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CENTRAL QUOTIENTS AND COVERINGS OF STEINBERG UNITARY LIE ALGEBRAS

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ABSTRACT. In this paper, we calculate the center and the universal covering algebra of the Steinberg unitary Lie algebra $\mathrm{stu}_n(\mathcal{A}, -, \gamma)$, where $(\mathcal{A}, -)$ is a unital nonassociative algebra with involution and $n \geq 3$.

0. **Introduction.** Let k be a field of characteristic $\neq 2$ or 3. Suppose that $n \geq 3$ and γ is an $n \times n$-diagonal matrix over k with nonzero diagonal entries $\gamma_1, \ldots, \gamma_n$. If $(\mathcal{A}, -)$ is an arbitrary unital nonassociative algebra with involution over k, the *Steinberg unitary Lie algebra* is the Lie algebra $\mathrm{stu}_n(\mathcal{A}, -, \gamma)$ over k generated by the symbols $u_{ij}(a)$, $a \in \mathcal{A}$, $1 \leq i \neq j \leq n$, subject to the relations:

(stu1) $$u_{ij}(a) = u_{ji}(-\gamma_i\gamma_j^{-1}\bar{a}),$$

(stu2) $$a \longmapsto u_{ij}(a) \text{ is a } k\text{-linear mapping},$$

(stu3) $$[u_{ij}(a), u_{jk}(b)] = u_{ik}(ab), \quad \text{for distinct } i, j, k,$$

(stu4) $$[u_{ij}(a), u_{kl}(b)] = 0, \quad \text{for distinct } i, j, k, l.$$

where $a, b \in \mathcal{A}$, $1 \leq i, j, k, l \leq n$ [AF].

Recall that if $\mathcal{G}$ is a perfect Lie algebra, a *central quotient* (respectively a *covering*) of $\mathcal{G}$ is a pair $(\mathcal{F}, \pi)$ where $\mathcal{F}$ is a perfect Lie algebra and π is an epimorphism of $\mathcal{G}$ onto $\mathcal{F}$ (respectively $\mathcal{F}$ onto $\mathcal{G}$) so that the kernel of π is contained in the center of $\mathcal{G}$ (respectively $\mathcal{F}$). In this paper we consider the following two basic problems regarding the structure of the Lie algebra $\mathrm{stu}_n(\mathcal{A}, -, \gamma)$:

- The description of the central quotients of $\mathrm{stu}_n(\mathcal{A}, -, \gamma)$, and
- The description of the coverings of $\mathrm{stu}_n(\mathcal{A}, -, \gamma)$.

Now by the fundamental homomorphism theorem, the first problem is equivalent to the calculation of the center of $\mathrm{stu}_n(\mathcal{A}, -, \gamma)$. On the other hand, since $\mathrm{stu}_n(\mathcal{A}, -, \gamma)$ is perfect, there is a universal covering of $\mathrm{stu}_n(\mathcal{A}, -, \gamma)$ which factors through any covering of $\mathrm{stu}_n(\mathcal{A}, -, \gamma)$ [Ga]. Also, the center of this universal covering is the second homology group $\mathrm{H}_2\big(\mathrm{stu}_n(\mathcal{A}, -, \gamma)\big)$ of $\mathrm{stu}_n(\mathcal{A}, -, \gamma)$. Thus the second problem is equivalent to the calculation of the space $\mathrm{H}_2\big(\mathrm{stu}_n(\mathcal{A}, -, \gamma)\big)$.

Let $\mathcal{J}$ be the kernel of the map $a \longrightarrow u_{ij}(a)$ from $\mathcal{A}$ to $\mathrm{stu}_n(\mathcal{A}, -, \gamma)$. Then $\mathcal{J}$ is independent of the choice of $i \neq j$ [AF, Lemma 2.1], and we say that $(\mathcal{A}, -)$ is *n-faithful* if

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$\mathcal{J} = (0)$. In [AF, Lemma 2.1], it is shown that the quotient algebra $(\mathcal{B}, -) = (\mathcal{A}, -)/\mathcal{J}$ is n-faithful and that the canonical map $(\mathcal{A}, -) \longrightarrow (\mathcal{B}, -)$ induces an isomorphism of $\mathrm{stu}_n(\mathcal{A}, -, \gamma)$ onto $\mathrm{stu}_n(\mathcal{B}, -, \gamma)$. These facts reduce our problems to the case when the algebra $(\mathcal{A}, -)$ is n-faithful.

Now if $(\mathcal{A}, -)$ is associative, then $\mathrm{stu}_n(\mathcal{A}, -, \gamma)$ is a covering of the elementary unitary Lie algebra $\mathrm{eu}_n(\mathcal{A}, -, \gamma)$ generated by the matrices $ae_{ij} - \gamma_i\gamma_j^{-1}\bar{a}e_{ji}$, $a \in \mathcal{A}$, $1 \leq i \neq j \leq n$. Thus, if $(\mathcal{A}, -)$ is associative then $(\mathcal{A}, -)$ is n-faithful. Moreover, if $n \geq 4$ the converse is true [AF, Lemma 2.2]. That is, if $n \geq 4$, then

$$(\mathcal{A}, -) \text{ is } n\text{-faithful} \iff (\mathcal{A}, -) \text{ is associative.}$$

The case $n = 3$ allows a larger class of coordinate algebras. Indeed, it is shown in [AF, Section 5] that

$$(\mathcal{A}, -) \text{ is 3-faithful} \iff (\mathcal{A}, -) \text{ is structurable.}$$

Recall that $(\mathcal{A}, -)$ is said to be *structurable* if it satisfies the operator identity

$$[V_{a,b}, V_{c,d}] = V_{\{a,b,c\},d} - V_{c,\{b,a,d\}},$$

where $V_{a,b} \in \mathrm{End}(\mathcal{A})$ is defined by $V_{a,b}c := \{a, b, c\} := (a\bar{b})c + (c\bar{b})a - (c\bar{a})b$. There are a number of nonassociative algebras with involution satisfying this operator identity—for example alternative algebras with involution—and an extensive theory of structurable algebras has been developed (see for example [K], [A1], [Sh], [Sm], [AF] and [F]). In any case our problems are reduced to the cases when $(\mathcal{A}, -)$ is associative and $n \geq 4$ or $(\mathcal{A}, -)$ is structurable and $n = 3$.

In this paper, we prove two main results regarding the two problems. For the first problem, we prove:

THEOREM A. *Suppose that* $(\mathcal{A}, -)$ *is n-faithful and* $\mathrm{char}(k) \nmid n$. *Then,*

$$\mathrm{center}\big(\mathrm{stu}_n(\mathcal{A}, -, \gamma)\big) \cong \mathrm{HF}(\mathcal{A}, -).$$

The space $\mathrm{HF}(\mathcal{A}, -)$, which we call the *full skew-dihedral homology group* of $(\mathcal{A}, -)$, is defined precisely in Section 1 below. It is a subspace of the vector space $\langle \mathcal{A}, \mathcal{A} \rangle$ generated by the symbols $\langle a, b \rangle$, $a, b \in \mathcal{A}$, subject to the relations that $\langle \cdot, \cdot \rangle$ is bilinear and skew-symmetric and

$$\langle a, b \rangle = \langle \bar{a}, \bar{b} \rangle \quad \text{and} \quad \langle ab, c \rangle + \langle bc, a \rangle + \langle ca, b \rangle = 0.$$

In the case when $(\mathcal{A}, -)$ is associative, $\mathrm{HF}(\mathcal{A}, -)$ contains as a subspace the first skew-dihedral homology group ${}_{-1}\mathrm{HD}_1(\mathcal{A}, -)$ studied in [L], [KLS] and [G]. We note that Theorem A is false if $\mathrm{char}(k) \mid n$, which we prove by calculating the center of $\mathrm{stu}_n(\mathcal{A}, -, \gamma)$ in that case as well.

Regarding the second problem, we prove

THEOREM B. *Suppose that* $(\mathcal{A}, -)$ *is n-faithful. Then,*

$$\mathrm{H}_2\big(\mathrm{stu}_n(\mathcal{A}, -, \gamma)\big) \cong \begin{cases} (0) & \text{if } n \neq 4, \\ \mathrm{L}(\mathcal{A}, -) & \text{if } n = 4. \end{cases}$$

Here $\mathrm{L}(\mathcal{A}, -)$ is the vector space generated by the symbols $\ell(a,b)$, $a, b \in \mathcal{A}$, subject to the relations that $\ell(\cdot,\cdot)$ is bilinear and skew-symmetric and

$$\ell(\bar{a}b, c) + \ell(\bar{b}c, a) + \ell(\bar{c}a, b) = 0 \quad \text{and} \quad \ell\big(c(ab - \overline{ab}) + (ba - \overline{ba})c, d\big) = 0.$$

In the proof of Theorem B we consider only the cases $n = 3$ and $n = 4$ since the theorem was proved in the case $n \geq 5$ in [G, Theorem 2.37]. The theorem says that if $n \neq 4$ then $\mathrm{stu}_n(\mathcal{A}, -, \gamma)$ is centrally closed. If $n = 4$ we prove the theorem by explicitly calculating the universal covering of $\mathrm{stu}_4(\mathcal{A}, -, \gamma)$. It is interesting to note that the three cases of the theorem ($n = 3, n = 4$ and $n \geq 5$), are proved using three quite different methods.

The organization of the paper is as follows. In Section 1, we introduce the space $\mathrm{HF}(\mathcal{A}, -)$ and obtain its basic properties. In Section 2, we introduce some elements $t(a,b)$ and $h(a,b)$, $a, b \in \mathcal{A}$, of $\mathrm{stu}_n(\mathcal{A}, -, \gamma)$ which play a key role in our discussion. In Sections 3 and 4, we prove Theorem A. Finally in the last two sections, we prove Theorem B. We conclude the final section by using our results to calculate $\mathrm{H}_2\big(\mathrm{peu}_n(\mathcal{A}, -, \gamma)\big)$ when $(\mathcal{A}, -)$ is associative.

Throughout the paper we will assume that k is a field of characteristic not 2 or 3, $n \geq 3$, and $\gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$, where $\gamma_1, \ldots, \gamma_n \neq 0 \in k$. All algebras, except Lie algebras, are assumed to be unital.

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1. **Skew-dihedral homology for structurable algebras.** Suppose in this section that $(\mathcal{A}, -)$ is a structurable k-algebra. We will introduce the full skew-dihedral homology group $\mathrm{HF}(\mathcal{A}, -)$ of $(\mathcal{A}, -)$.

Recall that a *derivation* of $(\mathcal{A}, -)$ is a derivation of $\mathcal{A}$ which commutes with $-$. For $a, b \in \mathcal{A}$, we define $D_{a,b} \in \mathrm{End}(\mathcal{A})$ by

$$D_{a,b}c := \frac{1}{3}\big[[a,b] + [\bar{a},\bar{b}], c\big] + (c,b,a) - (c,\bar{a},\bar{b}), \tag{1.1}$$

where $[a,b] := ab - ba$ and $(a,b,c) := (ab)c - a(bc)$, for $a, b, c \in \mathcal{A}$. Then $D_{a,b} \in \mathrm{Der}(\mathcal{A}, -)$, where $\mathrm{Der}(\mathcal{A}, -)$ is the Lie algebra of derivations of $(\mathcal{A}, -)$ [A1, Corollary 9]. We call any linear combination of derivations of the form $D_{a,b}$ an *inner derivation*. The inner derivations satisfy the identities:

$$D_{a,b} + D_{b,a} = 0, \tag{1.2}$$

$$D_{a,b} - D_{\bar{a},\bar{b}} = 0, \tag{1.3}$$

$$D_{ab,c} + D_{bc,a} + D_{ca,b} = 0, \tag{1.4}$$

$$[D, D_{a,b}] = D_{Da,b} + D_{a,Db}, \tag{1.5}$$

for any $a, b, c \in \mathcal{A}$, and any $D \in \mathrm{Der}(\mathcal{A}, -)$ [A1, Section 3]. Thus the space spanned by the $D_{a,b}$'s, denoted by $\mathrm{Inder}(\mathcal{A}, -)$ or $D_{\mathcal{A},\mathcal{A}}$, is an ideal of $\mathrm{Der}(\mathcal{A}, -)$.

Write $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ where

$$\mathcal{A}_+ = \{a \in \mathcal{A} \mid \bar{a} = a\}, \quad \mathcal{A}_- = \{a \in \mathcal{A} \mid \bar{a} = -a\}.$$

The following fact will be used later.

LEMMA 1.6. *For any $a, b, c \in \mathcal{A}$, one has*

$$(c, b, a) - (c, \bar{a}, \bar{b}) = -(c, a, b) + (c, \bar{b}, \bar{a}) \tag{1.7}$$

$$(c, b, a) - (c, \bar{a}, \bar{b}) = (b, a, c) - (\bar{a}, \bar{b}, c) \tag{1.8}$$

PROOF. By [A1, Proposition 1], we have

$$(s, a, b) = -(a, s, b) = (a, b, s) \tag{1.9}$$

for $a, b \in \mathcal{A}$, $s \in \mathcal{A}_-$; and

$$(d, e, f) - (f, d, e) = (e, d, f) - (f, e, d) \tag{1.10}$$

for $d, e, f \in \mathcal{A}_+$. But, by (1.9), it follows that (1.10) also holds if $f \in \mathcal{A}_-$. Hence,

$$(d, e, x) - (x, d, e) = (e, d, x) - (x, e, d) \tag{1.11}$$

for $x \in \mathcal{A}$ and $d, e \in \mathcal{A}_+$.

To prove (1.7) and (1.8), we may assume that $a, b \in \mathcal{A}_+ \cup \mathcal{A}_-$. If $a, b \in \mathcal{A}_+$, then (1.7) is trivial and (1.8) follows from (1.11). If $a \in \mathcal{A}_+$ and $b \in \mathcal{A}_-$ or $a \in \mathcal{A}_-$ and $b \in \mathcal{A}_+$, then (1.7) and (1.8) follow from (1.9). Finally, if $a, b \in \mathcal{A}_-$, then (1.7) is trivial and (1.8) follows from (1.9). ∎

We can now introduce the full skew-dihedral homology group of $(\mathcal{A}, -)$. Let I be the k-subspace of $\mathcal{A} \otimes_k \mathcal{A}$ spanned by the elements

$$a \otimes b + b \otimes a, \quad a \otimes b - \bar{a} \otimes \bar{b}, \quad ab \otimes c + bc \otimes a + ca \otimes b, \tag{1.12}$$

for all $a, b, c \in \mathcal{A}$. Then we can form the factor space

$$\langle \mathcal{A}, \mathcal{A} \rangle = (\mathcal{A} \otimes_k \mathcal{A})/I. \tag{1.13}$$

For notation we let $\langle a, b \rangle$ denote the coset $a \otimes b + I$ of $\langle \mathcal{A}, \mathcal{A} \rangle$. Then it follows that

$$\langle a, b \rangle + \langle b, a \rangle = 0, \text{ and} \tag{1.14}$$

$$\langle a, b \rangle - \langle \bar{a}, \bar{b} \rangle = 0, \text{ and} \tag{1.15}$$

$$\langle ab, c \rangle + \langle bc, a \rangle + \langle ca, b \rangle = 0, \tag{1.16}$$

for all $a, b, c \in \mathcal{A}$. Also, $\langle \cdot, \cdot \rangle$ is k-bilinear. Because of (1.2)–(1.4) there exists a surjective linear map $\rho: \langle \mathcal{A}, \mathcal{A} \rangle \longrightarrow \mathrm{Inder}(\mathcal{A}, -)$ such that

$$\rho(\langle a, b \rangle) = D_{a,b}.$$

The *full skew-dihedral homology group* of $(\mathcal{A}, -)$ is the subspace $\mathrm{HF}(\mathcal{A}, -)$ of $\langle \mathcal{A}, \mathcal{A} \rangle$ defined by

$$\mathrm{HF}(\mathcal{A}, -) = \ker \rho = \left\{ \sum \langle a_i, b_i \rangle \in \langle \mathcal{A}, \mathcal{A} \rangle \;\middle|\; \sum D_{a_i, b_i} = 0 \right\}.$$

EXAMPLE 1.17. Suppose that $(\mathcal{A}, -)$ is associative. Recall that the *first skew-dihedral homology group* ${}_{-1}\mathrm{HD}_1(\mathcal{A}, -)$ of $(\mathcal{A}, -)$ is the space defined by

$${}_{-1}\mathrm{HD}_1(\mathcal{A}, -) = \left\{ \sum \langle a_i, b_i \rangle \in \langle \mathcal{A}, \mathcal{A} \rangle \;\middle|\; \sum([a_i, b_i] + [\bar{a}_i, \bar{b}_i]) = 0 \right\}.$$

(See [L], [KLS] and [G].) Then, ${}_{-1}\mathrm{HD}_1(\mathcal{A}, -)$ is a subspace of $\mathrm{HF}(\mathcal{A}, -)$. (This is the reason for our use of the term full skew-dihedral homology group for $\mathrm{HF}(\mathcal{A}, -)$.) Furthermore, the elements $[a, b] + [\bar{a}, \bar{b}]$, $a, b \in \mathcal{A}$, satisfy the relations (1.14)–(1.16). Hence, there exists a unique linear map χ from $\langle \mathcal{A}, \mathcal{A} \rangle$ to $\mathcal{A}_- \cap [\mathcal{A}, \mathcal{A}]$ so that $\chi(\langle a, b \rangle) = [a, b] + [\bar{a}, \bar{b}]$. This map is surjective, and since

$$\mathrm{HF}(\mathcal{A}, -) = \left\{ \sum \langle a_i, b_i \rangle \in \langle \mathcal{A}, \mathcal{A} \rangle \;\middle|\; \sum([a_i, b_i] + [\bar{a}_i, \bar{b}_i]) \in \mathrm{center}(\mathcal{A}) \right\},$$

we have $\mathrm{HF}(\mathcal{A}, -) = \chi^{-1}\big(\mathrm{center}(\mathcal{A}) \cap \mathcal{A}_- \cap [\mathcal{A}, \mathcal{A}]\big)$. Thus, χ restricts to a linear homomorphism of $\mathrm{HF}(\mathcal{A}, -)$ onto $\mathrm{center}(\mathcal{A}) \cap \mathcal{A}_- \cap [\mathcal{A}, \mathcal{A}]$ with kernel ${}_{-1}\mathrm{HD}_1(\mathcal{A}, -)$. Hence, as vector spaces, we have

$$\mathrm{HF}(\mathcal{A}, -) / {}_{-1}\mathrm{HD}_1(\mathcal{A}, -) \cong \mathrm{center}(\mathcal{A}) \cap \mathcal{A}_- \cap [\mathcal{A}, \mathcal{A}],$$

and so

$$\mathrm{HF}(\mathcal{A}, -) \cong {}_{-1}\mathrm{HD}_1(\mathcal{A}, -) \oplus \mathrm{center}(\mathcal{A}) \cap \mathcal{A}_- \cap [\mathcal{A}, \mathcal{A}].$$

EXAMPLE 1.18. Suppose that $(\mathcal{A}, -)$ is associative and commutative. The space of $\Omega^1_{\mathcal{A}}/d\mathcal{A}$ of *Kähler differentials modulo exact forms* is defined to be the vector space $\mathcal{A} \otimes \mathcal{A}$ modulo the space spanned by the elements of the form $a \otimes b + b \otimes a$ and $ab \otimes c + bc \otimes a + ca \otimes b$, $a, b, c \in \mathcal{A}$ [KL]. Thus, $\langle \mathcal{A}, \mathcal{A} \rangle$ is a quotient space of $\Omega^1_{\mathcal{A}}/d\mathcal{A}$ and

$${}_{-1}\mathrm{HD}_1(\mathcal{A}, -) = \mathrm{HF}(\mathcal{A}, -) = \langle \mathcal{A}, \mathcal{A} \rangle.$$

In particular, if $-$ is the identity map, then these three spaces are each equal to $\Omega^1_{\mathcal{A}}/d\mathcal{A}$.

Returning to the general case of a structurable algebra, we next want to define a multiplication on $\langle \mathcal{A}, \mathcal{A} \rangle$ by

$$[\langle a, b \rangle, \langle c, d \rangle] = \langle D_{a,b}c, d \rangle + \langle c, D_{a,b}d \rangle. \tag{1.19}$$

To see that this multiplication is well-defined we need:

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LEMMA 1.20. *For any $a, b, c, d \in \mathcal{A}$,*

$$\langle (b,a,c),d\rangle - \langle (d,b,a),c\rangle + \langle (c,d,b),a\rangle - \langle (a,c,d),b\rangle = 0 \tag{1.21}$$

$$\langle D_{a,b}c,d\rangle + \langle c,D_{a,b}d\rangle + \langle D_{c,d}a,b\rangle + \langle a,D_{c,d}b\rangle = 0. \tag{1.22}$$

PROOF. From (1.14) and (1.16), the left hand side of (1.21) is

$$\begin{aligned}&\langle (ba)c - b(ac),d\rangle - \langle (db)a - d(ba),c\rangle + \langle (cd)b - c(db),a\rangle - \langle (ac)d - a(cd),b\rangle\\ &\quad = \big(\langle (ba)c,d\rangle + \langle d(ba),c\rangle\big) - \big(\langle b(ac),d\rangle + \langle (ac)d,b\rangle\big) - \big(\langle (db)a,c\rangle\\ &\qquad + \langle c(db),a\rangle\big) + \big(\langle a(cd),b\rangle + \langle (cd)b,a\rangle\big)\\ &\quad = -\langle cd,ba\rangle + \langle db,ac\rangle + \langle ac,db\rangle - \langle ba,cd\rangle = 0.\end{aligned}$$

So (1.21) holds.

Let $D'_{a,b}c := \big[[a,b],c\big] + 3(c,b,a)$ in which case $D_{a,b} = \frac{1}{3}(D'_{a,b} - D'_{\bar b,\bar a})$. Now, by (1.14) and (1.16), we obtain

(1.23)

$$\begin{aligned}\langle D'_{a,b}c,d\rangle + \langle c,D'_{a,b}d\rangle &= \langle [a,b]c,d\rangle - \langle c[a,b],d\rangle + \langle c,[a,b]d\rangle\\ &\qquad - \langle c,d[a,b]\rangle + 3\langle (c,b,a),d\rangle + 3\langle c,(d,a,b)\rangle\\ &= (\langle [a,b]c,d\rangle - \langle c,d[a,b]\rangle) - (\langle c[a,b],d\rangle - \langle c,[a,b]d\rangle)\\ &\qquad + 3\langle (c,b,a),d\rangle + 3\langle c,(d,a,b)\rangle\\ &= -\langle cd,[a,b]\rangle + \langle dc,[a,b]\rangle + 3\langle (c,b,a),d\rangle\\ &\qquad + 3\langle c,(d,b,a)\rangle\\ &= -\langle [c,d],[a,b]\rangle + 3\langle (c,b,a),d\rangle + 3\langle c,(d,b,a)\rangle.\end{aligned}$$

Thus,

$$\langle D'_{\bar b,\bar a}c,d\rangle + \langle c,D'_{\bar b,\bar a}d\rangle = -\langle [c,d],[\bar b,\bar a]\rangle + 3\langle (c,\bar a,\bar b),d\rangle + 3\langle c,(d,\bar a,\bar b)\rangle. \tag{1.24}$$

Combining (1.23) and (1.24) gives us

$$\begin{aligned}\langle D_{a,b}c,d\rangle + \langle c,D_{a,b}d\rangle = -\frac{1}{3}\langle [c,d],[a,b] + [\bar a,\bar b]\rangle &+ \langle (c,b,a) - (c,\bar a,\bar b),d\rangle\\ &+ \langle c,(d,b,a) - (d,\bar a,\bar b)\rangle.\end{aligned}$$

So the left hand side of (1.22) is

$$\begin{aligned}\langle D_{a,b}c,d\rangle + \langle c,D_{a,b}d\rangle + \langle D_{c,d}a,b\rangle + \langle a,D_{c,d}b\rangle &= -\frac{1}{3}\langle [c,d],[a,b] + [\bar a,\bar b]\rangle\\ + \langle (c,b,a) - (c,\bar a,\bar b),d\rangle + \langle c,(d,b,a) - (d,\bar a,\bar b)\rangle &- \frac{1}{3}\langle [a,b],[c,d]\\ + [\bar c,\bar d]\rangle + \langle (a,d,c) - (a,\bar c,\bar d),b\rangle + \langle a,(b,d,c) - (b,\bar c,\bar d)\rangle.\end{aligned}$$

But from (1.15), one has

$$\langle [a,b],[c,d] + [\bar c,\bar d]\rangle = \langle \overline{[a,b]},\overline{[c,d]} + \overline{[\bar c,\bar d]}\rangle = \langle [\bar a,\bar b],[\bar c,\bar d] + [c,d]\rangle.$$

This yields

$$\langle [a,b],[c,d]+[\bar c,\bar d]\rangle = \frac{1}{2}\langle [a,b]+[\bar a,\bar b],[c,d]+[\bar c,\bar d]\rangle$$

and

$$\langle [c,d],[a,b]+[\bar a,\bar b]\rangle = \frac{1}{2}\langle [c,d]+[\bar c,\bar d],[a,b]+[\bar a,\bar b]\rangle.$$

Therefore the left hand side of (1.22) becomes

$$\langle (c,b,a)-(c,\bar a,\bar b),d\rangle + \langle c,(d,b,a)-(d,\bar a,\bar b)\rangle \\ + \langle (a,d,c)-(a,\bar c,\bar d),b\rangle + \langle a,(b,d,c)-(b,\bar c,\bar d)\rangle,$$

which by Lemma 1.6 equals

$$\langle (b,a,c)-(\bar a,\bar b,c),d\rangle + \langle c,(d,b,a)-(d,\bar a,\bar b)\rangle + \langle -(a,c,d)+(a,\bar d,\bar c),b\rangle \\ + \langle a,-(c,d,b)+(\bar d,\bar c,b)\rangle = \langle (b,a,c),d\rangle - \langle (d,b,a),c\rangle \\ - \langle (a,c,d),b\rangle + \langle (c,d,b),a\rangle - \langle (\bar a,\bar b,c),d\rangle + \langle (d,\bar a,\bar b),c\rangle \\ + \langle (a,\bar d,\bar c),b\rangle - \langle (\bar d,\bar c,b),a\rangle.$$

Now since

$$\langle (a,\bar d,\bar c),b\rangle = \langle -\overline{(c,d,\bar a)},b\rangle = -\langle (c,d,\bar a),\bar b\rangle$$

and similarly $\langle (\bar d,\bar c,b),a\rangle = -\langle (\bar b,c,d),\bar a\rangle$, this equals

$$\big(\langle (b,a,c),d\rangle - \langle (d,b,a),c\rangle - \langle (a,c,d),b\rangle + \langle (c,d,b),a\rangle\big) \\ -\big(\langle (\bar a,\bar b,c),d\rangle + \langle (d,\bar a,\bar b),c\rangle - \langle (c,d,\bar a),\bar b\rangle) + \langle (\bar b,c,d),\bar a\rangle\big)$$

which is 0 by (1.21). This completes the proof. ∎

From (1.22), we see that the multiplication on $\langle \mathcal{A},\mathcal{A}\rangle$ defined by (1.19) is well-defined and anti-commutative. Next we claim the linear space $\langle \mathcal{A},\mathcal{A}\rangle$ with the multiplication defined as above is a Lie algebra.

PROPOSITION 1.25. *The linear space $\langle \mathcal{A},\mathcal{A}\rangle$ is a Lie algebra under (1.19). Moreover the map ρ is a Lie algebra epimorphism and hence* $\mathrm{HF}(\mathcal{A},-)$ *is a central ideal of $\langle \mathcal{A},\mathcal{A}\rangle$.*

PROOF. The second statement follows from the first statement and (1.5). To prove the first statement we only need to check the Jacobi identity. Let $a,b,c,d,e,f\in\mathcal{A}$. Then

$$\big[[\langle a,b\rangle,\langle c,d\rangle],\langle e,f\rangle\big] + \big[[\langle c,d\rangle,\langle e,f\rangle],\langle a,b\rangle\big] + \big[[\langle e,f\rangle,\langle a,b\rangle],\langle c,d\rangle\big] \\ = \big[[\langle a,b\rangle,\langle c,d\rangle],\langle e,f\rangle\big] - \big[\langle a,b\rangle,[\langle c,d\rangle,\langle e,f\rangle]\big] + \big[\langle c,d\rangle,[\langle a,b\rangle,\langle e,f\rangle]\big].$$

By definition this equals

$$[\langle D_{a,b}c,d\rangle + \langle c,D_{a,b}d\rangle,\langle e,f\rangle] - [\langle a,b\rangle,\langle D_{c,d}e,f\rangle + \langle e,D_{c,d}f\rangle] \\ + [\langle c,d\rangle,\langle D_{a,b}e,f\rangle + \langle e,D_{a,b}f\rangle].$$

Using (1.5) and our definition of the bracket again this becomes

$$\begin{aligned}\langle D_{D_{a,b}c,d}e,f\rangle &+ \langle e, D_{D_{a,b}c,d}f\rangle + \langle D_{c,D_{a,b}d}e,f\rangle + \langle e, D_{c,D_{a,b}d}f\rangle - \langle D_{a,b}(D_{c,d}e),f\rangle \\ &- \langle D_{c,d}e, D_{a,b}f\rangle - \langle D_{a,b}e, D_{c,d}f\rangle - \langle e, D_{a,b}(D_{c,d}f)\rangle \\ &+ \langle D_{c,d}(D_{a,b}e),f\rangle + \langle D_{a,b}e, D_{c,d}f\rangle + \langle D_{c,d}e, D_{a,b}f\rangle \\ &+ \langle e, D_{c,d}(D_{a,b}f)\rangle = \langle D_{D_{a,b}c,d}e + D_{c,D_{a,b}d}e,f\rangle \\ &+ \langle e, D_{D_{a,b}c,d}f + D_{c,D_{a,b}d}f\rangle - \langle D_{a,b}(D_{c,d}e) - D_{c,d}(D_{a,b}e),f\rangle \\ &- \langle e, D_{a,b}(D_{c,d}f) - D_{c,d}(D_{a,b}f)\rangle = \langle [D_{a,b}, D_{c,d}]e,f\rangle \\ &+ \langle e, [D_{a,b}, D_{c,d}]f\rangle - \langle [D_{a,b}, D_{c,d}]e,f\rangle - \langle e, [D_{a,b}, D_{c,d}]f\rangle = 0. \qquad \blacksquare\end{aligned}$$

2. **The elements** $t(a,b)$ **and** $h(a,b)$ **of** $\mathrm{stu}_n(\mathcal{A},-,\gamma)$. We suppose in this section that $n \geq 3$. We also suppose that $(\mathcal{A},-)$ is structurable if $n = 3$ and associative if $n \geq 4$. In other words, we assume that $(\mathcal{A},-)$ is n-faithful. We will introduce some important elements of $\mathrm{stu}_n(\mathcal{A},-,\gamma)$ and describe their properties for use in later sections.

First let

$$T_{ij}(a,b) = [u_{ij}(a), u_{ji}(b)]$$

for $1 \leq i \neq j \leq n$ and $a, b \in \mathcal{A}$. Put

$$\mathcal{T} := \sum_{1\leq i<j\leq n} T_{ij}(\mathcal{A},\mathcal{A}).$$

The following proposition is proved in [AF, Lemma 1.1].

PROPOSITION 2.1. *$\mathcal{T}$ is a subalgebra of the Lie algebra $\mathrm{stu}_n(\mathcal{A},-,\gamma)$ containing the center of $\mathrm{stu}_n(\mathcal{A},-,\gamma)$ and $[\mathcal{T}, u_{ij}(\mathcal{A})] \subseteq u_{ij}(\mathcal{A})$. Moreover,*

$$\mathrm{stu}_n(\mathcal{A},-,\gamma) = \mathcal{T} \oplus \coprod_{1\leq i<j\leq n} u_{ij}(\mathcal{A}).$$

Indeed, one can easily check that the following identities hold.

$$[T_{ij}(a,b), u_{ik}(c)] = u_{ik}(a(bc) - \bar{b}(\bar{a}c)), \tag{2.2}$$

$$[T_{ij}(a,b), u_{ij}(c)] = u_{ij}\big(a(bc) - \bar{b}(\bar{a}c) + c(ba - \bar{a}\bar{b})\big), \tag{2.3}$$

for $a, b, c \in \mathcal{A}$ and distinct i, j, k. We also have

PROPOSITION 2.4. *For any $a, b, c \in \mathcal{A}$, and distinct i, j, k, we have*

$$T_{ij}(a,b) = -T_{ji}(b,a) = T_{ji}(\bar{a},\bar{b}), \tag{2.5}$$

$$T_{ij}(ab,c) = T_{ik}(a,bc) + T_{kj}(b,ca), \tag{2.6}$$

$$T_{kj}(1,c) = -T_{jk}(1,c) = T_{kj}(c,1), \quad \textit{and} \tag{2.7}$$

$$T_{ij}(1,a) = 0, \quad \textit{if } a \in \mathcal{A}_+. \tag{2.8}$$

PROOF. This is proved in [G, Proposition 2.17]. We note that in [G] algebras were assumed to be associative. However, associativity was not used in the proof of (2.5)–(2.9). ∎

Next, as in [G] (where $(\mathcal{A}, -)$ was assumed to be associative), we define

$$t(a,b) = T_{1j}(a,b) - T_{1j}(1,ba). \tag{2.9}$$

The elements $t(a,b)$ are analogs of elements used in [KL] and [BGKN] in the study of the Steinberg Lie algebra $\mathrm{st}_n(\mathcal{A})$. As in the discussion following [G, Proposition 2.17], one easily checks using Proposition 2.4 that $t(a,b)$ does not depend on the choice of j, $2 \leq j \leq n$. Moreover, one checks using (2.2) and (2.3) that

$$[t(a,b), u_{1i}(c)] = u_{1i}\big(a(bc) - (ba)c + (\bar{a}\bar{b})c - \bar{b}(\bar{a}c)\big), \quad \text{if } i \geq 2, \tag{2.10}$$

$$[t(a,b), u_{ij}(c)] = u_{ij}\big((b,a,c) - (\bar{a},\bar{b},c)\big), \quad \text{if } i,j \geq 2, \quad i \neq j, \tag{2.11}$$

for $a,b,c \in \mathcal{A}$. We also have the following basic properties:

PROPOSITION 2.12. *For $a,b,c \in \mathcal{A}$, we have*

$$t(a,b) + t(b,a) = 0, \tag{2.13}$$

$$t(a,b) - t(\bar{a},\bar{b}) = 0, \quad \textit{and} \tag{2.14}$$

$$t(ab,c) + t(bc,a) + t(ca,b) = T_{12}\big(1,(c,a,b)\big) + T_{13}\big(1,(b,c,a)\big). \tag{2.15}$$

PROOF. Since $T_{12}(ab,c) = T_{13}(a,bc) + T_{32}(b,ca)$ by (2.6), we get

$$t(ab,c) + T_{12}\big(1,c(ab)\big) = t(a,bc) + T_{13}\big(1,(bc)a\big) + T_{32}(b,ca).$$

Also, since

$$T_{32}(b,ca) = T_{31}(b,ca) + T_{12}\big(1,(ca)b\big) = -T_{13}(ca,b) + T_{12}\big(1,(ca)b\big),$$

we have

$$\begin{aligned} t(ab,c) + T_{12}\big(1,c(ab)\big) &= t(a,bc) + T_{13}\big(1,(bc)a\big) - T_{13}(ca,b) + T_{12}\big(1,(ca)b\big) \\ &= t(a,bc) + T_{13}\big(1,(bc)a\big) - t(ca,b) - T_{13}\big(1,b(ca)\big) \\ &\qquad + T_{12}\big(1,(ca)b\big). \end{aligned}$$

This gives us

$$t(ab,c) - t(a,bc) + t(ca,b) = T_{12}\big(1,(c,a,b)\big) + T_{13}\big(1,(b,c,a)\big).$$

Setting $a = 1$ and using the fact that $t(1,bc) = 0$ we have $t(b,c) + t(c,b) = 0$ which proves (2.13). Also, we have

$$t(ab,c) + t(bc,a) + t(ca,b) = T_{12}\big(1,(c,a,b)\big) + T_{13}\big(1,(b,c,a)\big)$$

which is (2.15). Using (2.9), we have

$$\begin{aligned} t(a,b) &= T_{1j}(a,b) - T_{1j}(1,ba) = -T_{1j}(\bar{b},\bar{a}) + T_{1j}(\overline{ba},\bar{1}) \\ &= -T_{1j}(\bar{b},\bar{a}) + T_{1j}(\bar{a}\bar{b},1) = -\big(T_{1j}(\bar{b},\bar{a}) - T_{1j}(1,\bar{a}\bar{b})\big) = -t(\bar{b},\bar{a}) = t(\bar{a},\bar{b}), \end{aligned}$$

which yields (2.14). ∎

We will also need the following:

LEMMA 2.16. *For $a,b,c \in \mathcal{A}$ and $i \neq j$, we have*

$$T_{ij}\big(1,(a,b,c)\big) = T_{ij}\big(1,(c,a,b)\big).$$

PROOF. We may assume that $i = 1$ and $j = 2$. Replacing a,b,c by b,c,a in (2.15), we get

$$t(bc,a) + t(ca,b) + t(ab,c) = T_{12}\big(1,(a,b,c)\big) + T_{13}\big(1,(c,a,b)\big).$$

Combining this with (2.15) one sees that

$$T_{12}\big(1,(c,a,b)-(a,b,c)\big) + T_{13}\big(1,(b,c,a)-(c,a,b)\big) = 0.$$

Thus, $T_{12}(1,x) + T_{13}(1,y) = 0$. where $x = (c,a,b)-(a,b,c)$ and $y = (b,c,a)-(c,a,b)$. But putting $X = T_{12}(1,x) + T_{13}(1,y) = 0$, we have, by (2.2) and (2.3), that $[X,u_{12}(1)] = u_{12}\big(2(x-\bar{x}) + y - \bar{y}\big)$ and $[X,u_{13}(1)] = u_{13}\big(x - \bar{x} + 2(y-\bar{y})\big)$. Since $X = 0$ and $(\mathcal{A},-)$ is 3-faithful, we have $2(x-\bar{x}) + y - \bar{y} = 0$ and $x - \bar{x} + 2(y-\bar{y}) = 0$. Thus, $x - \bar{x} = -2(y-\bar{y}) = 4(x-\bar{x})$ and hence since the characteristic of k is not 3, we have $x - \bar{x} = 0$. Thus, $T_{12}(1,x) = \frac{1}{2}T_{12}(1,x-\bar{x}) = 0$ as required. ∎

For the rest of the section we suppose that the characteristic of k does not divide n. This assumption allows us to define

$$h(a,b) = t(a,b) - \frac{1}{n}\sum_{j=2}^{n} T_{1j}(1,[a,b]), \tag{2.17}$$

for $a,b \in \mathcal{A}$. By (2.8), we also have

$$h(a,b) = t(a,b) - \frac{1}{2n}\sum_{j=2}^{n} T_{1j}(1,[a,b]+[\bar{a},\bar{b}]). \tag{2.18}$$

PROPOSITION 2.19. *Suppose $a,b \in \mathcal{A}$ and put $D = \frac{3}{n}D_{a,b}$. Then*

$$[h(a,b),u_{ij}(c)] = u_{ij}(Dc), \tag{2.20}$$

$$[h(a,b),T_{ij}(c,d)] = T_{ij}(Dc,d) + T_{ij}(c,Dd), \tag{2.21}$$

$$[h(a,b),t(c,d)] = t(Dc,d) + t(c,Dd), \quad \textit{and} \tag{2.22}$$

$$[h(a,b),h(c,d)] = h(Dc,d) + h(c,Dd) \tag{2.23}$$

for $c,d \in \mathcal{A}$ and $i \neq j$.

PROOF. Since D is a derivation, it suffices to show (2.20). Also, if $i,j \neq 1$, we have $u_{ij}(c) = [u_{i1}(1),u_{1j}(c)]$, and hence it suffices to prove (2.20) when $i = 1$ or $j = 1$. Thus, by (stu1), we may assume that $i = 1$. In that case, putting $s = [a,b] + [\bar{a},\bar{b}]$, we have, using (2.2), (2.3) and (2.10),

$$\begin{aligned}
[h(a,b),u_{1j}(c)] &= \left[t(a,b) - \frac{1}{n}\sum_{k=2}^{n} T_{1k}(1,[a,b]), u_{1j}(c)\right] \\
&= u_{1j}\big(a(bc) - (ba)c + (\bar{a}\bar{b})c - \bar{b}(\bar{a}c)\big) - \frac{1}{n}u_{1j}(sc+cs) \\
&\quad - \frac{1}{n}\sum_{\substack{k=2\\k\neq j}}^{n} u_{1j}(sc) \\
&= u_{1j}\left(a(bc) - (ba)c + (\bar{a}\bar{b})c - \bar{b}(\bar{a}c) - \frac{n-1}{n}sc - \frac{1}{n}cs\right) \\
&= u_{1j}\left(sc - (a,b,c) + (\bar{b},\bar{a},c) - \frac{n-1}{n}sc - \frac{1}{n}cs\right) \\
&= u_{1j}\left(\frac{1}{n}[s,c] - (a,b,c) + (\bar{b},\bar{a},c)\right).
\end{aligned}$$

But if $(\mathcal{A}, -)$ is associative, then $Dc = \frac{1}{n}[s,c]$ and so (2.20) holds. So we may assume that $n = 3$. But in that case, using (1.8) and (1.2), we have

$$\begin{aligned}
\frac{1}{n}[s,c] - (a,b,c) + (\bar{b},\bar{a},c) &= \frac{1}{3}\big[[a,b] + [\bar{a},\bar{b}], c\big] - (a,b,c) + (\bar{b},\bar{a},c) \\
&= -\frac{1}{3}\big[[b,a] + [\bar{b},\bar{a}], c\big] - (c,a,b) + (c,\bar{b},\bar{a}) \\
&= -D_{b,a}c = D_{a,b}c,
\end{aligned}$$

as required for (2.20). ∎

The basic properties of $h(a,b)$ that we will use are:

PROPOSITION 2.24. *For $a,b,c \in \mathcal{A}$, the following identities hold:*

$$h(a,b) + h(b,a) = 0, \quad \textit{and} \tag{2.25}$$

$$h(a,b) - h(\bar{a},\bar{b}) = 0, \tag{2.26}$$

$$h(ab,c) + h(bc,a) + h(ca,b) = 0. \tag{2.27}$$

PROOF. (2.25) and (2.26) are clear from (2.18) and the corresponding properties of $t(a,b)$. From (2.15) and Lemma 2.16 we have

$$\begin{aligned}
h(ab,c) + h(bc,a) + h(ca,b) &= t(ab,c) + t(bc,a) + t(ca,b) \\
&\quad - \frac{1}{n}\sum_{j=2}^{n} T_{1j}(1,[ab,c] + [bc,a] + [ca,b]) \\
&= T_{12}\big(1,(c,a,b)\big) + T_{13}\big(1,(b,c,a)\big) \\
&\quad - \frac{1}{n}\sum_{j=2}^{n} T_{1j}\big(1,(a,b,c) + (b,c,a) + (c,a,b)\big).
\end{aligned}$$

If $(\mathcal{A}, -)$ is associative, this is 0. So we may assume that $n = 3$. Then, by Lemma 2.16, this expression is

$$T_{12}\big(1,(c,a,b)\big) + T_{13}\big(1,(b,c,a)\big) - \frac{1}{3}T_{12}\big(1,3(a,b,c)\big) - \frac{1}{3}T_{13}\big(1,3(a,b,c)\big) = 0,$$

as desired. ∎

Let $h(\mathcal{A},\mathcal{A})$ denote the space spanned by the elements $h(a,b)$, $a, b \in \mathcal{A}$. Then, by (2.23), $h(\mathcal{A},\mathcal{A})$ is a subalgebra of $\mathrm{stu}_n(\mathcal{A},-,\gamma)$. Also, by Proposition 2.24, the elements $h(a,b)$, $a,b \in \mathcal{A}$, satisfy the relations (1.14)–(1.16). Thus, there exists a unique linear map $\eta_h\colon \langle \mathcal{A},\mathcal{A}\rangle \longrightarrow h(\mathcal{A},\mathcal{A})$ so that

$$\eta_h(\langle a,b\rangle) = \frac{n}{3}h(a,b).$$

Then, by (2.23), η_h is a homomorphism of Lie algebras. We will see in the next two sections that η_h is an isomorphism.

3. **The center of $\mathrm{stu}_n(\mathcal{A},-,\gamma)$ when $(\mathcal{A},-)$ is associative.** Suppose in this section that $(\mathcal{A},-)$ is associative and $n \geq 3$. We calculate the center of $\mathrm{stu}_n(\mathcal{A},-,\gamma)$.

Let $\mathrm{gl}_n(\mathcal{A})$ be the Lie algebra of all $n \times n$ matrices over $\mathcal{A}$. Let $\mathrm{eu}_n(\mathcal{A},-,\gamma)$ be the subalgebra of $\mathrm{gl}_n(\mathcal{A})$ generated by the elements

$$(3.1) \qquad ae_{ij} - \gamma_i\gamma_j^{-1}\bar{a}e_{ji}, \quad a \in \mathcal{A}, \quad 1 \leq i \neq j \leq n.$$

Here, as usual the elements e_{ij} are the matrix units. Let

$$\mathcal{Z} = \mathrm{center}\big(\mathrm{eu}_n(\mathcal{A},-\gamma)\big) \quad \text{and} \quad \mathrm{peu}_n(\mathcal{A},-,\gamma) = \mathrm{eu}_n(\mathcal{A},-,\gamma)/\mathcal{Z}.$$

$\mathrm{eu}_n(\mathcal{A},-,\gamma)$ (resp. $\mathrm{peu}_n(\mathcal{A},-,\gamma)$) is called the *elementary unitary Lie algebra* (resp. the *projective elementary unitary Lie algebra*).

Now the elements (3.1) satisfy the relations (stu1)–(stu4). Hence we have a unique Lie algebra homomorphism $\phi\colon \mathrm{stu}_n(\mathcal{A},-,\gamma) \longrightarrow \mathrm{eu}_n(\mathcal{A},-,\gamma)$ so that

$$(3.2) \qquad \phi\big(u_{ij}(a)\big) = ae_{ij} - \gamma_i\gamma_j^{-1}\bar{a}e_{ji}$$

for $a \in \mathcal{A}$, $1 \leq i \neq j \leq n$. Then we have the commutative diagram:

$$\begin{tikzcd} \mathrm{stu}_n(\mathcal{A},-,\gamma) \arrow[r, "\phi"] \arrow[dr, "\zeta"'] & \mathrm{eu}_n(\mathcal{A},-,\gamma) \arrow[d, "\omega"] \\ & \mathrm{peu}_n(\mathcal{A},-,\gamma) \end{tikzcd}$$

where ω is the canonical projection and $\zeta = \omega \circ \phi$.

Straightforward calculation yields the following

$$(3.3) \qquad \phi\big(T_{ij}(a,b)\big) = (ab - \bar{b}\bar{a})e_{ii} - (ba - \bar{a}\bar{b})e_{jj}$$

$$(3.4) \qquad \phi\big(T_{ij}(1,s)\big) = 2s(e_{ii} - e_{jj}), \quad \text{and}$$

$$(3.5) \qquad \phi\big(t(a,b)\big) = ([a,b] + [\bar{a},\bar{b}])e_{11}$$

for $a, b \in \mathcal{A}$, $s \in \mathcal{A}_-$, and $i \neq j$.

Let $t(\mathcal{A}, \mathcal{A})$ denote the space spanned by the elements $t(a, b)$, $a, b \in \mathcal{A}$. An easy computation, which we leave to the reader, using (2.10), (2.13) and (2.15), shows that

$$[t(a,b), t(c,d)] = t(s, [c,d]) = t([s,c], d) + t(c, [s,d]) \tag{3.6}$$

for $a, b, c, d \in \mathcal{A}$, and $s = [a,b] + [\bar{a}, \bar{b}]$. Thus, $t(\mathcal{A}, \mathcal{A})$ is a Lie subalgebra of $\mathrm{stu}_n(\mathcal{A}, -, \gamma)$. Next since $(\mathcal{A}, -)$ is associative, it follows from Lemma 2.12 that the elements $t(a, b)$ satisfy relations (1.14)–(1.16). Thus, there exists a unique linear map $\eta_t \colon \langle \mathcal{A}, \mathcal{A} \rangle \longrightarrow t(\mathcal{A}, \mathcal{A})$ so that

$$\eta_t(\langle a, b \rangle) = \frac{1}{3} t(a, b).$$

Since $D_{a,b}(x) = \frac{1}{3}\big[[a,b] + [\bar{a}, \bar{b}], x\big]$, it follows from (3.6) that η_t is a homomorphism of Lie algebras.

The next two propositions are proved in the first part of Section 2 of [G].

PROPOSITION 3.7. *(a) Each element of* $\mathrm{stu}_n(\mathcal{A}, -, \gamma)$ *can be uniquely expressed in the form*

$$T + \sum_{i=2}^{n} T_{1i}(1, s_i) + \sum_{1 \le i < j \le n} u_{ij}(x_{ij}), \tag{3.8}$$

where $T \in t(\mathcal{A}, \mathcal{A})$, $s_i \in \mathcal{A}_-$ *and* $x_{ij} \in \mathcal{A}$.

(b) $\mathrm{stu}_n(\mathcal{A}, -, \gamma) = t(\mathcal{A}, \mathcal{A}) \oplus \coprod_{i=2}^{n} T_{1i}(1, \mathcal{A}_-) \oplus \coprod_{1 \le i < j \le n} u_{ij}(\mathcal{A})$

(c) η_t *is an isomorphism of* $\langle \mathcal{A}, \mathcal{A} \rangle$ *onto* $t(\mathcal{A}, \mathcal{A})$.

PROOF. Except for uniqueness, (a) is proved in Lemma 2.27 of [G] (using (2.5), (2.6), (2.8) and (2.9)). Uniqueness follows from (3.2), (3.4) and (3.5). (b) follows from (a). (c) is part of the proof of [G, Theorem 2.33]. ∎

REMARK 3.9. The major tool used in the proof of Proposition 3.7 is the explicit construction of a Lie algebra $\mathcal{L} = \langle \mathcal{A}, \mathcal{A} \rangle \oplus \mathrm{eu}_n(\mathcal{A}, -, \gamma)$ which contains an isomorphic copy of $\mathrm{stu}_n(\mathcal{A}, -, \gamma)$ as a subalgebra. This is used to show that η_t is injective (see the proof of Theorem 2.33 of [G]).

PROPOSITION 3.10.

$$\ker \phi = \Big\{ \sum t(a_i, b_i) \;\Big|\; \sum([a_i, b_i] + [\bar{a}_i, \bar{b}_i]) = 0 \Big\} \cong {}_{-1}\mathrm{HD}_1(\mathcal{A}, -).$$

PROOF. The equality is proved in Lemma 2.30 of [G]. The isomorphism is the restriction of η_t. ∎

PROPOSITION 3.11. *We have*

$$\mathrm{eu}_n(\mathcal{A}, -, \gamma) = \{ X \in \mathrm{gl}_n(\mathcal{A}) \mid \gamma^{-1} \overline{X}^t \gamma = -X, \mathrm{tr}(X) \in [\mathcal{A}, \mathcal{A}] \}, \tag{3.12}$$

and

$$\mathcal{Z} = \{sI_n \mid s \in \mathcal{A}_- \cap \operatorname{center}(\mathcal{A}), ns \in [\mathcal{A}, \mathcal{A}]\}. \tag{3.13}$$

PROOF. Since $\mathrm{eu}_n(\mathcal{A}, -, \gamma) = \phi\big(\mathrm{stu}_n(\mathcal{A}, -, \gamma)\big)$, it follows from (3.2), (3.4), (3.5) and Proposition 3.7 (a) that the elements of $\mathrm{eu}_n(\mathcal{A}, -, \gamma)$ are the elements of the form

$$se_{11} + \sum_{i=2}^{n} s_i(e_{11} - e_{ii}) + \sum_{1 \le i < j \le n} (x_{ij}e_{ij} - \gamma_i\gamma_j^{-1}\bar{x}_{ij}e_{ji}),$$

where $s \in \mathcal{A}_- \cap [\mathcal{A}, \mathcal{A}]$, $s_i \in \mathcal{A}_-$ and $x_{ij} \in \mathcal{A}$. (3.12) follows easily from this observation.

Suppose next that $X \in \mathcal{Z}$. Then, $\gamma^{-1}\bar{X}^t\gamma = -X$ and $\operatorname{tr}(X) \in [\mathcal{A}, \mathcal{A}]$. Since X commutes with the elements $e_{ij} - \gamma_i\gamma_j^{-1}e_{ji}$ for $1 \le i \ne j \le n$, it follows that $X = sI_n$, for some $s \in \mathcal{A}_-$ so that $ns = \operatorname{tr}(X) \in [\mathcal{A}, \mathcal{A}]$. Since X commutes with the elements (3.1) it follows that $s \in \operatorname{center}(\mathcal{A})$. This proves the inclusion "$\subseteq$" in 3.13. The reverse inclusion follows from 3.12. ∎

PROPOSITION 3.14. *We have*

$$\begin{aligned} \operatorname{center}\big(\mathrm{stu}_n(\mathcal{A}, -, \gamma)\big) &= \ker\zeta \\ &= \Big\{\sum_i t(a_i, b_i) - \sum_{j=2}^{n} T_{1j}(1, s) \;\Big|\; s \in \mathcal{A}_- \cap \operatorname{center}(\mathcal{A}), a_i, b_i \in \mathcal{A}, \\ &\qquad \sum_i([a_i, b_i] + [\bar{a}_i, \bar{b}_i]) = 2ns\Big\} \end{aligned}$$

PROOF. We first prove that $\ker\zeta$ equals the last set. Let $X \in \ker\zeta$. Then, by Proposition 3.7 (a), X can be expressed in the form

$$X = \sum_i t(a_i, b_i) + \sum_{j=2}^{n} T_{1j}(1, s_j) + \sum_{1 \le i < j \le n} u_{ij}(x_{ij}),$$

where $a_i, b_i \in \mathcal{A}$, $s_j \in \mathcal{A}_-$, $x_{ij} \in \mathcal{A}$. But $\phi(X) \in \ker(\omega) = \mathcal{Z}$. Hence, by (3.13), $\phi(X) = 2sI_n$ for some $s \in \mathcal{A}_- \cap \operatorname{center}(\mathcal{A})$ so that $ns \in [\mathcal{A}, \mathcal{A}]$. It follows then from (3.2), (3.4) and (3.5) that $x_{ij} = 0$ for all i, j. Then, by (3.4) and (3.5),

$$\sum_i([a_i, b_i] + [\bar{a}_i, \bar{b}_i])e_{11} + \sum_{j=2}^{n} 2s_j(e_{11} - e_{jj}) = 2sI_n.$$

Thus, $\sum_i([a_i, b_i] + [\bar{a}_i, \bar{b}_i]) + 2\sum_{j=2}^{n} s_j = 2s$ and $-2s_j = 2s$ for $2 \le j \le n$. Consequently, $s_j = -s$ for $2 \le j \le n$ and $\sum_i([a_i, b_i] + [\bar{a}_i, \bar{b}_i]) = 2ns$. This proves the inclusion "$\subseteq$". The reverse inclusion follows from (3.4), (3.5) and (3.13).

It remains to prove that $\operatorname{center}\big(\mathrm{stu}_n(\mathcal{A}, -, \gamma)\big) = \ker\zeta$. Let $X \in \ker\zeta$. But then by what we have just proved, $X \in \mathcal{T}$. Thus, by Proposition 2.1, $[X, u_{ij}(\mathcal{A})] \subseteq u_{ij}(\mathcal{A})$ for all $i \ne j$. Hence, $[X, u_{ij}(\mathcal{A})]$ is contained in $\ker\zeta \cap u_{ij}(\mathcal{A})$ which is $\{0\}$. So $X \in \operatorname{center}\big(\mathrm{stu}_n(\mathcal{A}, -, \gamma)\big)$. Conversely, let $X \in \operatorname{center}\big(\mathrm{stu}_n(\mathcal{A}, -, \gamma)\big)$. Thus, $\phi(X) \in \operatorname{center}\big(\mathrm{eu}_n(\mathcal{A},, \gamma)\big) = \mathcal{Z}$ and so $\zeta(X) = \omega\big(\phi(X)\big) = 0$. ∎

If $\operatorname{char}(k) \mid n$, we now have the desired description of $\operatorname{center}\big(\mathrm{stu}_n(\mathcal{A}, -, \gamma)\big)$.

COROLLARY 3.15. *Suppose that* char(k) $\mid n$. *Then*

$$\begin{aligned}\operatorname{center}\bigl(\operatorname{stu}_n(\mathcal{A},-,\gamma)\bigr) &= \Bigl\{\sum_i t(a_i,b_i)\Bigm|\sum_i([a_i,b_i]+[\bar a_i,\bar b_i])=0\Bigr\}\\ &\qquad\oplus\Bigl\{\sum_{j=2}^{n}T_{1j}(1,s)\Bigm| s\in\mathcal{A}_-\cap\operatorname{center}(\mathcal{A})\Bigr\}\\ &\cong {}_{-1}\mathrm{HD}_1(\mathcal{A},-)\oplus\mathcal{A}_-\cap\operatorname{center}(\mathcal{A}).\end{aligned}$$

PROOF. This follows immediately from Propositions 3.7 and 3.14. ∎

For the rest of this section we assume that char(k) $\not| \, n$. Thus, as noted at the end of Section 2, we have a Lie algebra homomorphism $\eta_h\colon\langle\mathcal{A},\mathcal{A}\rangle\longrightarrow h(\mathcal{A},\mathcal{A})$ so that $\eta_h(\langle a,b\rangle)=\frac{n}{3}h(a,b)$. Moreover, we have the following analog of Proposition 3.7.

PROPOSITION 3.16. *(a) Each element of* $\operatorname{stu}_n(\mathcal{A},-,\gamma)$ *can be expressed uniquely in the form*

$$H+\sum_{i=2}^{n}T_{1i}(1,s_i)+\sum_{1\le i<j\le n}u_{ij}(x_{ij}), \tag{3.17}$$

where $H\in h(\mathcal{A},\mathcal{A})$, $s_i\in\mathcal{A}_-$ *and* $x_{ij}\in\mathcal{A}$.

(b) $\operatorname{stu}_n(\mathcal{A},-,\gamma)=h(\mathcal{A},\mathcal{A})\oplus\coprod_{j=2}^{n}T_{1j}(1,\mathcal{A}_-)\oplus\coprod_{1\le i<j\le n}u_{ij}(\mathcal{A})$.

(c) η_h *is an isomorphism of* $\langle\mathcal{A},\mathcal{A}\rangle$ *onto* $h(\mathcal{A},\mathcal{A})$.

PROOF. (a) The fact that every element can be expressed in the form 3.17 follows from Proposition 3.7 (a) and 2.18. For uniqueness suppose that

$$\sum_i h(a_i,b_i)+\sum_{j=2}^{n}T_{1j}(1,s_j)+\sum_{1\le i<j\le n}u_{ij}(x_{ij})=0.$$

Then, by Proposition 3.7 (a) and 2.18, we have $\sum_i t(a_i,b_i)=0$. Thus, by Proposition 3.7 (c), $\sum_i\langle a_i,b_i\rangle=0$ and hence, by Example 1.17, $\sum_i([a_i,b_i]+[\bar a_i,\bar b_i])=0$ and so, by (2.18), $\sum_i h(a_i,b_i)=\sum_i t(a_i,b_i)$. Thus, uniqueness follows from Proposition 3.7 (a).

(b) follows from (a).

(c) Only the injectivity of η_h needs to be proved. For this suppose that $\sum_i\langle a_i,b_i\rangle=0$. Then, as in the proof of (a), we get $\sum_i h(a_i,b_i)=\sum_i t(a_i,b_i)$. But by Proposition 3.7 (a), $\sum_i t(a_i,b_i)=0$. Hence, $\sum_i h(a_i,b_i)=0$. ∎

THEOREM 3.18. *Suppose that* $(\mathcal{A},-)$ *is associative and* char(k) $\not| \, n$. *Then,*

$$\operatorname{center}\bigl(\operatorname{stu}_n(\mathcal{A},-,\gamma)\bigr)=\Bigl\{\sum_i h(a_i,b_i)\Bigm|\sum_i D_{a_i,b_i}=0\Bigr\}\cong\mathrm{HF}(\mathcal{A},-).$$

PROOF. By Proposition 3.16 (c), it is enough to prove the equality. For this, suppose that $X\in\operatorname{center}\bigl(\operatorname{stu}_n(\mathcal{A},-,\gamma)\bigr)$. Then, by Proposition 3.14, we have, $X=\sum_i t(a_i,b_i)-\sum_{j=2}^{n}T_{1j}(1,s)$ for some $a_i,b_i\in\mathcal{A}$, $s\in\mathcal{A}_-\cap\operatorname{center}(\mathcal{A})$ so that $\sum_i([a_i,b_i]+[\bar a_i,\bar b_i])=2ns$.

Then, $s = \frac{1}{2n}\sum_i([a_i,b_i] + [\bar{a}_i,\bar{b}_i])$ and so, by (2.18), $X = \sum_i h(a_i,b_i)$. But $\sum_i([a_i,b_i] + [\bar{a}_i,\bar{b}_i]) = 2ns \in \mathrm{center}(\mathcal{A})$ and so $\sum_i D_{a_i,b_i} = 0$. This proves the inclusion "$\subseteq$". The reverse inclusion follows from Proposition 3.14 and 2.18. ∎

REMARK 3.19. Theorem 3.18 is false if $\mathrm{char}(k) \mid n$. For example suppose that $\mathrm{char}(k) \mid n$ and $(\mathcal{A},-) = (k\oplus k, \mathrm{ex})$, where ex denotes the exchange involution. Then, an easy calculation, using only 1.14 and 1.15 shows that $\langle \mathcal{A},\mathcal{A}\rangle = \{0\}$. Thus, $\mathrm{HF}(\mathcal{A},-) = \{0\}$. However, by Corollary 3.15, we have $\mathrm{center}\big(\mathrm{stu}_n(\mathcal{A},-,\gamma)\big) \cong \mathcal{A}_-$, which is 1-dimensional.

4. **The center of** $\mathrm{stu}_3(\mathcal{A},-,\gamma)$. In this section we assume that $n = 3$ and that $(\mathcal{A},-)$ is a structurable algebra. We compute the center of $\mathrm{stu}_3(\mathcal{A},-,\gamma)$. Our approach will be similar to the approach in the previous section where $(\mathcal{A},-)$ was assumed to be associative. That approach used the three Lie algebras $\mathrm{eu}_3(\mathcal{A},-,\gamma)$, $\mathrm{peu}_3(\mathcal{A},-,\gamma)$, and an explicit construction of $\mathrm{stu}_3(\mathcal{A},-,\gamma)$ as a subalgebra of a Lie algebra constructed using $\langle \mathcal{A},\mathcal{A}\rangle$ (see Remark 3.9). In the present generality, we no longer have an analog of $\mathrm{eu}_3(\mathcal{A},-,\gamma)$. However, we do have an analog of the Lie algebra $\mathrm{peu}_3(\mathcal{A},-,\gamma)$ as well as an explicit construction of the Lie algebra $\mathrm{stu}_3(\mathcal{A},-,\gamma)$ using $\langle \mathcal{A},\mathcal{A}\rangle$. We begin by introducing these Lie algebras which we denote respectively by $\mathcal{K}(\mathcal{A},-,\gamma)$ and $\tilde{\mathcal{K}}(\mathcal{A},-,\gamma)$.

We define an involution J_γ on the algebra $M_3(\mathcal{A})$ of 3×3 matrices over $\mathcal{A}$ by

$$J_\gamma(X) = \gamma^{-1}\bar{X}^t\gamma,$$

where X^t is the transpose of the matrix X. We put

$$\mathcal{P} := \mathcal{P}(\mathcal{A},^-,\gamma) := \{X \in M_3(\mathcal{A}) : J_\gamma(X) = -X, \mathrm{tr}(X) = 0\}, \tag{4.1}$$

where $\mathrm{tr}(X) = \sum_{i=1}^3 x_{ii}$ for $X = (x_{ij}) \in M_3(\mathcal{A})$. A general element of $\mathcal{P}$ is then of the form

$$(e_{11}-e_{22})r + (e_{11}-e_{33})s + a[23] + b[31] + c[12],$$

where $a,b,c \in \mathcal{A}$ and $r,s \in \mathcal{A}_-$. Here we use the notation

$$a[ij] := ae_{ij} - \gamma_i\gamma_j^{-1}\bar{a}e_{ji}$$

for $a \in \mathcal{A}$, $1 \le i \ne j \le 3$. So we have

$$\mathcal{P} = (e_{11}-e_{22})\mathcal{A}_- \oplus (e_{11}-e_{33})\mathcal{A}_- \oplus \mathcal{A}[23] \oplus \mathcal{A}[31] \oplus \mathcal{A}[12]. \tag{4.2}$$

Next define $\Delta_{X,Y} \in \mathrm{Inder}(\mathcal{A},-)$ for $X = (x_{ij})$, $Y = (y_{ij}) \in M_3(\mathcal{A})$ by

$$\Delta_{X,Y} := \frac{1}{2}\sum_{i,j=1}^{3} D_{x_{ij},y_{ji}}.$$

Now we can define an algebra

$$\mathcal{K} := \mathcal{K}(\mathcal{A},-,\gamma) := \mathrm{Inder}(\mathcal{A},-) \oplus \mathcal{P} \tag{4.3}$$

with the multiplication given by

$$(4.4)\qquad [D_1+X_1,D_2+X_2]=([D_1,D_2]+\Delta_{X_1,X_2})+(D_1X_2-D_2X_1+[X_1,X_2]_0),$$

where D_iX_j is obtained by D_i acting on the entries of X_j, and

$$[X_1,X_2]_0=X_1X_2-X_2X_1-\frac{1}{3}\mathrm{tr}(X_1X_2-X_2X_1)I.$$

Then we have (see [A2, Corollary 4.11]):

PROPOSITION 4.5. *$\mathcal{K}(\mathcal{A},-,\gamma)$ is a Lie algebra.*

REMARK 4.6. The Lie algebra $\mathcal{K}(\mathcal{A},-,\gamma)$ was first studied by H. Asano and K. Yamaguti [AY], G. B. Seligman [Se], and then in its present form in [A2]. A version of this Lie algebra construction which works in arbitrary characteristic is given in [AF].

Next we put

$$(4.7)\qquad \tilde{\mathcal{K}}:=\tilde{\mathcal{K}}(\mathcal{A},-,\gamma):=\langle\mathcal{A},\mathcal{A}\rangle\oplus\mathcal{P},$$

and define an anti-commutative bilinear product [,] on $\tilde{\mathcal{K}}$ so that

$$[\langle a,b\rangle,\langle c,d\rangle]=\langle D_{a,b}c,d\rangle+\langle c,D_{a,b}d\rangle,$$
$$[\langle a,b\rangle,X]=D_{a,b}X,$$
$$[X,Y]=E_{X,Y}+[X,Y]_0,$$

where

$$E_{X,Y}:=\frac{1}{2}\sum_{i,j=1}^{3}\langle x_{ij},y_{ji}\rangle,$$

for $X=(x_{ij})$ and $Y=(y_{ij})$. The fact that there is a unique well-defined product with these properties follows from (1.22) and the existence of the map $\rho\colon\langle\mathcal{A},\mathcal{A}\rangle\longrightarrow\mathrm{Inder}(\mathcal{A},-)$ defined in Section 1. Moreover, we have a k-linear map $\sigma\colon\tilde{\mathcal{K}}(\mathcal{A},-,\gamma)\longrightarrow\mathcal{K}(\mathcal{A},-,\gamma)$ defined by

$$\sigma\Big(\sum_i\langle a_i,b_i\rangle+X\Big)=\sum_i D_{a_i,b_i}+X,$$

for $a_i,b_i\in\mathcal{A}$ and $X\in\mathcal{P}$. By the definition of the products, σ is an algebra homomorphism. Moreover, σ is surjective with kernel $\mathrm{HF}(\mathcal{A},-)$.

PROPOSITION 4.8. *$\tilde{\mathcal{K}}$ is a Lie algebra under the product* [,] *defined as above.*

PROOF. Let $J(x,y,z)=\big[[x,y],z\big]+\big[[y,z],x\big]+\big[[z,x],y\big]$ for $x,y,z\in\tilde{\mathcal{K}}$. Then J is an alternating function of its variables. We want to show that $J(x,y,z)=0$ for all $x,y,z\in\tilde{\mathcal{K}}$. Now $\sigma J(x,y,z)=0$, and so $J(x,y,z)\in\langle\mathcal{A},\mathcal{A}\rangle$ for all $x,y,z\in\tilde{\mathcal{K}}$. We denote by κ the projection of $\tilde{\mathcal{K}}$ onto $\langle\mathcal{A},\mathcal{A}\rangle$ relative to the decomposition $\tilde{\mathcal{K}}=\langle\mathcal{A},\mathcal{A}\rangle\oplus\mathcal{P}$. Then $\kappa\big(J(x,y,z)\big)=J(x,y,z)$. Hence, it suffices to show that $\kappa\big(J(x,y,z)\big)=0$. Thus, it suffices to show that κ applied to each of the following is (0):

$$J(\langle\mathcal{A},\mathcal{A}\rangle,\langle\mathcal{A},\mathcal{A}\rangle,\langle\mathcal{A},\mathcal{A}\rangle),\quad J(\langle\mathcal{A},\mathcal{A}\rangle,\langle\mathcal{A},\mathcal{A}\rangle,\mathcal{P}),$$
$$J(\langle\mathcal{A},\mathcal{A}\rangle,\mathcal{P},\mathcal{P})\quad\text{and}\quad J(\mathcal{P},\mathcal{P},\mathcal{P}).$$

But $J(\langle\mathcal{A},\mathcal{A}\rangle,\langle\mathcal{A},\mathcal{A}\rangle,\langle\mathcal{A},\mathcal{A}\rangle)=(0)$ by Proposition 1.25. Also $J(\langle\mathcal{A},\mathcal{A}\rangle,\langle\mathcal{A},\mathcal{A}\rangle,\mathcal{P})\subseteq\mathcal{P}$ and hence $\kappa\big(J(\langle\mathcal{A},\mathcal{A}\rangle,\langle\mathcal{A},\mathcal{A}\rangle,\mathcal{P})\big)=(0)$. Thus, it remains to show that

$$\kappa\big(J(\langle\mathcal{A},\mathcal{A}\rangle,\mathcal{P},\mathcal{P})\big)=(0),\quad\text{and}\tag{4.9}$$

$$\kappa\big(J(\mathcal{P},\mathcal{P},\mathcal{P})\big)=(0).\tag{4.10}$$

We first consider (4.9). If $X=(x_{ij})$, $Y=(y_{ij})\in\mathcal{P}$,

$$\begin{aligned}J(\langle a,b\rangle,X,Y)&=[D_{a,b}X,Y]-[D_{a,b}Y,X]-\big[\langle a,b\rangle,[X,Y]\big]\\&=E_{D_{a,b}X,Y}+E_{X,D_{a,b}Y}-[\langle a,b\rangle,E_{X,Y}]\pmod{\mathcal{P}}.\end{aligned}$$

So now we have to show

$$[\langle a,b\rangle,E_{X,Y}]=E_{D_{a,b}X,Y}+E_{X,D_{a,b}Y},\tag{4.11}$$

for $X,Y\in\mathcal{P}$. Indeed,

$$\begin{aligned}[\langle a,b\rangle,E_{X,Y}]&=\Big[\langle a,b\rangle,\frac{1}{2}\sum_{i,j=1}^{3}\langle x_{ij},y_{ji}\rangle\Big]\\&=\frac{1}{2}\sum_{i,j=1}^{3}(\langle D_{a,b}x_{ij},y_{ji}\rangle+\langle x_{ij},D_{a,b}y_{ji}\rangle)=E_{D_{a,b}X,Y}+E_{X,D_{a,b}Y}.\end{aligned}$$

For (4.10), we let $X,Y,Z\in\mathcal{P}$, then

$$\begin{aligned}\big[[X,Y],Z\big]&=\big[E_{X,Y}+[X,Y]_0,Z\big]\\&=[E_{X,Y},Z]+E_{[X,Y]_0,Z}+\big[[X,Y]_0,Z\big]_0=E_{[X,Y]_0,Z}\pmod{\mathcal{P}}.\end{aligned}$$

Hence $\kappa\big(\big[[X,Y],Z\big]\big)=E_{[X,Y]_0,Z}$. But if $a\in\mathcal{A}$, $Z\in\mathcal{P}$, we obtain

$$E_{aI,Z}=\frac{1}{2}\sum_{i,j=1}^{3}\langle(aI)_{ij},z_{ji}\rangle=\frac{1}{2}\sum_{i=1}^{3}\langle a,z_{ii}\rangle=\frac{1}{2}\langle a,\mathrm{tr}(Z)\rangle=0.$$

It follows that $E_{[X,Y]_0,Z}=E_{XY-YX,Z}$ and thus (4.10) is equivalent to

$$E_{XY-YX,Z}+E_{YZ-ZY,X}+E_{ZX-XZ,Y}=0,\tag{4.12}$$

for $X,Y,Z\in\mathcal{P}$. In fact, if $X=(x_{ij})$, $Y=(y_{ij})$, $Z=(z_{ij})$,

$$\begin{aligned}&E_{XY-YX,Z}+E_{YZ-ZY,X}+E_{ZX-XZ,Y}\\&\quad=\frac{1}{2}\sum_{1\le i,j,k\le3}\langle x_{ik}y_{kj}-y_{ik}x_{kj},z_{ji}\rangle+\frac{1}{2}\sum_{1\le i,j,k\le3}\langle y_{ik}z_{kj}-z_{ik}y_{kj},x_{ji}\rangle\\&\qquad+\frac{1}{2}\sum_{1\le i,j,k\le3}\langle z_{ik}x_{kj}-x_{ik}z_{kj},y_{ji}\rangle\\&\quad=\frac{1}{2}\sum_{1\le i,j,k\le3}\langle x_{ik}y_{kj},z_{ji}\rangle+\frac{1}{2}\sum_{1\le i,j,k\le3}\langle y_{ik}z_{kj},x_{ji}\rangle+\frac{1}{2}\sum_{1\le i,j,k\le3}\langle z_{ik}x_{kj},y_{ji}\rangle\\&\qquad-\Big(\frac{1}{2}\sum_{1\le i,j,k\le3}\langle y_{ik}x_{kj},z_{ji}\rangle+\frac{1}{2}\sum_{1\le i,j,k\le3}\langle z_{ik}y_{kj},x_{ji}\rangle\\&\qquad\quad+\frac{1}{2}\sum_{1\le i,j,k\le3}\langle x_{ik}z_{kj},y_{ji}\rangle\Big)\\&\quad=F_1-F_2,\end{aligned}$$

where

$$\begin{aligned} F_1 &= \frac{1}{2}\sum_{1\le i,j,k\le 3}\langle x_{ik}y_{kj}, z_{ji}\rangle + \frac{1}{2}\sum_{1\le i,j,k\le 3}\langle y_{ik}z_{kj}, x_{ji}\rangle + \frac{1}{2}\sum_{1\le i,j,k\le 3}\langle z_{ik}x_{kj}, y_{ji}\rangle \\ &= \frac{1}{2}\sum_{1\le i,j,k\le 3}\langle x_{ik}y_{kj}, z_{ji}\rangle + \frac{1}{2}\sum_{1\le i,j,k\le 3}\langle y_{kj}z_{ji}, x_{ik}\rangle + \frac{1}{2}\sum_{1\le i,j,k\le 3}\langle z_{ji}x_{ik}, y_{kj}\rangle \\ &= \frac{1}{2}\sum_{1\le i,j,k\le 3}\big(\langle x_{ik}y_{kj}, z_{ji}\rangle + \langle y_{kj}z_{ji}, x_{ik}\rangle + \langle z_{ji}x_{ik}, y_{kj}\rangle\big) = 0. \end{aligned}$$

Similarly, F_2 has the obvious meaning and, as for F_1, we get $F_2 = 0$ as desired. ∎

By the universal property of $\mathrm{stu}_3(\mathcal{A}, -, \gamma)$ there is a unique Lie algebra homomorphism $\nu\colon \mathrm{stu}_3(\mathcal{A}, -, \gamma) \longrightarrow \mathcal{K}(\mathcal{A}, -, \gamma)$ so that

$$\nu\big(u_{ij}(a)\big) = a[ij]$$

for $a \in \mathcal{A}$, $1 \le i \ne j \le 3$.

PROPOSITION 4.13. *Each element of* $\mathrm{stu}_3(\mathcal{A}, -, \gamma)$ *can be expressed uniquely in the form*

$$H + T_{12}(1, s_2) + T_{13}(1, s_3) + u_{12}(x_{12}) + u_{23}(x_{23}) + u_{13}(x_{13}), \tag{4.14}$$

where $H \in h(\mathcal{A}, \mathcal{A})$, $s_2, s_3 \in \mathcal{A}_-$ *and* $x_{12}, x_{23}, x_{13} \in \mathcal{A}$. *Thus,*

$$\begin{aligned} \mathrm{stu}_3(\mathcal{A}, -, \gamma) = h(\mathcal{A}, \mathcal{A}) \oplus T_{12}(1, \mathcal{A}_-) \oplus T_{13}(1, \mathcal{A}_-) \oplus u_{23}(\mathcal{A}) \\ \oplus u_{31}(\mathcal{A}) \oplus u_{12}(\mathcal{A}). \end{aligned} \tag{4.15}$$

Moreover,

$$\nu\big(T_{1j}(1, s)\big) = 2s(e_{11} - e_{jj}). \tag{4.16}$$

for $s \in \mathcal{A}_-$, $j = 2, 3$, *and*

$$\nu\big(h(a, b)\big) = D_{a,b} \tag{4.17}$$

for $a, b \in \mathcal{A}$.

PROOF. First of all

$$\begin{aligned} \nu\big(T_{ij}(a, b)\big) &= \Delta_{a[ij], b[ji]} + \big[a[ij], b[ji]\big]_0 \\ &= \frac{1}{2}(D_{a,b} + D_{\bar a, \bar b}) + e_{ii}(ab - \bar b\bar a) + e_{jj}(\bar a\bar b - ba) \\ &\qquad - \frac{1}{3}(ab - \bar b\bar a + \bar a\bar b - ba)I \\ &= D_{a,b} + e_{ii}(ab - \bar b\bar a) + e_{jj}(\bar a\bar b - ba) - \frac{1}{3}([a, b] + [\bar a, \bar b])I. \end{aligned} \tag{4.18}$$

for $a, b \in \mathcal{A}$, $i \ne j$. In particular,

$$\nu\big(T_{ij}(1, c)\big) = (c - \bar c)(e_{ii} - e_{jj}). \tag{4.19}$$

for $c \in \mathcal{A}$, and so we have (4.16). Also, using (4.18) and (4.19), we have

$$
\begin{aligned}
\nu\bigl(t(a,b)\bigr) &= \nu\bigl(T_{12}(a,b) - T_{12}(1,ba)\bigr)\\
&= D_{a,b} + e_{11}(ab - \bar{b}\bar{a}) + e_{22}(\bar{a}\bar{b} - ba)\\
&\quad - \frac{1}{3}([a,b] + [\bar{a},\bar{b}])I - (e_{11} - e_{22})(ba - \bar{a}\bar{b})\\
&= D_{a,b} + \frac{1}{3}(e_{11} - e_{22})([a,b] + [\bar{a},\bar{b}])\\
&\quad + \frac{1}{3}(e_{11} - e_{33})([a,b] + [\bar{a},\bar{b}]).
\end{aligned} \tag{4.20}
$$

Hence, by (4.20) and (4.19), we have

$$
\nu\bigl(h(a,b)\bigr) = \nu\Bigl(t(a,b) - \frac{1}{3}T_{12}(1,[a,b]) - \frac{1}{3}T_{13}(1,[a,b])\Bigr) = D_{a,b},
$$

which proves (4.17).

Next, the fact that every element of $\mathrm{stu}_3(\mathcal{A},-,\gamma)$ can be expressed in the form (4.14) follows easily using Proposition 2.1, (2.5), (2.6), (2.8), (2.9) and (2.18). Uniqueness of this form follows then from (4.16) and (4.17). This concludes the proof of the first statement. The second statement follows from the first. ■

By the universal property of $\mathrm{stu}_3(\mathcal{A},-,\gamma)$ there is a (unique) Lie algebra homomorphism $\tau\colon \mathrm{stu}_3(\mathcal{A},-,\gamma) \longrightarrow \tilde{\mathcal{K}}(\mathcal{A},-,\gamma)$ such that

$$
\tau\bigl(u_{ij}(a)\bigr) = a[ij]
$$

for $a \in \mathcal{A}$, $1 \le i \ne j \le 3$.

PROPOSITION 4.21. *The map η_h (defined in Section 2) is a Lie algebra isomorphism of $\langle \mathcal{A},\mathcal{A}\rangle$ onto $h(\mathcal{A},\mathcal{A})$. Moreover, τ is an isomorphism of $\mathrm{stu}_3(\mathcal{A},-,\gamma)$ onto $\tilde{\mathcal{K}}$.*

PROOF. Exactly as in the proof of Proposition 4.13 (with Δ replaced by E and $D_{a,b}$ replaced by $\langle a,b\rangle$), we establish the formulas

$$
\tau\bigl(T_{1j}(1,s)\bigr) = 2s(e_{11} - e_{jj}). \tag{4.22}
$$

for $s \in \mathcal{A}_-$, $j = 2,3$, and

$$
\tau\bigl(h(a,b)\bigr) = \langle a,b\rangle. \tag{4.23}
$$

Thus, $\tau\bigl(h(\mathcal{A},\mathcal{A})\bigr) = \langle \mathcal{A},\mathcal{A}\rangle$ and τ restricted to $h(\mathcal{A},\mathcal{A})$ is an inverse of η_h. This proves the first statement, and the second statement follows using (4.22), (4.23) and Proposition 4.13. ■

THEOREM 4.24. *We have*

$$
\mathrm{center}\bigl(\mathrm{stu}_3(\mathcal{A},-,\gamma)\bigr) = \ker\nu = \Bigl\{\sum_i h(a_i,b_i) \Bigm| \sum_i D_{a_i,b_i} = 0\Bigr\} \cong \mathrm{HF}(\mathcal{A},-).
$$

PROOF. Let $\mathcal{Z}$ denote the center of $\mathrm{stu}_3(\mathcal{A}, -, \gamma)$. Since $\ker \nu \subseteq \mathcal{T}$ and $[\mathcal{T}, u_{ij}(\mathcal{A})] \subseteq u_{ij}(\mathcal{A})$ for all $i \neq j$, it follows that $\ker \nu \subseteq \mathcal{Z}$. On the other hand, one easily checks that $\mathcal{K}(\mathcal{A}, -, \gamma)$ has zero center. Hence $\mathcal{Z} = \ker \nu$. This proves the first equality. The second equality follows from Proposition 4.13. The final isomorphism is the restriction of η_h. ∎

REMARK 4.25. If $(\mathcal{A}, -)$ is associative, it follows from the first equalities in Theorem 4.24 and Proposition 3.14 that

$$\mathcal{K}(\mathcal{A}, -, \gamma) \cong \mathrm{stu}_3(\mathcal{A}, -, \gamma)/\mathrm{center}\big(\mathrm{stu}_3(\mathcal{A}, -, \gamma)\big) \cong \mathrm{peu}_3(\mathcal{A}, -).$$

(See [AF], p. 9.)

We conclude the section by noting that Theorem 4.24 together with Theorem 3.18 of the previous section establish Theorem A in the introduction.

5. **Coverings of** $\mathrm{stu}_3(\mathcal{A}, -, \gamma)$. We begin this section by recalling a few facts about central extensions of Lie algebras. A good reference for these facts is [Ga, pp. 13–15].

Suppose that $\mathcal{G}$ is a Lie algebra. A *central extension* of $\mathcal{G}$ is a pair $(\hat{\mathcal{G}}, \pi)$, where $\pi\colon \hat{\mathcal{G}} \longrightarrow \mathcal{G}$ is a surjective Lie algebra homomorphism whose kernel lies in the center of $\hat{\mathcal{G}}$. A *covering* of $\mathcal{G}$ is a central extension $(\hat{\mathcal{G}}, \pi)$ of $\mathcal{G}$ so that $\hat{\mathcal{G}}$ is perfect, *i.e.*, $[\hat{\mathcal{G}}, \hat{\mathcal{G}}] = \hat{\mathcal{G}}$. A *universal covering* of $\mathcal{G}$ is a covering $(\hat{\mathcal{G}}, \pi)$ of $\mathcal{G}$ such that for every central extension $(\tilde{\mathcal{G}}, \tau)$ of $\mathcal{G}$ there is a homomorphism $\psi\colon \hat{\mathcal{G}} \longrightarrow \tilde{\mathcal{G}}$ so that $\tau \circ \psi = \pi$. If $(\hat{\mathcal{G}}, \pi)$ is a universal covering of $\mathcal{G}$, then

$$\mathrm{H}_2(\mathcal{G}) \cong \ker \pi,$$

where $\mathrm{H}_2(\mathcal{G})$ is the second homology group of $\mathcal{G}$. (See [Ga, p. 13] and [KL, p. 123].) Now it is clear that if $\mathcal{G}$ has a universal covering then $\mathcal{G}$ is perfect (in fact this is true if $\mathcal{G}$ has any covering at all). Conversely, it is shown in [Ga] that if $\mathcal{G}$ is perfect, then $\mathcal{G}$ has a unique universal covering (up to isomorphism).

A perfect Lie algebra $\mathcal{G}$ is said to be *centrally closed* (or *simply connected*) if every central extension of $\mathcal{G}$ splits. We then have the following characterization of universal coverings (see [Ga, pp. 14–15]):

LEMMA 5.1. *Suppose that $(\hat{\mathcal{G}}, \pi)$ is a covering of $\mathcal{G}$. Then, $(\hat{\mathcal{G}}, \pi)$ is a universal covering of $\mathcal{G}$ if and only if $\hat{\mathcal{G}}$ is centrally closed.*

Our goal in the rest of the paper is to compute $\mathrm{H}_2\big(\mathrm{stu}_n(\mathcal{A}, -, \gamma)\big)$. If $n \geq 5$, it is shown in [G, Theorem 2.37] that $\mathrm{H}_2\big(\mathrm{stu}_n(\mathcal{A}, -, \gamma)\big) = \{0\}$. In this section, we will show that if $n = 3$ then we also have $\mathrm{H}_2\big(\mathrm{stu}_3(\mathcal{A}, -, \gamma)\big) = \{0\}$. In the final section, we will treat the case when $n = 4$.

For the rest of the section then we will assume that $n = 3$ and that $(\mathcal{A}, -)$ is a structurable algebra. For brevity, we put

$$\mathcal{G} = \mathrm{stu}_3(\mathcal{A}, -, \gamma).$$

In order to prove that $\mathrm{H}_2(\mathcal{G}) = \{0\}$, it is necessary and sufficient, by Lemma 5.1, to prove that $\mathcal{G}$ is centrally closed.

LEMMA 5.2. *Let* $\mathcal{M} = u_{13}(\mathcal{A}) \oplus u_{23}(\mathcal{A})$. *Then,*

$$[[\mathcal{M},\mathcal{M}],\mathcal{M}] \subseteq \mathcal{M} \tag{5.3}$$

and

$$\mathcal{G} = \mathcal{M} \oplus [\mathcal{M},\mathcal{M}]. \tag{5.4}$$

PROOF. From (stu3) and Proposition 2.1 clearly $[\mathcal{M},\mathcal{M}] = \mathcal{T} \oplus u_{12}(\mathcal{A})$ and thus (5.3)–(5.4) follow. ∎

Now put $D = ad\big(u_{12}(1)\big)$ and $\delta = \gamma_1\gamma_2^{-1}$. Then

$$D\big(u_{13}(a) + u_{23}(b)\big) = -\delta u_{23}(a) + u_{13}(b) = u_{13}(b) - \delta u_{23}(a),$$

and hence

$$D^2\big(u_{13}(a) + u_{23}(b)\big) = -\delta u_{13}(a) - \delta u_{23}(b).$$

So we have $D^2|_{\mathcal{M}} = -\delta I$. In otherwords

$$(D^2 + \delta I)|_{\mathcal{M}} = 0. \tag{5.5}$$

We claim

LEMMA 5.6. *The restriction of* $D(D^2 + 4\delta I)$ *to* $[\mathcal{M},\mathcal{M}]$ *is zero.*

PROOF. For this we can extend the base field and assume that $-\delta = \alpha^2$ for some $\alpha \neq 0 \in k$. Hence, by (5.5), $D^2 - \alpha^2 I|_{\mathcal{M}} = 0$. It follows that $D|_{\mathcal{M}}$ is diagonalizable with eigenvalues $\pm\alpha$ so that we can write

$$\mathcal{M} = \mathcal{M}_\alpha \oplus \mathcal{M}_{-\alpha},$$

where $\mathcal{M}_\alpha$, $\mathcal{M}_{-\alpha}$ are the eigenspaces for $D|_{\mathcal{M}}$. But then

$$[\mathcal{M},\mathcal{M}] = [\mathcal{M}_\alpha,\mathcal{M}_\alpha] + [\mathcal{M}_\alpha,\mathcal{M}_{-\alpha}] + [\mathcal{M}_{-\alpha},\mathcal{M}_{-\alpha}].$$

Therefore,

$$(D - 2\alpha I)D(D + 2\alpha I)|_{[\mathcal{M},\mathcal{M}]} = 0$$

which gives the desired result. ∎

It follows from (5.4)–(5.6) that

$$D(D^2 + \delta I)(D^2 + 4\delta I) = 0. \tag{5.7}$$

Moreover, the polynomials λ, $\lambda^2 + \delta$ and $\lambda^2 + 4\delta$ are pairwise relatively prime. Thus, we have

$$\mathcal{G} = \mathcal{G}_{(\lambda^2+\delta)} \oplus \mathcal{G}_{(\lambda)} \oplus \mathcal{G}_{(\lambda^2+4\delta)}, \tag{5.8}$$

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where $\mathcal{G}_{(p(\lambda))}$ is the null space of $p(D)$,

$$\mathcal{M} = \mathcal{G}_{(\lambda^2+\delta)}, \tag{5.9}$$

and

$$[\mathcal{M}, \mathcal{M}] = \mathcal{G}_{(\lambda)} \oplus \mathcal{G}_{(\lambda^2+4\delta)}. \tag{5.10}$$

Now, in order to prove that $\mathcal{G}$ is centrally closed, we assume that $(\tilde{\mathcal{G}}, \pi)$ is a central extension of $\mathcal{G}$ with kernel $\mathcal{V}$. Thus, $\mathcal{V}$ is contained in the center of $\tilde{\mathcal{G}}$. Our goal is to show that the homomorphism π splits.

Choose $\tilde{x} \in \tilde{\mathcal{G}}$ so that $\pi(\tilde{x}) = u_{12}(1)$. Let $\tilde{D} = ad(\tilde{x})$. Then, by (5.7),

$$\tilde{D}(\tilde{D}^2 + \delta I)(\tilde{D}^2 + 4\delta I)\tilde{\mathcal{G}} \subseteq \mathcal{V}.$$

Since $\mathcal{V}$ is central and so $\tilde{D}\mathcal{V} = 0$, we have

$$\tilde{D}^2(\tilde{D}^2 + \delta I)(\tilde{D}^2 + 4\delta I) = 0.$$

But the polynomials λ^2, $\lambda^2 + \delta$ and $\lambda^2 + 4\delta$ are pairwise relatively prime. So we obtain

$$\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_{(\lambda^2+\delta)} \oplus \tilde{\mathcal{G}}_{(\lambda^2)} \oplus \tilde{\mathcal{G}}_{(\lambda^2+4\delta)}, \tag{5.11}$$

where $\tilde{\mathcal{G}}_{p(\lambda)}$ is the null space of $p(\tilde{D})$.

Now clearly $\pi\big(\tilde{\mathcal{G}}_{(\lambda^2+\delta)}\big) \subseteq \mathcal{G}_{(\lambda^2+\delta)}$ and $\pi\big(\tilde{\mathcal{G}}_{(\lambda^2+4\delta)}\big) \subseteq \mathcal{G}_{(\lambda^2+4\delta)}$. Moreover, it follows from (5.8) that $\mathcal{G}_{(\lambda)} = \mathcal{G}_{(\lambda^2)}$. Hence, we have

$$\pi\big(\tilde{\mathcal{G}}_{(\lambda^2)}\big) \subseteq \mathcal{G}_{(\lambda^2)} = \mathcal{G}_{(\lambda)}.$$

As π is onto it follows from (5.8) and (5.11) that

$$\pi(\tilde{\mathcal{G}}_{(\lambda^2+4\delta)}) = \mathcal{G}_{(\lambda^2+4\delta)}, \tag{5.12}$$

$$\pi(\tilde{\mathcal{G}}_{(\lambda^2+\delta)}) = \mathcal{G}_{(\lambda^2+\delta)}, \tag{5.13}$$

$$\pi(\tilde{\mathcal{G}}_{(\lambda^2)}) = \mathcal{G}_{(\lambda)}. \tag{5.14}$$

So now we put $\tilde{\mathcal{M}} = \tilde{\mathcal{G}}_{(\lambda^2+\delta)}$, then, by (5.9) and (5.13), we have

$$\pi(\tilde{\mathcal{M}}) = \mathcal{M} = u_{13}(\mathcal{A}) \oplus u_{23}(\mathcal{A}). \tag{5.15}$$

LEMMA 5.16. $\big[[\tilde{\mathcal{M}}, \tilde{\mathcal{M}}], \tilde{\mathcal{M}}\big] \subseteq \tilde{\mathcal{M}}$.

PROOF. For this we can assume again that $-\delta = \alpha^2$, where $\alpha \neq 0 \in k$. Thus, $\tilde{D}^2 + \delta I|_{\tilde{\mathcal{M}}} = 0$ implies $(\tilde{D} - \alpha I)(\tilde{D} + \alpha I)|_{\tilde{\mathcal{M}}} = 0$. so we have

$$\tilde{\mathcal{M}} = \tilde{\mathcal{M}}_\alpha \oplus \tilde{\mathcal{M}}_{-\alpha},$$

where $\tilde{\mathcal{M}}_{\pm\alpha}$ are the eigenspaces of $\tilde{D}|_{\tilde{\mathcal{M}}}$. It follows that

$$\begin{aligned}\big[[\tilde{\mathcal{M}},\tilde{\mathcal{M}}],\tilde{\mathcal{M}}\big] = \big[[\tilde{\mathcal{M}}_\alpha,\tilde{\mathcal{M}}_\alpha],\tilde{\mathcal{M}}_\alpha\big] + \big[[\tilde{\mathcal{M}}_\alpha,\tilde{\mathcal{M}}_\alpha],\tilde{\mathcal{M}}_{-\alpha}\big] + \big[[\tilde{\mathcal{M}}_\alpha,\tilde{\mathcal{M}}_{-\alpha}],\tilde{\mathcal{M}}_\alpha\big]\\ + \big[[\tilde{\mathcal{M}}_\alpha,\tilde{\mathcal{M}}_{-\alpha}],\tilde{\mathcal{M}}_{-\alpha}\big] + \big[[\tilde{\mathcal{M}}_{-\alpha},\tilde{\mathcal{M}}_{-\alpha}],\tilde{\mathcal{M}}_\alpha\big]\\ + \big[[\tilde{\mathcal{M}}_{-\alpha},\tilde{\mathcal{M}}_{-\alpha}],\tilde{\mathcal{M}}_{-\alpha}\big].\end{aligned}$$

For any $x,y,z \in \tilde{\mathcal{M}}$, it follows from (5.3) and (5.15) that there is a $w \in \tilde{\mathcal{M}}$ such that $\big[[x,y],z\big] - w \in \mathcal{V}$. If $x,y,z \in \tilde{\mathcal{M}}_\alpha$, we have

$$\tilde{D}w = \tilde{D}\big[[x,y],z\big] = 3\alpha\big[[x,y],z\big] \in \tilde{\mathcal{M}},$$

and so $\big[[x,y],z\big] = (3\alpha)^{-1}\tilde{D}w \in \tilde{D}\tilde{\mathcal{M}} \subseteq \tilde{\mathcal{M}}$. A similar argument covers the other cases for x,y,z and so we get (5.16). ∎

LEMMA 5.17. *We can choose preimages $\tilde{u}_{ij}(a)$ of $u_{ij}(a)$ in $\tilde{\mathcal{G}}$ under π so that*
(i) $a \longmapsto \tilde{u}_{ij}(a)$ is linear,
(ii) $\tilde{u}_{ij}(a) = -\gamma_i\gamma_j^{-1}\tilde{u}_{ji}(\bar{a})$,
(iii) $[\tilde{u}_{12}(a),\tilde{u}_{23}(b)] = \tilde{u}_{13}(ab)$, and
(iv) $[\tilde{u}_{21}(a),\tilde{u}_{13}(b)] = \tilde{u}_{23}(ab)$,
for distinct i,j and $a,b \in \mathcal{A}$.

PROOF. By (5.15), using a basis for $\mathcal{A}$, we can choose $\tilde{u}_{13}(a)$, $\tilde{u}_{23}(b) \in \tilde{\mathcal{M}}$ so that

$$\pi\big(\tilde{u}_{13}(a)\big) = u_{13}(a), \quad \pi\big(\tilde{u}_{23}(b)\big) = u_{23}(b),$$

and (i) holds for $(i,j) = (1,3)$ and $(2,3)$. Then choose $\tilde{u}_{31}(a)$ and $\tilde{u}_{32}(b)$ so that (ii) holds for $(i,j) = (1,3)$ and $(2,3)$. Put $\tilde{u}_{12}(a) = [\tilde{u}_{13}(a),\tilde{u}_{32}(1)]$ and $\tilde{u}_{21}(a) = -\gamma_2\gamma_1^{-1}\tilde{u}_{12}(\bar{a})$. Thus we have (i) and (ii).

Now

$$[\tilde{u}_{12}(a),\tilde{u}_{23}(b)] = \big[[\tilde{u}_{13}(a),\tilde{u}_{32}(1)],\tilde{u}_{23}(b)\big] = -\gamma_3\gamma_2^{-1}\big[[\tilde{u}_{13}(a),\tilde{u}_{23}(1)],\tilde{u}_{23}(b)\big]$$

and so by Lemma 5.16, $[\tilde{u}_{12}(a),\tilde{u}_{23}(b)] \in \tilde{\mathcal{M}}$. On the other hand,

$$[\tilde{u}_{12}(a),\tilde{u}_{23}(b)] = \tilde{u}_{13}(ab) + v(a,b)$$

for some $v(a,b) \in \mathcal{V}$. Hence,

$$v(a,b) = [\tilde{u}_{12}(a),\tilde{u}_{23}(b)] - \tilde{u}_{13}(ab) \in \tilde{\mathcal{M}} = \tilde{\mathcal{G}}_{(\lambda^2+\delta_3)}.$$

But of course $\tilde{D}v(a,b) = 0$ and so $v(a,b) \in \tilde{\mathcal{G}}_{(\lambda^2)}$. It follows that

$$v(a,b) \in \tilde{\mathcal{G}}_{(\lambda^2+\delta)} \cap \tilde{\mathcal{G}}_{(\lambda^2)} = (0),$$

and so we have (iii). Similarly,

$$[\tilde{u}_{21}(a),\tilde{u}_{13}(b)] = -\gamma_2\gamma_1^{-1}[\tilde{u}_{12}(\bar{a}),\tilde{u}_{13}(b)] = -\gamma_2\gamma_1^{-1}\big[[\tilde{u}_{13}(\bar{a}),\tilde{u}_{32}(1)],\tilde{u}_{13}(b)\big],$$

and so as above $[\tilde{u}_{21}(a),\tilde{u}_{13}(b)] \in \tilde{\mathcal{M}}$. Again,

$$[\tilde{u}_{21}(a),\tilde{u}_{13}(b)] = \tilde{u}_{23}(ab) + v'(a,b)$$

for some $v'(a,b) \in \mathcal{V}$. Then, $v'(a,b) \in \tilde{\mathcal{M}}$ and $v'(a,b) = 0$ as above. This completes the proof. ∎

Now we are in a position to prove the following:

THEOREM 5.18. *If* $(\mathcal{A}, -)$ *is a structurable algebra, then* $\mathrm{stu}_3(\mathcal{A}, -, \gamma)$ *is centrally closed and hence*

$$\mathrm{H}_2\big(\mathrm{stu}_3(\mathcal{A}, -, \gamma)\big) = \{0\}.$$

PROOF. As we have already noted, it is enough to show that the central extension $(\tilde{\mathcal{G}}, \pi)$ of $\mathcal{G}$ splits.

Let $\tilde{u}_{ij}(a)$ be the preimage of $u_{ij}(a)$ in $\tilde{\mathcal{G}}$ under π chosen as in Lemma 5.17. Also, replacing $\mathcal{M} = u_{13}(\mathcal{A}) \oplus u_{23}(\mathcal{A})$ by $\mathcal{M} = u_{12}(\mathcal{A}) \oplus u_{32}(\mathcal{A})$ (correspondingly, $D = ad\big(u_{12}(1)\big)$ by $D = ad\big(u_{13}(1)\big)$) in the procedure (5.2) through (5.17), we can choose $\tilde{u}'_{ij}(a)$ as the preimage of $u_{ij}(a)$ in $\tilde{\mathcal{G}}$ under π satisfying

$$\begin{gathered} a \longmapsto \tilde{u}'_{ij}(a) \text{ is linear,} \\ \tilde{u}'_{ij}(a) = -\gamma_i\gamma_j^{-1}\tilde{u}'_{ji}(\bar{a}), \\ [\tilde{u}'_{13}(a), \tilde{u}'_{32}(b)] = \tilde{u}'_{12}(ab), \quad \text{and} \\ [\tilde{u}'_{31}(a), \tilde{u}'_{12}(b)] = \tilde{u}'_{32}(ab). \end{gathered}$$

Note that $\pi\big(\tilde{u}_{ij}(a)\big) = u_{ij}(a) = \pi\big(\tilde{u}'_{ij}(a)\big)$, so we have $\tilde{u}_{ij}(a) - \tilde{u}'_{ij}(a) \in \mathcal{V}$. Therefore, if we choose $\tilde{u}''_{12}(a) = \tilde{u}'_{12}(a)$, $\tilde{u}''_{13}(a) = \tilde{u}_{13}(a)$ and $\tilde{u}''_{23}(a) = \tilde{u}_{23}(a)$, and then choose $\tilde{u}''_{21}(a)$, $\tilde{u}''_{31}(a)$ and $\tilde{u}''_{32}(a)$ so that (stu1) holds, then the elements $\tilde{u}''_{ij}(a)$ for $a \in \mathcal{A}$, $1 \leq i \neq j \leq 3$ satisfy the defining relations (stu1)–(stu3). Actually we get (stu3) first for $(i,j,k) = (1,2,3), (2,1,3)$ and $(1,3,2)$. The other cases of (stu3) follow from these using (stu1) and (stu2). Then by the universal property of $\mathrm{stu}_3(\mathcal{A}, -, \gamma)$, there exists a (unique) Lie algebra homomorphism

$$\psi\colon \mathrm{stu}_3(\mathcal{A}, -, \gamma) \longrightarrow \tilde{\mathcal{G}}$$

such that $\psi\big(u_{ij}(a)\big) = \tilde{u}''_{ij}(a)$. Evidently, $\pi \circ \psi = id$ which implies that the original homomorphism splits. So $\mathrm{stu}_3(\mathcal{A}, -, \gamma)$ is centrally closed. ∎

In view of Theorem 4.24, we obtain the following corollary of Theorem 5.18:

COROLLARY 5.19. *The pair* $\big(\mathrm{stu}_3(\mathcal{A}, -, \gamma), \nu\big)$ *(defined in Section 4) is a universal covering of* $\mathcal{K}(\mathcal{A}, -, \gamma)$ *and hence*

$$\mathrm{H}_2\big(\mathcal{K}(\mathcal{A}, -, \gamma)\big) \cong \mathrm{HF}(\mathcal{A}, -).$$

REMARK 5.20. Theorem 5.18 generalizes a result in [G] in which it is assumed that $(\mathcal{A}, -)$ is associative and $\mathcal{A}_-$ contains an invertible central element.

REMARK 5.21. If $\gamma = \mathrm{diag}(1, -1, 1)$, then $\mathcal{K}(\mathcal{A}, -, \gamma) \cong \mathcal{K}(\mathcal{A}, -)$, where $\mathcal{K}(\mathcal{A}, -)$ is the $\mathbb{Z}$-graded Lie algebra obtained from $(\mathcal{A}, -)$ using the Kantor construction [A2, Theorem 2.2]. Thus, it follows from Corollary 5.19 that

$$\mathrm{H}_2\big(\mathcal{K}(\mathcal{A}, -)\big) \cong \mathrm{HF}(\mathcal{A}, -). \tag{5.22}$$

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In particular, if $\mathcal{A}$ is a linear Jordan algebra and $-$ is the identity involution, then $\mathcal{K}(\mathcal{A},-) = \mathcal{K}(\mathcal{A})$, where $\mathcal{K}(\mathcal{A})$ is the classical Tits-Kantor-Koecher Lie algebra constructed from $\mathcal{A}$ [J, Section 8.5], and hence (5.22) gives a determination of $\mathrm{H}_2\big(\mathcal{K}(\mathcal{A})\big)$.

REMARK 5.23. Suppose that $\mathcal{B}$ is an alternative algebra. Let $\mathrm{st}_3(\mathcal{B})$ be the Steinberg Lie algebra of $\mathcal{B}$ (see for example [AF, Section 7] or [BGKN, Section 2] for the definition of $\mathrm{st}_3(\mathcal{B})$), and let $\mathrm{psl}_3(\mathcal{B})$ be the quotient of $\mathrm{st}_3(\mathcal{B})$ by its center. The second homology group of the Lie algebra $\mathrm{psl}_3(\mathcal{B})$ has been calculated in [BGKN, Section 2]. The same result can also be obtained from Corollary 5.19 using the fact that $\mathrm{st}_3(\mathcal{B})$ is isomorphic to $\mathrm{stu}_3(\mathcal{B}\oplus\mathcal{B}^{\mathrm{op}}, \mathrm{ex}, I_3)$, where $\mathcal{B}^{\mathrm{op}}$ is the opposite algebra of $\mathcal{B}$ and ex is the exchange involution [AF, Theorem 7.1].

6. **Coverings of** $\mathrm{stu}_4(\mathcal{A},-,\gamma)$**.** Throughout this section we assume that $(\mathcal{A},-)$ is associative and (except in the final corollary and example) that n=4. We will compute $\mathrm{H}_2\big(\mathrm{stu}_4(\mathcal{A},-,\gamma)\big)$.

Let J be the subspace of $\mathcal{A}\otimes_k\mathcal{A}$ spanned by the following elements:

$$a\otimes b + b\otimes a,$$
$$\bar{a}b\otimes c + \bar{b}c\otimes a + \bar{c}a\otimes b,$$
$$\big(c(ab-\overline{ab}) + (ba-\overline{ba})c\big)\otimes d,$$

for $a,b,c,d\in\mathcal{A}$. Define $\mathrm{L}(\mathcal{A},-) := \frac{\mathcal{A}\otimes_k\mathcal{A}}{J}$ to be the quotient space and write $\ell(a,b) = a\otimes b + J$. Then

$$\ell(a,b) + \ell(b,a) = 0, \tag{6.1}$$
$$\ell(\bar{a}b,c) + \ell(\bar{b}c,a) + \ell(\bar{c}a,b) = 0, \tag{6.2}$$
$$\ell\big(c(ab-\overline{ab}) + (ba-\overline{ba})c, d\big) = 0. \tag{6.3}$$

Next we collect some identities which can be easily derived from (6.1)–(6.3).

LEMMA 6.4. *For $a,b,c,d\in\mathcal{A}$, we have*

$$\ell(a,b) = \ell(a,\bar{b}), \tag{6.5}$$
$$\ell(ab,c) + \ell(\bar{b}c,a) + \ell(ac,b) = 0, \tag{6.6}$$
$$\ell\big((c-\bar{c})a + a(c-\bar{c}), b\big) = 0, \tag{6.7}$$
$$\ell\big(a,(c-\bar{c})b\big) - \ell\big((c-\bar{c})a, b\big) = 0, \tag{6.8}$$
$$\ell([c,d]a + [\bar{c},\bar{d}]a, b) = 0. \tag{6.9}$$

PROOF. Taking $b=c=1$ in (6.2) gives $\ell(\bar{a},1)=0$ and so $\ell(1,a)=0$. Now taking $c=1$ in (6.2) and using (6.1) yields (6.5). Substituting $\bar{a}$ by a in (6.2) and then using (6.5) gives (6.6). (6.7) follows from (6.3). Replacing c by $\bar{c}$ in (6.6), we have

$$\ell(ab,\bar{c}) + \ell(\bar{b}\bar{c},a) + \ell(a\bar{c},b) = 0.$$

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Combining this with (6.6) and using (6.5) and (6.7) gives (6.8). Finally, by (6.7)

$$\ell\big(c(ab-\overline{ab})+(ab-\overline{ab})c, d\big)=0,$$

and combining this with (6.3) gives (6.9). ∎

REMARK 6.10. If $\mathcal{A}$ is commutative, then (6.3) can be deduced from (6.1) and (6.2). Let

$$\mathcal{G}=\mathrm{stu}_4(\mathcal{A},-,\gamma).$$

Our goal is to show that $\mathrm{H}_2(\mathcal{G})=L(\mathcal{A},-)$.

By [AF, p. 3], we know that $\mathcal{G}$ is $\mathbb{Z}_2^4$-graded Lie algebra such that $\deg\big(u_{ij}(a)\big)=\epsilon_i+\epsilon_j$, where $\epsilon_i=(0,\cdots,1,\cdots,0)$ with 1 in the i-th place. Moreover, by Proposition 2.1,

$$\mathcal{G}=\mathrm{stu}_4(\mathcal{A},-,\gamma)=\mathcal{G}_0\oplus\coprod_{1\le i<j\le 4}\mathcal{G}_{\epsilon_i+\epsilon_j}$$

where

$$\mathcal{G}_0=\mathcal{T}=\sum_{1\le i<j\le 4}[u_{ij}(\mathcal{A}),u_{ji}(\mathcal{A})]\qquad\text{and}\qquad \mathcal{G}_{\epsilon_i+\epsilon_j}=u_{ij}(\mathcal{A}).$$

Thus, by Proposition 3.7 (b), we have

$$\mathcal{G}_0=t(\mathcal{A},\mathcal{A})\oplus T_{12}(1,\mathcal{A}_-)\oplus T_{13}(1,\mathcal{A}_-)\oplus T_{14}(1,\mathcal{A}_-). \tag{6.11}$$

We now define an anti-commutative bilinear bracket on the vector space

$$\hat{\mathcal{G}}=\mathrm{L}(\mathcal{A},-)\oplus\mathrm{stu}_4(\mathcal{A},-,\gamma)$$

by

$$[\mathrm{L}(\mathcal{A},-),\hat{\mathcal{G}}]=(0), \tag{6.12}$$

$$[x,y]=\text{ the product }[x,y]\text{ in }\mathcal{G}\text{ for }x\in\mathcal{G}_\alpha,\quad y\in\mathcal{G}_\beta,\quad \alpha+\beta\neq\epsilon, \tag{6.13}$$

$$[u_{12}(a),u_{34}(b)]=\ell(a,b), \tag{6.14}$$

$$[u_{24}(a),u_{13}(b)]=-\gamma_3^{-1}\gamma_2\ell(a,b),\quad\text{and} \tag{6.15}$$

$$[u_{32}(a),u_{14}(b)]=-\ell(a,b), \tag{6.15}$$

where

$$\epsilon=\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4.$$

Then $\hat{\mathcal{G}}$ is a $\mathbb{Z}_2^4$-graded algebra with $\hat{\mathcal{G}}_0=\mathcal{G}_0$, $\hat{\mathcal{G}}_{\epsilon_i+\epsilon_j}=\mathcal{G}_{\epsilon_i+\epsilon_j}$ and $\hat{\mathcal{G}}_\epsilon=\mathrm{L}(\mathcal{A},-)$.

PROPOSITION 6.17. *$\hat{\mathcal{G}}$ is a Lie algebra.*

PROOF. Since $\mathcal{G}$ is a Lie algebra, to prove the Jacobi identity in $\hat{\mathcal{G}}$ it suffices to check $J(x,y,z)=0$ for $\deg(x)+\deg(y)+\deg(z)=\epsilon$. We can also assume that $\deg(x)$, $\deg(y)$ and $\deg(z)$ are not equal to ϵ. This leaves only two possibilities:

Case 1: $\deg(x)=\epsilon_i+\epsilon_j$, $\deg(y)=\epsilon_k+\epsilon_l$ for distinct i,j,k,l, and $\deg(z)=0$;

Case 2: $\deg(x)=\epsilon_i+\epsilon_j$, $\deg(y)=\epsilon_i+\epsilon_k$, $\deg(z)=\epsilon_i+\epsilon_l$ for distinct i,j,k,l.

For Case 1, we assume that $\{i,j\} = \{1,2\}$. We omit the other 2 subcases when $\{i,j\} = \{1,3\}$ or $\{1,4\}$ since they are very similar (although not identical). Thus, we suppose that $x = u_{12}(a)$ and $y = u_{34}(b)$, where $a, b \in \mathcal{A}$. By (6.11), we can assume that either $z = t(c,d)$, where $c, d \in \mathcal{A}$, or $z = T_{1j}(c)$, where $2 \leq j \leq 4$ and $c \in \mathcal{A}$. When $z = t(c,d)$, then, by (2.10), (2.11) and (6.9), we have

$$J(x,y,z) = \big[[t(c,d), u_{12}(a)], u_{34}(b)\big] = \ell([c,d]a + [\bar{c},\bar{d}]a, b) = 0.$$

When $z = T_{12}(1,c)$, by (2.3) and (6.7),

$$J(x,y,z) = \big[[T_{12}(1,c), u_{12}(a)], u_{34}(b)\big] = \ell\big((c-\bar{c})a + a(c-\bar{c}), b\big) = 0.$$

When $z = T_{13}(1,c)$, by (2.2), (2.5) and (6.8),

$$\begin{aligned} J(x,y,z) &= \big[[u_{34}(b), T_{13}(1,c)], u_{12}(a)\big] + [T_{13}(1,c), u_{12}(a)], u_{34}(b)\big] \\ &= \big[u_{34}\big((c-\bar{c})b\big), u_{12}(a)\big] + \big[u_{12}\big((c-\bar{c})a\big), u_{34}(b)\big] \\ &= -\ell\big(a, (c-\bar{c})b\big) + \ell\big((c-\bar{c})a, b\big) = 0. \end{aligned}$$

When $z = T_{14}(1,c)$, by (2.2), (2.5), (6.7) and (6.8),

$$\begin{aligned} J(x,y,z) &= \big[[u_{34}(b), T_{14}(1,c)], u_{12}(a)\big] + \big[[T_{14}(1,c), u_{12}(a)], u_{34}(b)\big] \\ &= \ell\big(a, b(c-\bar{c})\big) + \ell\big((c-\bar{c})a, b\big) = 0. \end{aligned}$$

For Case 2, we assume that $i = 1$. We omit the other 3 subcases since they are very similar. So we suppose that $x = u_{12}(a)$, $y = u_{13}(b)$ and $z = u_{14}(c)$, where $a, b \in \mathcal{A}$. Then, using (6.5) and (6.6),

$$\begin{aligned} J(x,y,z) &= \big[[u_{12}(a), u_{13}(b)], u_{14}(c)\big] + \big[[u_{13}(b), u_{14}(c)], u_{12}(a)\big] \\ &\quad + \big[[u_{14}(c), u_{12}(a)], u_{13}(b)\big] \\ &= \gamma_1\gamma_3^{-1}[u_{32}(\bar{b}a), u_{14}(c)] - \gamma_1\gamma_3^{-1}[u_{34}(\bar{b}c), u_{12}(a)] \\ &\quad + \gamma_1\gamma_2^{-1}[u_{24}(\bar{a}c), u_{13}(b)] \\ &= \gamma_1\gamma_3^{-1}\big(-\ell(\bar{b}a, c) + \ell(a, \bar{b}c) - \ell(\bar{a}c, b)\big) \\ &= \gamma_1\gamma_3^{-1}\big(-\ell(\bar{a}b, c) - \ell(\bar{b}c, \bar{a}) - \ell(\bar{a}c, b)\big) = 0 \end{aligned}$$

■

Define $\pi \colon \hat{\mathcal{G}} \longrightarrow \mathcal{G}$ by $\pi\big(\mathrm{L}(\mathcal{A}, -)\big) = (0)$ and $\pi|_{\mathcal{G}} = \mathrm{id}$. Then, it follows from Proposition 6.17 that $(\hat{\mathcal{G}}, \pi)$ is a covering of $\mathcal{G}$. We will show that $(\hat{\mathcal{G}}, \pi)$ is the universal covering of $\mathcal{G}$. To do this, we define a Lie algebra $\mathcal{G}^\sharp$ to be the Lie algebra generated by the symbols $u^\sharp_{ij}(a)$, $a \in \mathcal{A}$, $1 \leq i \neq j \leq 4$ and the vector space $\mathrm{L}(\mathcal{A}, -)$, subject to the relations:

(stu1$^\sharp$) $$u^\sharp_{ij}(a) = u^\sharp_{ji}(-\gamma_i\gamma_j^{-1}\bar{a}),$$

(stu2$^\sharp$) $$a \longmapsto u^\sharp_{ij}(a) \text{ is a } k\text{-linear mapping},$$

(stu3$^\sharp$) $$[u^\sharp_{ij}(a), u^\sharp_{jk}(b)] = u^\sharp_{ik}(ab), \qquad \text{for distinct } i, j, k,$$

(stu4$^\sharp$) $$[\mathrm{L}(\mathcal{A}, -), u^\sharp_{ij}(a)] = 0, \qquad \text{for distinct } i, j,$$

(stu5$^\sharp$) $$[u^\sharp_{12}(a), u^\sharp_{34}(b)] = \ell(a,b),$$

(stu6$^\sharp$) $$[u^\sharp_{24}(a), u^\sharp_{13}(b)] = -\gamma_3^{-1}\gamma_2\ell(a,b),$$

(stu7$^\sharp$) $$[u^\sharp_{32}(a), u^\sharp_{14}(b)] = -\ell(a,b).$$

where $a, b \in \mathcal{A}$, $1 \leq i, j, k \leq 4$. Clearly, there is a unique Lie algebra homomorphism $\psi \colon \mathcal{G}^\sharp \longrightarrow \hat{\mathcal{G}}$ such that $\psi\big(u^\sharp_{ij}(a)\big) = u_{ij}(a)$. We claim that ψ is actually an isomorphism.

LEMMA 6.18. *$\psi \colon \mathcal{G}^\sharp \longrightarrow \hat{\mathcal{G}}$ is an isomorphism.*

PROOF. Let $T^\sharp_{ij}(a,b) = [u^\sharp_{ij}(a), u^\sharp_{ji}(b)]$. Then one can easily check that for $a, b \in \mathcal{A}$ and distinct i, j, k, one has

(i) $T^\sharp_{ij}(a,b) = -T^\sharp_{ji}(b,a) = T^\sharp_{ji}(\bar{a},\bar{b})$,

(ii) $T^\sharp_{ij}(ab,c) = T^\sharp_{ik}(a,bc) + T^\sharp_{kj}(b,ca)$,

(iii) $T^\sharp_{kj}(1,c) = -T^\sharp_{jk}(1,c) = T^\sharp_{kj}(c,1)$, and

(iv) $T^\sharp_{ij}(1,a) = 0$, if $a \in \mathcal{A}_+$.

Indeed, the proof of (i)–(iv) is the same as the proof of [G, Proposition 2.17]. Put $t^\sharp(a,b) = T^\sharp_{1j}(a,b) - T^\sharp_{1j}(1,ba)$ for $a, b \in \mathcal{A}$, $2 \leq j \leq 4$. Then, as in the discussion following [G, Proposition 2.17], it follows that $t^\sharp(a,b)$ does not depend on the choice of j. Also, one easily checks (as in [AF, Lemma 1.1]) that

$$\mathcal{G}^\sharp = \mathcal{T}^\sharp + \sum_{1 \leq i < j \leq 4} u^\sharp_{ij}(\mathcal{A}),$$

where

$$\mathcal{T}^\sharp := \sum_{1 \leq i < j \leq 4} [u^\sharp_{ij}(\mathcal{A}), u^\sharp_{ji}(\mathcal{A})].$$

It then follows from (i)–(iv) above that

$$\mathcal{T}^\sharp = t^\sharp(\mathcal{A},\mathcal{A}) + T^\sharp_{12}(1,\mathcal{A}_-) + T^\sharp_{13}(1,\mathcal{A}_-) + T^\sharp_{14}(1,\mathcal{A}_-)$$

where $t^\sharp(\mathcal{A},\mathcal{A})$ is the linear span of the elements $t^\sharp(a,b)$. So by Proposition 3.7 (a) it suffices to show that the restriction of ψ to $t^\sharp(\mathcal{A},\mathcal{A})$ is injective.

Now the argument given in the proof of Proposition 2.12 shows that the elements $t^\sharp(a,b)$, $a, b \in \mathcal{A}$, satisfy the relations (1.14)–(1.16). Thus, it follows from Proposition 3.7 (c) that there exists a linear map from $t(\mathcal{A},\mathcal{A})$ to $t^\sharp(\mathcal{A},\mathcal{A})$ so that $t(a,b) \longrightarrow t^\sharp(a,b)$ for $a, b \in \mathcal{A}$. This map is the inverse of the restriction of ψ to $t^\sharp(\mathcal{A},\mathcal{A})$. ∎

THEOREM 6.19. *$(\hat{\mathcal{G}}, \pi)$ is the universal covering of* $\mathrm{stu}_4(\mathcal{A}, -, \gamma)$ *and hence*

$$\mathrm{H}_2\big(\mathrm{stu}_4(\mathcal{A}, -, \gamma)\big) \cong \mathrm{L}(\mathcal{A}, -).$$

PROOF. Suppose that

$$0 \longrightarrow \mathcal{V} \longrightarrow \tilde{\mathcal{G}} \xrightarrow{\tau} \mathrm{stu}_4(\mathcal{A}, -, \gamma) \longrightarrow 0$$

is a central extension of $\mathrm{stu}_4(\mathcal{A}, -, \gamma)$. We must show that there exists a Lie algebra homomorphism $\lambda \colon \hat{\mathcal{G}} \longrightarrow \tilde{\mathcal{G}}$ so that $\tau \circ \lambda = \pi$. Thus, by Lemma 6.18, it suffices to show that there exists a Lie algebra homomorphism $\xi \colon \mathcal{G}^\sharp \longrightarrow \tilde{\mathcal{G}}$ so that $\tau \circ \xi = \pi \circ \psi$.

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Using a basis for $\mathcal{A}$, we choose a preimage $\tilde{u}_{ij}(a)$ of $u_{ij}(a)$ under τ, $1 \leq i \neq j \leq 4$, $a \in \mathcal{A}$, so that the elements $\tilde{u}_{ij}(a)$ satisfy the relations (stu1$^\sharp$) and (stu2$^\sharp$). For distinct i,j,k, let

$$[\tilde{u}_{ij}(a), \tilde{u}_{jk}(b)] = \tilde{u}_{ik}(ab) + v^j_{ik}(a,b)$$

where $v^j_{ik}(a,b) \in \mathcal{V}$. Take $l \notin \{i,j,k\}$. Then

$$\big[\tilde{u}_{li}(c), [\tilde{u}_{ij}(a), \tilde{u}_{jk}(b)]\big] = [\tilde{u}_{li}(c), \tilde{u}_{ik}(ab)].$$

But the left hand side is, by the Jacobi identity,

$$\big[[\tilde{u}_{li}(c), \tilde{u}_{ij}(a)], \tilde{u}_{jk}(b)\big] + \big[\tilde{u}_{ij}(a), [\tilde{u}_{li}(c), \tilde{u}_{jk}(b)]\big] = [\tilde{u}_{lj}(ca), \tilde{u}_{jk}(b)]$$

as $[\tilde{u}_{li}(c), \tilde{u}_{jk}(b)] \in \mathcal{V}$. Thus

$$[\tilde{u}_{li}(c), \tilde{u}_{ik}(ab)] = [\tilde{u}_{lj}(ca), \tilde{u}_{jk}(b)]. \tag{6.20}$$

In particular, $[\tilde{u}_{li}(c), \tilde{u}_{ik}(b)] = [\tilde{u}_{lj}(c), \tilde{u}_{jk}(b)]$. It follows that $v^i_{lk}(c,b) = v^j_{lk}(c,b)$ which shows that v^i_{lk} is independent of the choice of i. Setting $v_{lk}(c,b) = v^i_{lk}(c,b)$, we have

$$[\tilde{u}_{li}(c), \tilde{u}_{ik}(b)] = \tilde{u}_{lk}(cb) + v_{lk}(c,b). \tag{6.21}$$

Taking $c = 1$, we have

$$[\tilde{u}_{li}(1), \tilde{u}_{ik}(b)] = \tilde{u}_{lk}(b) + v_{lk}(1,b). \tag{6.22}$$

Now for $l > k$, we replace $\tilde{u}_{lk}(b)$ by $\tilde{u}_{lk}(b) + v_{lk}(1,b)$, and then we rechoose $\tilde{u}_{kl}(b) = -\gamma_k\gamma_l^{-1}\tilde{u}_{lk}(\bar{b})$. Then, the new elements $\tilde{u}_{ij}(b)$ still satisfy the relations (stu1$^\sharp$) and (stu2$^\sharp$). Moreover, we have for $l > k$ that

$$[\tilde{u}_{li}(1), \tilde{u}_{ik}(b)] = \tilde{u}_{lk}(b). \tag{6.23}$$

We next check that (6.23) holds for $l < k$. In fact, using (6.20) and $k > l$ we have

$$\begin{aligned}[\tilde{u}_{li}(1), \tilde{u}_{ik}(b)] &= [\tilde{u}_{lj}(b), \tilde{u}_{jk}(1)] = [-\gamma_l\gamma_j^{-1}\tilde{u}_{jl}(\bar{b}), -\gamma_j\gamma_k^{-1}\tilde{u}_{kj}(1)] \\ &= -\gamma_l\gamma_k^{-1}[\tilde{u}_{kj}(1), \tilde{u}_{jl}(\bar{b})] = -\gamma_l\gamma_k^{-1}\tilde{u}_{kl}(\bar{b}) = \tilde{u}_{lk}(b).\end{aligned}$$

It follows from (6.20) and (6.23) that

$$[\tilde{u}_{lj}(a), \tilde{u}_{jk}(b)] = [\tilde{u}_{li}(1), \tilde{u}_{ik}(ab)] = \tilde{u}_{lk}(ab) \tag{6.24}$$

for $a, b \in \mathcal{A}$ and distinct l,j,k. Thus, the elements $\tilde{u}_{ij}(a)$ satisfy (stu3$^\sharp$).

Next for distinct i,j,k,l,

$$\begin{aligned}[\tilde{u}_{ij}(ab), \tilde{u}_{kl}(cd)] &= \big[[\tilde{u}_{ik}(a), \tilde{u}_{kj}(b)], \tilde{u}_{kl}(cd)\big] \\ &= -\gamma_k\gamma_j^{-1}[\tilde{u}_{ik}(a), \tilde{u}_{jl}(\bar{b}cd)] - [\tilde{u}_{kj}(b), \tilde{u}_{il}(acd)].\end{aligned} \tag{6.25}$$

Meanwhile,

$$\begin{aligned}[\tilde{u}_{ij}(ab), \tilde{u}_{kl}(cd)] &= \big[[\tilde{u}_{il}(a), \tilde{u}_{lj}(b)], \tilde{u}_{kl}(cd)\big] \\ &= [\tilde{u}_{kj}(cdb), \tilde{u}_{il}(a)] - \gamma_i\gamma_l^{-1}[\tilde{u}_{lj}(b), \tilde{u}_{ki}(cd\bar{a})].\end{aligned} \tag{6.26}$$

Also, using (6.25),

$$
\begin{aligned}
(6.27) \qquad [\tilde{u}_{ij}(ab), \tilde{u}_{kl}(cd)] &= -[\tilde{u}_{kl}(cd), \tilde{u}_{ij}(ab)] \\
&= \gamma_i \gamma_l^{-1}[\tilde{u}_{ki}(c), \tilde{u}_{lj}(\bar{d}ab)] + [\tilde{u}_{il}(d), \tilde{u}_{kj}(cab)].
\end{aligned}
$$

Taking $a = b = d = 1$ in (6.26) and (6.27) and subtracting gives us $2[\tilde{u}_{il}(1), \tilde{u}_{kj}(c)] = 0$. It follows that

$$(6.28) \qquad [\tilde{u}_{ij}(1), \tilde{u}_{kl}(a)] = [\tilde{u}_{ij}(a), \tilde{u}_{kl}(1)] = 0$$

for $a \in \mathcal{A}$ and all distinct i, j, k, l. But then taking $b = d = 1$ in (6.25), we get

$$(6.29) \qquad [\tilde{u}_{ik}(a), \tilde{u}_{jl}(c)] = -\gamma_k^{-1}\gamma_j[\tilde{u}_{ij}(a), \tilde{u}_{kl}(c)],$$

while taking $a = d = 1$ in (6.25), we obtain

$$(6.30) \qquad [\tilde{u}_{kj}(b), \tilde{u}_{il}(c)] = -[\tilde{u}_{ij}(b), \tilde{u}_{kl}(c)].$$

It follows from (6.29) and (6.30) that (6.25) and (6.26) become

$$
\begin{aligned}
(6.31) \qquad [\tilde{u}_{ij}(ab), \tilde{u}_{kl}(cd)] &= [\tilde{u}_{ij}(a), \tilde{u}_{kl}(\bar{b}cd)] + [\tilde{u}_{ij}(b), \tilde{u}_{kl}(acd)] \\
&= -[\tilde{u}_{ij}(cdb), \tilde{u}_{kl}(a)] - [\tilde{u}_{ij}(a\bar{d}\bar{c}), \tilde{u}_{kl}(\bar{b})].
\end{aligned}
$$

Letting $b = c = 1$ in (6.31) gives

$$(6.32) \qquad [\tilde{u}_{ij}(a), \tilde{u}_{kl}(d)] = -[\tilde{u}_{ij}(d), \tilde{u}_{kl}(a)],$$

while letting $d = 1$ in (6.31) gives

$$(6.33) \qquad [\tilde{u}_{ij}(ab), \tilde{u}_{kl}(c)] = [\tilde{u}_{ij}(a), \tilde{u}_{kl}(\bar{b}c)] + [\tilde{u}_{ij}(b), \tilde{u}_{kl}(ac)].$$

Finally, we have, using (2.3),

$$(6.34) \qquad \left[\tilde{u}_{ij}\big((ab - \overline{ab})c + c(ba - \overline{ba})\big), \tilde{u}_{kl}(d)\right] = \left[[\tilde{T}_{ij}(a,b), \tilde{u}_{ij}(c)], \tilde{u}_{kl}(d)\right] = 0,$$

where $\tilde{T}_{ij}(a,b) = [\tilde{u}_{ij}(a), \tilde{u}_{ji}(b)]$.

Now put

$$\tilde{\ell}(a,b) = [\tilde{u}_{12}(a), \tilde{u}_{34}(b)]$$

for $a, b \in \mathcal{A}$. Then, by (6.32),(6.33) and (6.34), we have

$$(6.35) \qquad \tilde{\ell}(a,b) + \tilde{\ell}(b,a) = 0$$

$$(6.36) \qquad \tilde{\ell}(\bar{a}b, c) = \tilde{\ell}(\bar{a}, \bar{b}c) + \tilde{\ell}(b, \bar{a}c), \qquad \text{and}$$

$$(6.37) \qquad \tilde{\ell}\big((ab - \overline{ab})c + c(ba - \overline{ba}), d\big) = 0,$$

for $a, b, c, d \in \mathcal{A}$. Putting $a = b = 1$ in (6.36) gives $\tilde{\ell}(1, c) = 0$, and so putting $c = 1$ in (6.36) gives $\tilde{\ell}(a, b) = \tilde{\ell}(a, \bar{b})$. But then $\tilde{\ell}(\bar{a}, \bar{b}c) = \tilde{\ell}(a, \bar{b}c)$ and $\tilde{\ell}(b, \bar{a}c) = \tilde{\ell}(b, \bar{c}a)$. Thus, (6.36) becomes

$$(6.38) \qquad \tilde{\ell}(\bar{a}b, c) + \tilde{\ell}(\bar{b}c, a) + \tilde{\ell}(\bar{c}a, b) = 0$$

for $a, b, c \in \mathcal{A}$. Let $\tilde{\mathrm{L}}(\mathcal{A}, -)$ be the space spanned by the elements $\tilde{\ell}(a,b)$, $a, b \in \mathcal{A}$. It follows from (6.35), (6.37) and (6.38) that there exists a linear map from $\mathrm{L}(\mathcal{A}, -)$ onto $\tilde{\mathrm{L}}(\mathcal{A}, -)$ so that $\ell(a,b) \longmapsto \tilde{\ell}(a,b)$. But since $[\tilde{u}_{12}(a), \tilde{u}_{34}(b)] \in \mathcal{V}$ for $a, b \in \mathcal{A}$, we have $[\tilde{\mathrm{L}}(\mathcal{A}, -), \tilde{u}_{ij}(a)] = (0)$. Moreover, using (6.29), we have

$$\begin{aligned}[\tilde{u}_{24}(a), \tilde{u}_{13}(b)] &= \gamma_2\gamma_4^{-1}\gamma_1\gamma_3^{-1}[\tilde{u}_{42}(\bar{a}), \tilde{u}_{31}(\bar{b})] = -\gamma_4^{-1}\gamma_1[\tilde{u}_{43}(\bar{a}), \tilde{u}_{21}(\bar{b})] \\ &= -\gamma_3^{-1}\gamma_2[\tilde{u}_{34}(a), \tilde{u}_{12}(b)] = \gamma_3^{-1}\gamma_2\tilde{\ell}(b,a) = -\gamma_3^{-1}\gamma_2\tilde{\ell}(a,b),\end{aligned}$$

and, by (6.30),

$$[\tilde{u}_{32}(a), \tilde{u}_{14}(b)] = -[\tilde{u}_{12}(a), \tilde{u}_{34}(b)] = -\tilde{\ell}(a,b).$$

Thus, the elements $u_{ij}(a)$ and $\tilde{\ell}(a,b)$ satisfy the relations (stu1$^\sharp$)–(stu7$^\sharp$), and so there exists a Lie algebra homomorphism $\xi\colon \mathcal{G}^\sharp \longrightarrow \tilde{\mathcal{G}}$ so that $\xi\big(u_{ij}^\sharp(a)\big) = \tilde{u}_{ij}(a)$ for $1 \leq i \neq j \leq 4$ and $a \in \mathcal{A}$. But then $\tau \circ \xi\big(u_{ij}^\sharp(a)\big) = \tau\big(u_{ij}(a)\big) = u_{ij}(a)$ and $\pi \circ \psi\big(u_{ij}^\sharp(a)\big) = \pi\big(u_{ij}(a)\big) = u_{ij}(a)$, and thus $\tau \circ \xi = \pi \circ \psi$ are required. ■

We note that Theorem 6.19, together with Theorem 5.18 in the previous section and Theorem 2.27 of [G] (which treats the case $n \geq 5$) establish Theorem B of the introduction.

As a consequence of Theorem 6.19, we also obtain the following result which was proved in [G,Theorem 2.46]:

COROLLARY 6.39. *Suppose that* $\mathcal{A}_- * \mathcal{A} = \mathcal{A}$, *where* $*$ *is the Jordan product* $a * b = \frac{1}{2}(ab + ba)$ *and* $\mathcal{A}_- * \mathcal{A} = \{\sum_i a_i * b_i \mid a_i \in \mathcal{A}_-, b_i \in \mathcal{A}\}$. *Then* $\mathrm{stu}_4(\mathcal{A}, -, \gamma)$ *is centrally closed.*

PROOF. It follows from (6.7) that $\mathrm{L}(\mathcal{A}, -) = (0)$ and hence $\hat{\mathcal{G}} = \mathcal{G}$. ■

Clearly, if $(\mathcal{A}, -)$ is an associative commutative algebra with identity involution, then $\mathrm{L}(\mathcal{A}, -) \cong \Omega^1_{\mathcal{A}}/d\mathcal{A}$. This explains a counterexample in [G, Example 2.50] which says that $\mathrm{stu}_4(\mathcal{A}, -, \gamma)$ is in general not centrally closed. It also gives us the following corollary of Theorem 6.19:

COROLLARY 6.40. *If* $(\mathcal{A}, -)$ *is an associative commutative algebra with identity involution, then* $\mathrm{H}_2\big(\mathrm{stu}_4(\mathcal{A}, -, \gamma)\big) \cong \Omega^1_{\mathcal{A}}/d\mathcal{A}$.

We conclude the paper by using the results of Sections 3, 5 and the present section to calculate $\mathrm{H}_2\big(\mathrm{peu}_n(\mathcal{A}, -, \gamma)\big)$.

THEOREM 6.41. *Suppose that* $(\mathcal{A}, -)$ *is associative and* $n \geq 3$. *Then,*

$$\mathrm{H}_2\big(\mathrm{peu}_n(\mathcal{A}, -, \gamma)\big) \cong \begin{cases} \mathrm{HF}(\mathcal{A}, -) & \text{if } n \neq 4 \text{ and } \mathrm{char}(k) \nmid n, \\ {}_{-1}\mathrm{HD}_1(\mathcal{A}, -) \oplus \mathcal{A}_- \cap \mathrm{center}(\mathcal{A}) & \text{if } n \neq 4 \text{ and } \mathrm{char}(k) \mid n, \\ \mathrm{L}(\mathcal{A}, -) \oplus \mathrm{HF}(\mathcal{A}, -) & \text{if } n = 4. \end{cases}$$

PROOF. If $n \neq 4$, this follows from Theorem B, Propositions 3.14 and 3.15 and Theorem 3.18. So suppose that $n = 4$. Then, by Theorem 6.19, $\hat{\mathcal{G}}$ is centrally closed. Moreover, we have the coverings $\pi\colon \hat{\mathcal{G}} \longrightarrow \mathcal{G}$ and $\zeta\colon \mathcal{G} \longrightarrow \mathrm{peu}(\mathcal{A}, -, \gamma)$ defined in this section and Section 3 respectively, and, by Propositions 3.14 and Theorem 3.18, we have $\ker(\zeta\circ\pi) = \mathrm{L}(\mathcal{A},-)\oplus\mathrm{HF}(\mathcal{A},-)$. ∎

EXAMPLE 6.42. Suppose that $\mathcal{A} = k[t,t^{-1}]$, where t is an indeterminant and $\mathrm{char}(k) = 0$. Since the units of $\mathcal{A}$ are at^n, $0 \neq a \in k$, $n \in \mathbb{Z}$, it follows that any involution $-$ on $\mathcal{A}$ satisfies either

$$\bar{t} = t, \quad \bar{t} = -t \quad \text{or} \quad \bar{t} = at^{-1} \text{ for some } a \neq 0 \in k.$$

Now it is well known that $\Omega^1_{\mathcal{A}}/d\mathcal{A}$ is the 1-dimensional space spanned by the image of $t\otimes t^{-1}$ under the canonical map from $\mathcal{A}\otimes\mathcal{A}$ to $\Omega^1_{\mathcal{A}}/d\mathcal{A}$ (see the argument in [Ga, pp. 19–20]). Using this fact and the fact that $\mathrm{HF}(\mathcal{A},-)$ is a quotient of $\Omega^1_{\mathcal{A}}/d\mathcal{A}$ (see Example 1.18), it is easy to establish that

$${}_{-1}\mathrm{HD}_1(\mathcal{A},-) = \mathrm{HF}(\mathcal{A},-) \cong \begin{cases} k & \text{if } \bar{t} = t \text{ or } \bar{t} = -t, \\ (0) & \text{otherwise.} \end{cases}$$

It is also easy to establish that

$$\mathrm{L}(\mathcal{A},-) \cong \begin{cases} k & \text{if } \bar{t} = t, \\ (0) & \text{otherwise.} \end{cases}$$

We leave the details of those two calculations to the reader. Hence, by Theorem 6.41, we have

$$\mathrm{H}_2\big(\mathrm{peu}_n(\mathcal{A},-,\gamma)\big) \cong \begin{cases} (0) & \text{if } \bar{t} = at^{-1}, a \neq 0 \in k, \\ k & \text{if } \bar{t} = -t, \\ k & \text{if } \bar{t} = t \text{ and } n \neq 4, \\ k\oplus k & \text{if } \bar{t} = t \text{ and } n = 4. \end{cases}$$

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