt{3}(12x^2-4x+3)}{16x^4+72x^2-48x+21}, \frac{-32\sqrt{3}(2x-1)}{16x^4+72x^2-48x+21}\right].$$

Suppose that $I_3\mathcal{R}_0 \cap \mathcal{R}_0 \neq \emptyset$, then

$$\frac{8(4x^3-3x+3)}{16x^4+72x^2-48x+21} + i\frac{4\sqrt{3}(12x^2-4x+3)}{16x^4+72x^2-48x+21} = x + i\sqrt{3}/2$$

should have solutions for x. However, this is impossible by a simple computation. Thus, $I_3\mathcal{R}_0 \cap \mathcal{R}_0 = \emptyset$. Therefore, $I_3\mathcal{R}_0$ and $\mathcal{R}_0$ are linked. □

Similarly, we have the following result.

LEMMA 4.6. *The limit sets $I_3\mathcal{R}_1$ and $\mathcal{R}_1$ are linked and the limit sets $I_3\mathcal{R}_1$ and $\mathcal{R}_0$ intersect at one point.*

COROLLARY 4.7. *The union of $\mathcal{R}_i$ and $I_3\mathcal{R}_i$ ($i = 0, 1$) is connected.*

PROOF. Since $\mathcal{R}_0$ and $\mathcal{R}_1$ are infinite $\mathbb{R}$-circles, we obtain $\mathcal{R}_0 \cap \mathcal{R}_1 = \{q_\infty\}$. From Lemmas 4.5 and 4.6, $I_3\mathcal{R}_0 \cap \mathcal{R}_1 = \{P_{I_2I_3I_1I_3}\}$ and $I_3\mathcal{R}_1 \cap \mathcal{R}_0 = \{P_{I_1I_3I_2I_3}\}$. It is obvious that $I_3\mathcal{R}_0 \cap I_3\mathcal{R}_1 = \{I_3q_\infty\} = [0, 0] \in \mathcal{N}$. Now, there is a path in $\mathcal{R}_0 \cup \mathcal{R}_1 \cup I_3\mathcal{R}_0 \cup I_3\mathcal{R}_1$ between any two points in it. Thus, the union is connected. See Figure 1. □

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PROOF OF THEOREM 1.3. (1) This is a consequence of Lemmas 4.5 or 4.6.

(2) From Proposition 4.3, the limit set Λ contains an $\mathbb{R}$-circle. Then the Γ-orbit of the $\mathbb{R}$-circle is contained in Λ. Since Λ is the smallest closed nonempty invariant subset of $\partial\mathbf{H}^2_{\mathbb{C}}$ under the action of Γ, it is the closure of the Γ-orbit of the $\mathbb{R}$-circle. Thus, Λ is the closure of a countable union of $\mathbb{R}$-circles.

(3) Let n be a positive integer and $\gamma = \gamma_1\gamma_2\cdots\gamma_n \in \Gamma$, where $\gamma_i \in \{I_1, I_2, I_3\}$ for $i = 1, \ldots, n$. Let $\mathcal{U}_0 = \mathcal{R}_0 \cup \mathcal{R}_1$ and $\mathcal{U}_i = \gamma_1\cdots\gamma_i\mathcal{U}_0$. Since $\mathcal{R}_0 \cap \mathcal{R}_1 = \{q_\infty\}$, the subset $\mathcal{U}_i$ of Λ is connected for $i = 0, 1, \ldots, n$. For $i \in \{0, 1, \ldots, n-1\}$, we see that

$$\mathcal{U}_i \cap \mathcal{U}_{i+1} = \gamma_1\cdots\gamma_i\mathcal{U}_0 \cap \gamma_1\cdots\gamma_{i+1}\mathcal{U}_0 = \gamma_1\cdots\gamma_i(\mathcal{U}_0 \cap \gamma_{i+1}\mathcal{U}_0).$$

By Lemmas 4.5 and 4.6, $\mathcal{U}_0 \cap \gamma_{i+1}\mathcal{U}_0 \neq \emptyset$, so $\mathcal{U}_i \cap \mathcal{U}_{i+1} \neq \emptyset$. Thus, there is a path in Λ from q_∞ to γq_∞. From item (2), Λ is the closure of the Γ-orbit of an $\mathbb{R}$-circle. Hence, Λ is connected. □

REMARK 4.8. We note that Λ is not *slim* (see [2] for the definition). In other words, there are three distinct points of Λ lying in the same $\mathbb{C}$-circle.

Acknowledgement

We would like to thank the referee for comments which improved a previous version of this paper.

References

[1] M. Acosta, 'On the limit set of a spherical CR uniformization', *Algebr. Geom. Topol.* **22** (2022), 3305–3325.

[2] E. Falbel, A. Guilloux and P. Will, 'Slim curves, limit sets and spherical CR uniformisations', Preprint, 2022, arXiv:2205.08797.

[3] E. Falbel and J. R. Parker, 'The geometry of the Eisenstein–Picard modular group', *Duke Math. J.* **131** (2006), 249–289.

[4] E. Falbel and V. Zocca, 'A Poincaré's polyhedron theorem for complex hyperbolic geometry', *J. reine angew. Math.* **516** (1999), 133–158.

[5] W. M. Goldman, *Complex Hyperbolic Geometry*, Oxford Mathematical Monographs (Oxford University Press, Oxford, 1999).

[6] Y. Jiang, J. Wang and B. Xie, 'A uniformizable spherical CR structure on a two-cusped hyperbolic 3-manifold', *Algebr. Geom. Topol.* **23** (2023), 4143–4184.

[7] J. R. Parker, *Notes on Complex Hyperbolic Geometry* (University of Durham, Durham, 2003). https://maths.dur.ac.uk/users/j.r.parker/img/NCHG.pdf.

[8] A. Pratoussevitch, 'Traces in complex hyperbolic triangle groups', *Geom. Dedicata* **111** (2005), 159–185.

[9] R. E. Schwartz, 'Complex hyperbolic triangle groups', in: *Proceedings of the International Congress of Mathematicians, Beijing, 2002, Volume II: Invited Lectures* (ed. T. Li) (Higher Education Press, Beijing, 2002), 339–350.

[10] R. E. Schwartz, 'Real hyperbolic on the outside, complex hyperbolic on the inside', *Invent. Math.* **151** (2003), 221–295.

[11] K. Takeuchi, 'Arithmetic triangle groups', *J. Math. Soc. Japan* **29** (1977), 91–106.

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