BULL. AUSTRAL. MATH. SOC. 16D60
VOL. 58 (1998) [57–73]

PRIMITIVE IDEALS IN THE COORDINATE RING OF QUANTUM EUCLIDEAN SPACE

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A twisted group algebra $k^{\sigma}P$ on a free Abelian group P with finite rank and a Poisson structure on kP are studied. As an application, the primitive spectrum of $\mathcal{O}_q(\mathfrak{o}k^n)$, the coordinate ring of quantum Euclidean space, is described and a Poisson algebra A is constructed so that there is a bijection between the primitive spectrum of $\mathcal{O}_q(\mathfrak{o}k^n)$ and the symplectic spectrum of R.

0. INTRODUCTION

The purpose of this paper is to characterise all primitive ideals of $\mathcal{O}_q(\mathfrak{o}k^n)$, the coordinate ring of quantum Euclidean space and to construct a Poisson algebra A such that there is a natural bijection between the primitive ideals of $\mathcal{O}_q(\mathfrak{o}k^n)$ and the symplectic ideals of A, when the ground field k is an uncountably infinite algebraically closed field with characteristic zero and the parameter $q \in k^*$ is not a root of unity. This paper confirms S.P. Smith's suggestion for $\mathcal{O}_q(\mathfrak{o}k^n)$; namely that the primitive ideals of certain algebras related to quantum groups should correspond bijectively to the symplectic leaves of a naturally associated Poisson structure on the associated algebraic variety.

In Sections 1 and 2, we establish the structure of the twisted group algebra $k^{\sigma}P$ when σ is an antisymmetric bimultiplicative map on the free Abelian group P with finite rank, and the Poisson structure on kP induced by an antisymmetric bilinear map u on P. The idea of these sections was given to the author by T.J. Hodges. The authors thank him deeply for permission to use it here. In Section 3 that is a main part of this paper, we characterise the primitive ideals of $\mathcal{O}_q(\mathfrak{o}k^n)$; this arises from the work of Takeuchi [**11**]. The reader is referred to the articles [**10**] and [**11**] for further background of $\mathcal{O}_q(\mathfrak{o}k^n)$. The multiplicative rule of this algebra $\mathcal{O}_q(\mathfrak{o}k^n)$ is very similar to that of the quantised Weyl algebra which has been studied by various authors (see [**1, 2, 4**]), and so the techniques of proofs are similar to those of [**1**] and [**8**]. The final section constructs a Poisson algebra A such that there is a bijection between the set of primitive ideals of $\mathcal{O}_q(\mathfrak{o}k^n)$ and the set of symplectic ideals of B.

Received 24th November, 1997
The first author was partially supported by the Basic Science Research Institute Program, Korean Ministry of Education, Project No. BSRI-96-1427.

Henceforth, we assume throughout that k is an uncountably infinite algebraically closed field with characteristic zero, the parameter $q \in k^*$ is not a root of unity and P is a free Abelian group with finite rank unless stated otherwise.

1. Twisted Group Algebras

1.1. Since quantum tori are essentially just twisted group algebras (see **1.6**), we begin with a brief review of some fairly well-known results about ideals in twisted group algebras.

Let $\sigma \in Z^2(P, k^*)$ be a 2-cocycle on a free Abelian group P with finite rank. Then the twisted group algebra $k^\sigma P$ is the k-algebra with generators t_λ for $\lambda \in P$ with relations:

$$t_\lambda t_\mu = \sigma(\lambda, \mu) t_{\lambda+\mu}$$

In particular, if σ is bimultiplicative and antisymmetric, that is,

$$\sigma(\lambda_1 + \lambda_2, \mu) = \sigma(\lambda_1, \mu)\sigma(\lambda_2, \mu)$$
$$\sigma(\lambda, \mu) = \sigma(\mu, \lambda)^{-1},$$

then σ is a 2-cocycle on P, and thus the twisted group algebra $k^\sigma P$ is defined and satisfies the commutation relations:

$$t_\lambda t_\mu = \sigma^2(\lambda, \mu) t_\mu t_\lambda$$

Henceforth, we assume that σ is bimultiplicative and antisymmetric on P.

1.2. Define

$$P_\sigma = \{\lambda \in P \mid \sigma^2(\lambda, \mu) = 1 \quad \forall \mu \in P\}.$$

Clearly P_σ is a subgroup of P and free since every subgroup of a free Abelian group (with finite rank) is free.

Lemma. *The centre $Z(k^\sigma P)$ of $k^\sigma P$ is $Z(k^\sigma P) = \{\sum_\lambda a_\lambda t_\lambda \mid \lambda \in P_\sigma\}$, which is isomorphic to kP_σ.*

Proof: Put $Z = Z(k^\sigma P)$. For $f = \sum_\lambda a_\lambda t_\lambda \in k^\sigma P$, $f \in Z$ if and only if $t_\mu f = f t_\mu$ for all $\mu \in P$. Since $t_\mu f = \sum_\lambda \sigma^2(\mu, \lambda) a_\lambda t_\lambda t_\mu$, this will occur if and only if $\lambda \in P_\sigma$ for all λ in the support of f. ∎

Theorem 1.3. *There is a bijection preserving inclusions between the ideals of $k^\sigma P$ and the ideals of the centre $Z(k^\sigma P)$. That is, if I is an ideal of $k^\sigma P$ then $I = (I \cap Z(k^\sigma P))k^\sigma P$, and if J is an ideal of $Z(k^\sigma P)$ then $J = Jk^\sigma P \cap Z(k^\sigma P)$.*

Proof: Consider the action of P as automorphisms of $k^\sigma P$ defined by

$$\lambda(t_\mu) = \sigma^2(\lambda, \mu) t_\mu = t_\lambda t_\mu t_\lambda^{-1}.$$

Let $\mathcal{T}$ be a transversal for P_σ in P. Then the weight space decomposition of $k^\sigma P$ under this action is

$$(*) \qquad k^\sigma P = \bigoplus_{\nu \in \mathcal{T}} Z(k^\sigma P) t_\nu .$$

If I is an ideal of $k^\sigma P$ then I must be invariant under this action and so

$$I = \bigoplus_\nu I \cap Z(k^\sigma P) t_\nu = \bigoplus_\nu \big(I \cap Z(k^\sigma P)\big) t_\nu = \big(I \cap Z(k^\sigma P)\big) k^\sigma P.$$

If J is an ideal of $Z(k^\sigma P)$ and $x \in Jk^\sigma P \cap Z(k^\sigma P)$ then $x = \sum_i x_i f_i$ for some $x_i \in J$ and $f_i \in k^\sigma P$. Replace each f_i with an element written by the decomposition (*) and then x can be expressed by $x = \sum_{\nu \in \mathcal{T}} a_\nu t_\nu$ for some $a_\nu \in J$. Since $x \in Z(k^\sigma P)$, if $\nu \notin P_\sigma$ then $a_\nu = 0$ and so $x \in J$. Therefore we have that $J = Jk^\sigma P \cap Z(k^\sigma P)$. ∎

PROPOSITION 1.4. *The centre of the fractional algebra* Fract$(k^\sigma P)$ *is* Fract$\big(Z(k^\sigma P)\big)$.

PROOF: Observe that both $k^\sigma P$ and $Z = Z(k^\sigma P)$ are affine domains, thus there exist fractional algebras Fract$(k^\sigma P)$ and Fract(Z). Clearly Fract(Z) is contained in the centre of Fract$(k^\sigma P)$. For $x, y \in k^\sigma P$, if xy^{-1} is a central element of Fract$(k^\sigma P)$ then $xy = yx$ and $t_\lambda xy^{-1} t_\lambda^{-1} = xy^{-1}$ for all $t_\lambda \in k^\sigma P$, thus we have that $xt_\lambda y = yt_\lambda x$. Express y as elements of (*) in the proof of **1.3**. Let us call the number of nonzero $z_\nu \in Z$ in the expression $y = \sum z_\nu t_\nu$ the length of y. We may assume that y has the shortest length in the set $\{y' \mid xy^{-1} = x'y'^{-1}$ for some $x'\}$. If the length of y is greater than 1 then $0 \neq y - \alpha t_\lambda y t_\lambda^{-1}$ has shorter length than y for some nonzero scalar α and t_λ and we have that

$$x\big(y - \alpha t_\lambda y t_\lambda^{-1}\big) = xy - \alpha x t_\lambda y t_\lambda^{-1} = yx - \alpha y t_\lambda x t_\lambda^{-1} = y\big(x - \alpha t_\lambda x t_\lambda^{-1}\big).$$

Therefore $xy^{-1} = y^{-1}x = \big(x - \alpha t_\lambda x t_\lambda^{-1}\big)\big(y - \alpha t_\lambda y t_\lambda^{-1}\big)^{-1}$. This contradicts to the shortest length of y, so $y = z_\nu t_\nu$ and $xy^{-1} = (\sigma(\nu,\nu) x t_{-\nu}) z_\nu^{-1} \in$ Fract(Z). ∎

THEOREM 1.5. *Let $\{e_1, \dots, e_n\}$ be a basis of P and let H be the subsemigroup (with identity) of P generated by $e_1, \dots, e_n$. Given an antisymmetric bimultiplicative map σ, let R be a Noetherian k-algebra such that $k^\sigma H \subseteq R \subseteq k^\sigma P$. Then the multiplicative set $\mathcal{C}$ generated by $t_{e_i}, i = 1, \dots, n$, is an Ore set of R and the localisation $\mathcal{C}^{-1}R$ is isomorphic to $k^\sigma P$. If all prime ideals of R are completely prime then all maximal ideals of $Z(k^\sigma P)$ correspond bijectively to all primitive ideals of R disjoint from $\mathcal{C}$. In fact, the map $M \mapsto (Mk^\sigma P)^c$ from the maximal ideals of $Z(k^\sigma P)$ into the* *primitive ideals of* R *disjoint from* $\mathcal{C}$ *is bijective, where* $(Mk^{\sigma}P)^{c}$ *is the contraction of* $Mk^{\sigma}P$.

PROOF: Clearly, every element of $\mathcal{C}$ is invertible in $k^{\sigma}P$ and each element of $k^{\sigma}P$ is of the form $b^{-1}a$, $a \in k^{\sigma}H$, $b \in \mathcal{C}$, thus the localisation $\mathcal{C}^{-1}R$ is isomorphic to $k^{\sigma}P$ and $\mathcal{C}$ is a left Ore set. Similarly $\mathcal{C}$ is a right Ore set. Note that, by **1.3**, there is a bijection between the set of all maximal ideals of $k^{\sigma}P$ and the set of all maximal ideals of $Z(k^{\sigma}P)$.

Let M be a maximal ideal of $k^{\sigma}P$. Then the contraction M^{c} to R is a prime ideal disjoint from $\mathcal{C}$ and any prime ideal Q of R properly containing M^{c} contains an element of $\mathcal{C}$ since M is maximal. Since Q is completely prime, $t_{e_i} \in Q$ for some i and so $t_{e_1} \cdots t_{e_n} \in Q \cap \mathcal{C}$. Therefore the intersection of all prime ideals properly containing M^{c} is not equal to M^{c} and so M^{c} is primitive by [**7**, 9.1.8].

Conversely, let Q be a primitive ideal of R disjoint from $\mathcal{C}$. Then Q is contraction of a prime ideal M of $k^{\sigma}P$. It suffices to show that M is maximal. Let $\lambda_1, \ldots, \lambda_r$ be a basis of the subgroup P_{σ}. The elements t_{λ_i}, $i = 1, \ldots, r$, can be written as $t_{\lambda_i} = \beta_i b_i^{-1} a_i$ for some $\beta_i \in k^{*}$, $a_i, b_i \in \mathcal{C}$. Since t_{λ_i} are central elements of $k^{\sigma}P$ and Q is disjoint from $\mathcal{C}$, $a_i - \alpha_i b_i \in Q$ for some $\alpha_i \in k^{*}$ by [**7**, 9.1.7]. Hence M contains $t_{\lambda_i} - \alpha_i\beta_i$ for each $i = 1, \ldots, r$ and thus $M \cap Z(k^{\sigma}P)$ is maximal in $Z(k^{\sigma}P)$ and $M = \big(M \cap Z(k^{\sigma}P)\big)k^{\sigma}P$ is maximal in $k^{\sigma}P$ by **1.3**. ∎

1.6. (See [**2**, 2.1], [**6**] and [**7**, 1.5.10 (ii)]) Let $\lambda = (\lambda_{ij})$ be an $n \times n$ matrix of nonzero elements of k such that $\lambda_{ii} = 1$ and $\lambda_{ji} = \lambda_{ij}^{-1}$ for $1 \leqslant i, j \leqslant n$. The multiparameter coordinate ring of quantum affine n-space is the k-algebra $\mathcal{O}_{\lambda}(k^{n})$ generated by elements $x_1, \ldots, x_n$ subject only to the relations $x_i x_j = \lambda_{ij} x_j x_i$ for $1 \leqslant i, j \leqslant n$. Note that $\mathcal{O}_{\lambda}(k^{n})$ can be expressed as an n-fold iterated skew polynomial ring starting with the field k; hence, $\mathcal{O}_{\lambda}(k^{n})$ is an affine domain. In particular, if $\lambda_{ij} = q^{-1}$, $i < j$ then $\mathcal{O}_{\lambda}(k^{n})$ is called the coordinate ring of quantum affine n-space and denoted $\mathcal{O}_{q}(k^{n})$. As in [**2**, 2.1] and [**6**], we write $P(\lambda)$ for the localisation of $\mathcal{O}_{\lambda}(k^{n})$ with respect to the multiplicative set generated by $x_1, \ldots, x_n$, that is, $P(\lambda)$ is the k-algebra generated by $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$ subject to the relations $x_i x_j = \lambda_{ij} x_j x_i$.

Note that $P(\lambda)$ is the twisted group algebra $k^{\sigma}P$, where the free Abelian group P has basis $\{e_1, \ldots, e_n\}$ and an antisymmetric bimultiplicative map $\sigma \in Z^{2}(P, k^{*})$ is given by

$$\sigma(e_i, e_j) = \lambda_{ij}^{1/2}.$$

Conversely, all twisted group algebra $k^{\sigma}P$ with an antisymmetric bimultiplicative map σ can be presented as a $P(\lambda)$ for $\lambda = \big(\sigma^{2}(e_i, e_j)\big)$. In fact, $\phi : P(\lambda) \longrightarrow k^{\sigma}P$ defined by $\phi(x_i) = t_{e_i}$ for all $i = 1, \ldots, n$ is an isomorphism. Moreover, the subalgebra $\mathcal{O}_{\lambda}(k^{n})$ of $P(\lambda)$ is isomorphic to the twisted semigroup algebra $k^{\sigma}H$, where H is the subsemigroup of P generated by $e_1, \dots, e_n$.

2. Poisson Tori

2.1. Now let $u \in Z^2(P,k)$ be an antisymmetric bilinear map. That is,

$$u(\lambda_1 + \lambda_2, \mu) = u(\lambda_1, \mu) + u(\lambda_2, \mu)$$
$$u(\lambda, \mu) = -u(\mu, \lambda).$$

Then it is easily verified that the bracket

$$\{t_\lambda, t_\mu\} = u(\lambda, \mu) t_{\lambda+\mu}$$

defines a Poisson bracket on the group algebra kP. If kP has a Poisson structure then we always assume that it is induced by an antisymmetric bilinear map u.

Lemma 2.2. *Set*

$$Z_p(kP) = \{f \in kP \mid \{f,g\} = 0 \; \forall g \in kP\}.$$

Then $Z_p(kP) = kP_u$ where $P_u = \{\lambda \in P \mid u(\lambda,\mu) = 0 \; \forall \mu \in P\}$. The Poisson subalgebra $Z_p(kP)$ of kP, which has the trivial Poisson structure (that is, $\{f,g\} = 0 \; \forall f, g$), is called the Poisson centre.

Proof: Let $f = \sum_\lambda a_\lambda t_\lambda$. Then $f \in Z_p(kP)$ if and only if $\{t_\mu, f\} = 0$ for all $\mu \in P$. Since $\{t_\mu, f\} = \sum_\lambda u(\mu,\lambda) a_\lambda t_{\mu+\lambda}$, this will occur if and only if $\lambda \in P_u$ for all λ in the support of f. ∎

2.3. Recall that a Poisson ideal of a Poisson algebra A is an ideal I such that $\{f,g\} \in I$ for all $f \in I$ and $g \in A$.

Theorem. *There is a bijection preserving inclusions between the Poisson ideals of kP and the Poisson ideals of $Z_p(kP)$. That is, if I is a Poisson ideal of kP then $I = (I \cap Z_p(kP))kP$, and if J is a Poisson ideal of $Z_p(kP)$ then $J = (JkP) \cap Z_p(kP)$.*

Proof: Set $Z_p = Z_p(kP)$. Consider the action of P as linear endomorphisms of kP defined by

$$\lambda(t_\mu) = u(\lambda,\mu) t_\mu = \{t_\lambda, t_\mu\} t_\lambda^{-1}.$$

Let $\mathcal{T}$ be a transversal for P_u in P. Then the weight space decomposition of kP under this action is

$$kP = \bigoplus_{\nu \in \mathcal{T}} Z_p t_\nu. \qquad (**)$$

If I is a Poisson ideal of kP then I must be invariant under this action and so

$$I = \bigoplus_{\nu} I \cap Z_p t_\nu = \bigoplus_{\nu} (I \cap Z_p) t_\nu = (I \cap Z_p) kP.$$

If J is a Poisson ideal of Z_p and if $x \in (JkP) \cap Z_p$ then $x = \sum_i x_i f_i$ for some $x_i \in J$ and $f_i \in kP$. Replace each f_i with an element written by the decomposition (**) and then x can be expressed by $x = \sum_{\nu \in T} a_\nu t_\nu$ for some $a_\nu \in J$. Since $x \in Z_p$, if $\nu \notin P_u$ then $a_\nu = 0$ and so $x \in J$. Therefore we have that $J = (JkP) \cap Z_p$. ∎

2.4. Let A be a Poisson algebra over k and let Q be a prime Poisson ideal of A, which means prime in the commutative algebra A and Poisson in A. Then the Poisson bracket on A defines a Poisson bracket on Fract(A/Q) and we define Q to be symplectic if

$$\{a \in \operatorname{Fract}(A/Q) \mid \{a, b\} = 0 \ \forall b \in \operatorname{Fract}(A/Q)\}$$

reduces to the set of scalars (see [**5**, A.4.1]). A Poisson algebra A is called symplectic whenever the Poisson ideal $\langle 0 \rangle$ is symplectic.

PROPOSITION. *The set $Z = \{a \in \operatorname{Fract}(kP) \mid \{a, b\} = 0 \ \forall b \in \operatorname{Fract}(kP)\}$ is equal to the fractional algebra of the Poisson centre $Z_p(kP) = kP_u$ of kP.*

PROOF: This follows from the modified version of the proof in **1.4**. For completeness sake, we write out the proof. Clearly Fract(kP_u) is contained in Z. If x, y are elements of kP and $xy^{-1} \in Z$ then $\{x, t_\mu\} y = \{y, t_\mu\} x$ for all $t_\mu \in kP$. Express y as elements of (**) in the proof of **2.3**. Let us call the number of nonzero $z_\nu \in kP_u$ in the expression $y = \sum z_\nu t_\nu$ the length of y. We may assume that y has the shortest length in the set $\{y' \mid xy^{-1} = x'y'^{-1}$ for some $x'\}$. If the length of y is greater than 1 then $0 \neq y - \alpha\{y, t_\mu\} t_\mu^{-1}$ has shorter length than y for some scalar $\alpha \in k^*$ and $t_\mu \in kP$, and

$$x\big(y - \alpha\{y, t_\mu\} t_\mu^{-1}\big) = yx - \alpha x\{y, t_\mu\} t_\mu^{-1} = y\big(x - \alpha\{x, t_\mu\} t_\mu^{-1}\big).$$

Therefore $xy^{-1} = \big(x - \alpha\{x, t_\mu\} t_\mu^{-1}\big)\big(y - \alpha\{y, t_\mu\} t_\mu^{-1}\big)^{-1}$, which is a contradiction to the shortest length of y. Hence we have that $y = z_\nu t_\nu$ and $xy^{-1} = (xt_{-\nu}) z_\nu^{-1} \in$ Fract(kP_u). ∎

THEOREM 2.5. *Let $\{e_1, \dots, e_n\}$ be a basis of P and let H be the subsemigroup (with identity) of P generated by $\{e_1, \dots, e_n\}$. Let kP be the Poisson algebra induced by an antisymmetric bilinear map u, let A be a sub-Poisson and Noetherian subalgebra such that $kH \subseteq A \subseteq kP$ and let C be the multiplicative set generated by $t_{e_1}, \dots, t_{e_n}$. Then $C^{-1}A = kP$ and extensions of all symplectic ideals of A disjoint from C are maximal Poisson ideals of kP.*

PROOF: Clearly, the localisation $\mathcal{C}^{-1}A$ is isomorphic to kP. Let Q be a symplectic ideal of A disjoint from $\mathcal{C}$ and let $\lambda_1, \ldots, \lambda_r$ be a basis of the subgroup P_u. Then Q is contraction of a Poisson ideal M of kP and $t_{\lambda_i} = a_i b_i^{-1}$ for some $a_i, b_i \in \mathcal{C}$, $i = 1, \ldots, r$, and so $a_i - \alpha_i b_i \in Q$ for some $\alpha_i \in k^*$. Hence $t_{\lambda_i} - \alpha_i \in M \cap Z_p(kP)$. Therefore M is a maximal Poisson ideal of kP by **2.3**. ∎

COROLLARY 2.6. *Under the same conditions as* **2.5**, *let* $P = P_u \bigoplus P'$ *for some subgroup* P'. *Then there is a bijection between the set of symplectic ideals of* A *and the set of maximal Poisson ideals of* kP.

PROOF: Let M be a maximal Poisson ideal of kP and let $e_1, \ldots, e_r$ be a basis of P_u. Then $M = \langle t_{e_1} - \alpha_1, \ldots, t_{e_r} - \alpha_r \rangle kP$ for some $\alpha_i \in k^*$ by **2.3**. Let us prove that the contraction M^c to A is symplectic. Clearly, M^c is a prime ideal of A since M is prime in kP. Note that the antisymmetric bilinear map $u' = u|_{P' \times P'}$ gives a Poisson structure on kP' and $P'_{u'}$ is trivial. Since $kP' \cong kP/M = \overline{\mathcal{C}}(A/M^c)$ is symplectic by **2.4** and Fract(A/M^c) is isomorphic to Fract(kP'), M^c is a symplectic ideal of A. Hence the conclusion follows from **2.5**. ∎

LEMMA 2.7. *Given an antisymmetric bimultiplicative map* $\sigma \in Z^2(P, k^*)$ *and* $0 \neq q \in k$ *which is not a root of unity, define* $u : P \times P \longrightarrow k$ *by*

$$\sigma(\lambda, \mu) = q^{u(\lambda, \mu)} \quad \forall \lambda, \mu \in P.$$

Then u *is an antisymmetric bilinear map and*

$$Z(k^\sigma P) \cong Z_p(kP) = kP_u.$$

PROOF: Clearly, u is an antisymmetric bilinear map and $P_\sigma = P_u$ since q is not a root of unity, and so $Z(k^\sigma P) \cong Z_p(kP) = kP_u$ by **1.2** and **2.2**. ∎

3. PRIMITIVE IDEALS IN THE COORDINATE RING OF QUANTUM EUCLIDEAN SPACE

DEFINITION 3.1: (See [**9, 5, 10,** Section 4] and [**11,** 5.1]) Let $q \in k^*$. For each positive integer n, the coordinate ring of quantum Euclidean space $\mathcal{O}_q(\mathfrak{o}k^{2n})$ is the k-algebra generated by $2n$ variables $y_1, x_1, y_2, x_2, \cdots, y_n, x_n$ satisfying the following relations:

$$\begin{aligned} y_i y_j &= q y_j y_i && (i < j) \\ x_i y_j &= q y_j x_i && (i \neq j) \\ x_j x_i &= q x_i x_j && (i < j) \\ x_j y_j &= y_j x_j + (1 - q^2) \sum_{1 \leqslant l < j} q^{l-j} y_l x_l && (\text{all } j). \end{aligned}$$

The coordinate ring of quantum Euclidean space $\mathcal{O}_q(\mathfrak{o}k^{2n+1})$ is the k-algebra generated by $2n+1$ variables $z_0, y_1, x_1, y_2, x_2, \cdots, y_n, x_n$ satisfying the following relations:

$$\begin{aligned}
z_0 y_j &= q y_j z_0 && \text{(all } j) \\
z_0 x_j &= q^{-1} x_j z_0 && \text{(all } j) \\
y_i y_j &= q y_j y_i && (i < j) \\
x_i y_j &= q y_j x_i && (i \neq j) \\
x_j x_i &= q x_i x_j && (i < j) \\
x_j y_j &= y_j x_j + (1-q^2) \sum_{1 \leqslant l < j} q^{l-j} y_l x_l + q^{(1/2)-j}(1-q) z_0^2 && \text{(all } j).
\end{aligned}$$

Hereafter, we write $\mathcal{O}_q^n$ for $\mathcal{O}_q(\mathfrak{o}k^{2n})$.

Lemma 3.2. *The algebra $\mathcal{O}_q^n$ is a Noetherian domain and all its prime ideals are completely prime.*

Proof: By [**9**, **5**], $\mathcal{O}_q^n$ is an iterated skew polynomial ring

$$\mathcal{O}_q^n = k[y_1][x_1; \beta_1][y_2; \alpha_2][x_2; \beta_2, \delta_2] \cdots [y_n; \alpha_n][x_n; \beta_n, \delta_n],$$

for certain automorphisms α_i and β_i. Thus, it is a Noetherian domain. Moreover it is easy to check that α_i, β_i and left β_i-derivation δ_i satisfy the condition of [**3**, 2.3], and so all prime ideals of $\mathcal{O}_q^n$ are completely prime. ∎

Lemma 3.3. *In $\mathcal{O}_q^n$, set*

$$z_i = q^{-2} x_i y_i - y_i x_i = q^{-2}(1-q^2) \sum_{1 \leqslant l \leqslant i} q^{l-i} y_l x_l$$

for $i = 1, \ldots, n$. Then z_n is central, all z_i are normal and $y_i, y_{i-1}, x_i, x_{i-1}$ are normal modulo z_{i-1} for each $i \geqslant 1$ $(z_0 = 0)$. More precisely,

$$\begin{aligned}
z_j y_i &= y_i z_j & z_j x_i &= x_i z_j && (i \leqslant j) \\
z_j y_i &= q^2 y_i z_j & z_j x_i &= q^{-2} x_i z_j && (i > j) \\
q^2 z_i &= x_i y_i - q^2 y_i x_i & q z_i &= x_{i+1} y_{i+1} - y_{i+1} x_{i+1} && (i \geqslant 1) \\
q^2 z_i &= (1-q^2) y_i x_i + q z_{i-1} & z_j z_i &= z_i z_j && (i \geqslant 1).
\end{aligned}$$

Proof: This follows immediately from direct calculations. ∎

Definition 3.4: (See [**8**, 1.4]) Let $\wp_n = \{z_1, y_1, x_1, z_2, y_2, x_2, \ldots, z_n, y_1, x_n\}$ be a subset of $\mathcal{O}_q^n$. We shall say that $T \subseteq \wp_n$ is *admissible* if T satisfies the conditions:

(i) $y_i \in T$ or $x_i \in T$ if and only if $z_i \in T$ and $z_{i-1} \in T$, $2 \leqslant i \leqslant n$.

(ii) $y_1 \in T$ or $x_1 \in T$ if and only if $z_1 \in T$.

https://doi.org/10.1017/S0004972700032007 Published online by Cambridge University Press

The definition of an admissible set should be compared with that of a p-sequence in [**1**, 4.2]. In fact, if T is an admissible set then

$$S = T - \{z_i \mid y_i \in T \text{ or } x_i \in T\}$$

is p-sequence in $\mathcal{O}_q^n$. Note that the ideal generated by T is equal to the ideal generated by the p-sequence S as in [**1**, 4.3].

LEMMA 3.5. *For each prime ideal P of $\mathcal{O}_q^n$, $P \cap \wp_n$ is an admissible set.*

PROOF: This follows immediately from **3.3**. ∎

LEMMA 3.6. *Let T be an admissible set. Then the ideal $\langle T\rangle$ is completely prime and there is a subalgebra A_T of $\mathcal{O}_q^n$ such that A_T is a multiparameter coordinate ring of quantum affine space $\mathcal{O}_{\lambda_T}(k^m)$ for some matrix $\lambda_T = (\lambda_{ij})$, $\lambda_{ij} = 1, q^{\pm 1}$ or $q^{\pm 2}$ and*

$$A_T = \mathcal{O}_{\lambda_T}(k^m) \subseteq \mathcal{O}_q^n/\langle T\rangle \subseteq P(\lambda_T).$$

PROOF: The ideal $\langle T\rangle$ is completely prime as in [**1**, 4.5]. Put

$$T_y = \{y_j \mid y_j \notin T, x_j \in T\}$$
$$T_x = \{x_j \mid x_j \notin T, y_j \in T\}$$
$$S_T = \{y_j, z_j \mid z_j \notin T\} \cup \{y_j \mid z_j \in T,\ y_j \notin T,\ x_j \notin T\} \cup T_y \cup T_x,$$

and let A_T be the subalgebra of $\mathcal{O}_q^n$ generated by all elements of S_T and let m be the number of elements in S_T. Since there is no index i such that both y_i and x_i are in S_T, we have that, by **3.1** and **3.3**, $s_is_j = \lambda_{ij}s_js_i$ for any pair $s_i, s_j \in S_T$, and so $A_T = \mathcal{O}_{\lambda_T}(k^m)$ for the $m \times m$-matrix $\lambda_T = (\lambda_{ij})$ by **1.6**. Since

$$T - \{z_i \mid y_i \in T \text{ or } x_i \in T\}$$

is a normalising sequence of generators for $\langle T\rangle$ and each element of S_T is not in the ideal $\langle T\rangle$, we get immediately that $A_T \cap \langle T\rangle = 0$, hence A_T is embedded into $\mathcal{O}_q^n/\langle T\rangle$. The image in $\mathcal{O}_q^n/\langle T\rangle$ of the multiplicative set generated by all the elements in S_T is a right and left Ore set in $\mathcal{O}_q^n/\langle T\rangle$ as in [**1**, 4.8]. Let B_T denote the localisation of $\mathcal{O}_q^n/\langle T\rangle$ at this set. Since $(1-q^2)y_ix_i = q^2z_i - qz_{i-1}$ $(z_0 = 0)$ by **3.3** and all nonzero generators $\overline{y}_j \in B_T$ are invertible, we have that all $\overline{x}_j \in \mathcal{O}_q^n/\langle T\rangle$ are in B_T. Therefore,

$$A_T = \mathcal{O}_{\lambda_T}(k^m) \subseteq \mathcal{O}_q^n/\langle T\rangle \subseteq B_T = P(\lambda_T).$$

∎

It is cumbersome to use the standard overlining notation for images in factor rings of $\mathcal{O}_q^n$ and so we shall write, for example, x_i for the image of x_i in a factor ring if no confusion arises.

3.7. Let T be an admissible set such that $y_i \in T$ and $x_i \in T$ for some i. Then the index i is said to be *removable* in T.

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LEMMA. *If T is an admissible set of $\mathcal{O}_q^n$ with removable indices then there is an integer $m < n$ and an admissible set T' of $\mathcal{O}_q^m$ such that $\mathcal{O}_q^n/\langle T\rangle \cong \mathcal{O}_q^m/\langle T'\rangle$ and T' has no removable indices.*

PROOF: Suppose that j is removable in T. Then there is a natural epimorphism ϕ from $\mathcal{O}_q^{n-1}$ onto $\mathcal{O}_q^n/\langle T\rangle$ given by

$$\begin{aligned} &y_i \mapsto q^{-1}y_i, && x_i \mapsto x_i, && i < j \\ &y_i \mapsto y_{i+1}, && x_i \mapsto x_{i+1}, && i \geqslant j. \end{aligned}$$

Since $\ker(\phi)$ is prime by **3.6**, $\ker(\phi) \cap \wp_{n-1}$ is an admissible set of $\mathcal{O}_q^{n-1}$ by **3.5**. An induction on n completes the proof. ∎

3.8. From here to **3.11**, we shall work to find the centre of $P(\lambda_T) = B_T$ in **3.6** when T has no removable indices.

Let T be an admissible set of $\mathcal{O}_q^n$ without removable indices. Note that $(q^{-2}-1)y_1x_1^{-1} = z_1^{-1}y_1^2$ and z_n are central elements of Fract $\mathcal{O}_q^n$. For S_T as in the proof of **3.6**, put

$$\begin{aligned} &U_T = S_T - \{z_1, z_n\} && c_{-1} = z_1^{-1}y_1^2 && c_0 = z_n && \text{if } z_n \notin T \text{ and } z_1 \notin T \\ &U_T = S_T - \{z_1\} && c_{-1} = z_1^{-1}y_1^2 && c_0 = 0 && \text{if } z_n \in T \text{ and } z_1 \notin T \\ &U_T = S_T - \{z_n\} && c_{-1} = 0 && c_0 = z_n && \text{if } z_n \notin T \text{ and } z_1 \in T \\ &U_T = S_T && c_{-1} = 0 && c_0 = 0 && \text{if } z_n \in T \text{ and } z_1 \in T. \end{aligned}$$

LEMMA.

$$\begin{aligned} \operatorname{ind} U_T :=& \{i \mid z_i \notin U_T\} \\ =& \{i_1, i_1+1, i_1+2, \cdots, i_1+v_1\} \cup \{i_2, i_2+1, i_2+2, \cdots, i_2+v_2\} \\ & \cup \cdots \cup \{i_r, i_r+1, i_r+2, \cdots, i_r+v_r\} \end{aligned}$$

for some nonnegative integers v_i and positive integers $1 = i_1 < i_2 < \cdots < i_r$ satisfying $i_j - (i_{j-1} + v_{j-1}) \geqslant 2$, $i_r + v_r = n$.

PROOF: Since T has no removable indices, it follows immediately from the definition of admissible set. ∎

LEMMA 3.9. *Let T, U_T, i_j and v_j be as in* **3.8**.
(1) Let v_r be odd. Rewrite the elements of

$$\left(U_T \cap \{y_{i_r}, x_{i_r}, y_{i_r+1}, x_{i_r+1}, \ldots, y_n, x_n\}\right) \cup \{z_{i_r-1}\}$$

as $u_1, u_2, \ldots, u_p$, say, where $u_1 > u_2 > \ldots > u_p$ in the ordering

$$y_n > y_{n-1} > \ldots > y_1 > z_1 > x_1 > z_2 > x_2 > \ldots > z_n > x_n.$$

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Note that p is odd. Then

$$c_r = u_1^{\varepsilon_1} u_2^{-\varepsilon_2} \cdots u_{p-1}^{-\varepsilon_{p-1}} u_p^{\varepsilon_p}, \qquad \varepsilon_k = \begin{cases} 2 & u_k \neq z_{i_r-1} \\ 1 & u_k = z_{i_r-1} \end{cases}$$

is a central element of B_T. If v_r is even then put $c_r = 0$.

(2) Let v_j, $1 < j < r$, be even. Rewrite the elements of

$$\left(U_T \cap \{y_{i_j}, x_{i_j}, y_{i_j+1}, x_{i_j+1}, \ldots, y_{i_j+v_j}, x_{i_j+v_j}\}\right) \cup \{z_{i_j-1}\}$$

as $u_1, u_2, \ldots, u_p$, say, where $u_1 > u_2 > \ldots > u_p$ in the ordering

$$y_n > y_{n-1} > \ldots > y_1 > z_1 > x_1 > z_2 > x_2 > \ldots > z_n > x_n.$$

Note that p is even. Then

$$c_j = u_1^{\varepsilon_1} u_2^{-\varepsilon_2} \cdots u_{p-1}^{\varepsilon_{p-1}} u_p^{-\varepsilon_p}, \qquad \varepsilon_k = \begin{cases} 2 & u_k \neq z_{i_j-1} \\ 1 & u_k = z_{i_j-1} \end{cases}$$

is a central element of B_T. If v_j is odd then put $c_j = 0$.

PROOF: This follows by direct calculations using **3.1** and **3.3**. □

LEMMA 3.10. *Let T, U_T, i_j, v_j and c_i be as* **3.8** *and* **3.9**. *Put*

$$V_T = U_T - \{z_{i_j-1} \mid c_j \neq 0,\ j = 2, \ldots, r\}.$$

(1) Let v_1 be odd and let $V_T \cap \{z_1, z_2, \ldots, z_n\} \neq \emptyset$. Assume that i is the least index such that $z_i \in V_T$. Rewrite the elements of $\left(V_T \cap \{y_1, x_1, \ldots, y_{i-1}, x_{i-1}\}\right) \cup \{z_i\}$ as $u_1, u_2, \ldots, u_p$, say, where $u_1 > u_2 > \ldots > u_p$ in the ordering

$$y_n > y_{n-1} > \ldots > y_1 > z_1 > x_1 > z_2 > x_2 > \ldots > z_n > x_n.$$

Note that p is odd. Then

$$c_1 = u_1^{\varepsilon_1} u_2^{-\varepsilon_2} \cdots u_{p-1}^{-\varepsilon_{p-1}} u_p^{\varepsilon_p}, \qquad \varepsilon_k = \begin{cases} 2 & u_k \neq z_i \\ 1 & u_k = z_i \end{cases}$$

is a central element of B_T. Put $z = z_i$.

(2) Let v_1 be odd and let $V_T \cap \{z_1, z_2, \ldots, z_n\} = \emptyset$. Rewrite the elements of $V_T \cap \{y_1, x_1, \ldots, y_n, x_n\}$ as $u_1, u_2, \ldots, u_p$, say, where $u_1 > u_2 > \ldots > u_p$ in the ordering

$$y_n > y_{n-1} > \ldots > y_1 > z_1 > x_1 > z_2 > x_2 > \ldots > z_n > x_n.$$

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Note that p is odd. Then

$$c_1 = u_1 u_2^{-1} \cdots u_{p-1}^{-1} u_p$$

is a central element of B_T. Put $z = \begin{cases} y_1 & y_1 \in V_T \\ x_1 & x_1 \in V_T. \end{cases}$ *If v_1 is even then put $c_1 = 0$.*

PROOF: As in the proof of **3.11**, this follows by **3.1** and **3.3**. ∎

LEMMA 3.11. *Let B_T be as in* **3.6** *and let T, V_T, c_1 and z be as in* **3.10**. *Put*

$$\begin{aligned} W_T &= V_T - \{z\} \qquad && c_1 \neq 0 \\ W_T &= V_T && c_1 = 0. \end{aligned}$$

Then the centre of B_T is the subalgebra generated by $\{c_i^{\pm 1} \mid c_i \neq 0$, for $i = -1, 0, \dots, r\}$. If B_T is presented by $B_T = k^\sigma P$ (see **1.6***) then $P = P' \bigoplus P_\sigma$ and the rank of P_σ is equal to the number of nonzero c_i's, $i = -1, 0, \dots, r$.*

PROOF: Let S_T be as in the proof of **3.6**. Note that each element of $S_T - W_T$ is the divisor s of nonzero c_i, $i = -1, 0, 1, \dots, r$, such that the power of s is 1 or -1, for example, if $c_{-1} \neq 0$ then $z_1 \in S_T - W_T$ and if $c_r \neq 0$ then $z_{i_r - 1} \in S_T - W_T$, and that the number of elements in W_T is even. Rewrite the elements of W_T as $w_1, w_2, \dots, w_{2p}$, say, where $w_1 > w_2 > \dots > w_{2p}$ in the ordering

$$x_n > z_n > y_n > x_{n-1} > z_{n-1} > y_{n-1} > \dots > x_1 > z_1 > y_1.$$

By the McConnell-Pettit criterion [**6**], we see that the subalgebra W of B_T generated by all $w_i^{\pm 1}$ is simple. But in **3.13**, we shall give another proof for the simplicity of W in order to avoid routine and messy calculations in finding the determinant of a huge matrix. Since the subalgebra of B_T generated by $\{c_i^{\pm 1} \mid c_i \neq 0\}$ is contained in the centre of B_T and each element $s \in S_T - W_T$ is a divisor of each the nonzero c_i with power 1, we have that the centre of B_T is the subalgebra generated by the nonzero $c_i^{\pm 1}$ and $P = P' \bigoplus P_\sigma$, where P' and P_σ are the subgroups corresponding W and the centre $Z(B_T)$, respectively. ∎

PROPOSITION 3.12. *Let A be a simple algebra over a field k and let $B = A[y; \alpha][x; \beta]$ be an iterated skew polynomial ring, where*

$$\alpha : A \longrightarrow A, \qquad \beta : A[y; \alpha] \longrightarrow A[y; \alpha]$$

are automorphisms such that $\beta(A) = A$, $\beta(y) = dy$, $d \in k^*$ and for each pair i, j of nonnegative integers with $i + j \geqslant 1$, there is no $0 \neq a \in A$ satisfying the two conditions

$$d^j \alpha(a) = a, \qquad d^{-i} \beta(a) = a.$$

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Then the localisation $C = A[y^{\pm 1}, x^{\pm 1}]$ at the multiplicative set generated by y and x is simple.

PROOF: Note that all elements of C are uniquely expressed in a form $f = \sum a_{ij} y^i x^j$. Let us denote by length of f the number of nonzero $a_{ij} \in A$. For a nonzero ideal I of C, choose $0 \neq f \in I$ with the smallest length. Suppose that the length of f is greater than 1. We may assume that f is of the form $f = a + by^i x^j +$ (other terms) for some nonzero $a, b \in A$ and some nonnegative integers i, j with $i + j \geqslant 1$. Since $Ab(\alpha^i \beta^j(A)) = A$, we may also assume that $b = 1$. By our hypothesis, we have that $d^j \alpha(a) \neq a$ or $d^{-i}\beta(a) \neq a$, say $d^j \alpha(a) \neq a$. Then

$$fy - d^j y f = (a - d^j \alpha(a))y + \text{ (other terms)} \neq 0$$

and the length of $fy - d^j y f \in I$ is less than that of f. This is a contradiction. Hence the length of f is 1 and so f is invertible. ∎

COROLLARY 3.13. *Let A, B, C, and α, β be as in **3.12** and let $d \in k^*$ not be a root of unity. If $\beta = \alpha^r$ or $\alpha = \beta^r$ in A for some $r \geqslant 1$ then C is simple.*

PROOF: For some pair i, j of nonnegative integers and $i + j \geqslant 1$, and some nonzero $a \in A$, suppose that

$$d^j \alpha(a) = a, \qquad d^{-i}\beta(a) = a.$$

Suppose that $\beta = \alpha^r$. Then $a = d^{-i}\beta(a) = d^{-i}\alpha^r(a) = d^{-i-rj}a$, which is absurd because d is not a root of unity and $-i - rj < 0$. Hence C is simple by **3.12**. For the case $\alpha = \beta^r$, the proof is similar. ∎

(Proof for the simplicity of W in the proof of **3.11**.) Under the same notations as in the proof of **3.11**, note that $w_{2i-1}w_{2i} = q^k w_{2i} w_{2i-1}$ for some $k = \pm 1$ or ± 2 because y_j and z_j with the same index j cannot be $w_{2i-1} = z_j$, $w_{2i} = y_j$ by the construction of W_T. Then, by induction on p, the simplicity of W follows immediately from **3.13**. ∎

3.14. Let T be an admissible set $\mathcal{O}_q^n$ without removable indices. Call the rank of P_σ in **3.11** the *degree* of T, and denote it by deg(T).

If T is arbitrary admissible set of $\mathcal{O}_q^n$ then there are $m \leqslant n$ and an admissible set T' of $\mathcal{O}_q^m$ without removable indices by **3.7**. Denote $\deg(T) = \deg(T')$. Call an admissible set T *connected* if T satisfies the property: if $z_i \in T, z_j \in T$ and $i < j$ then $z_l \in T$ for all $i \leqslant l \leqslant j$ (see [**8**, 1.6 (2)]). Clearly every admissible set T' without removable indices is the disjoint union of connected admissible subsets without removable indices. By **3.9**, **3.10** and **3.11**, it is easy to find $\deg(T)$ for any admissible set T without removable indices.

THEOREM. *Let T be an admissible set of $\mathcal{O}_q^n$ and let $\mathrm{Prim}_T(\mathcal{O}_q^n)$ be the set of all primitive ideals P of $\mathcal{O}_q^n$ such that $P \cap \wp_n = T$. Then there is a bijection between*

$Prim_T(\mathcal{O}_q^n)$ and the set of all maximal ideals $Max(k[t_1^{\pm 1}, \dots, t_s^{\pm 1}])$, where $s = \deg T$, and $Prim(\mathcal{O}_q^n) = \bigsqcup_T Prim_T(\mathcal{O}_q^n)$.

PROOF: If $n = 1$ then $\mathcal{O}_q^1$ is the commutative polynomial ring with two variables, hence the theorem follows from Hilbert's Nullstelnsatz because every primitive ideal of a commutative ring is maximal. Because of induction on n, and **3.7**, we may assume that T has no removable indices. The theorem then follows immediately from **1.5**, **3.2**, **3.6** and **3.11**. ∎

4. Poisson strcture of the quantum Euclidean space

4.1. In **3.6**, if $T = \emptyset$ then the subalgebra $A_\emptyset$ of $\mathcal{O}_q^n$ is generated by

$$S_\emptyset = \{y_1, \dots, y_n, z_1, \dots, z_n\}$$

and thus the matrix $\lambda_\emptyset$ is

$$\lambda_\emptyset = (\lambda_{ij}) = \begin{pmatrix} 1 & q & q & \cdots & q & 1 & 1 & 1 & \cdots & 1 \\ q^{-1} & 1 & q & \cdots & q & q^{-2} & 1 & 1 & \cdots & 1 \\ q^{-1} & q^{-1} & 1 & \cdots & q & q^{-2} & q^{-2} & 1 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ q^{-1} & q^{-1} & q^{-1} & \cdots & 1 & q^{-2} & q^{-2} & q^{-2} & \cdots & 1 \\ 1 & q^2 & q^2 & \cdots & q^2 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & q^2 & \cdots & q^2 & 1 & 1 & 1 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

and $P(\lambda_\emptyset)$ can be presented by the twisted group algebra $k^\sigma P$ by **1.6**, where P has rank $2n$ and an antisymmetric bimultiplicative map σ is given by

$$\sigma(e_i, e_j) = \lambda_{ij}^{1/2}.$$

By **2.7**, kP is a Poisson algebra with Poisson bracket induced by the antisymmetric bilinear map $u \in Z^2(P, k)$ defined by

$$q^{u(e_i, e_j)} = \sigma^2(e_i, e_j) = \lambda_{ij}.$$

4.2. Let A_n be the commutative algebra $A_n = k[y_1, \dots, y_n, x_1, \dots, x_n]$ with $2n$ variables and for each $i = 1, \dots, n$, set, as in **3.3**,

$$z_i = q^{-2}(1 - q^2) \sum_{1 \leqslant l \leqslant i} q^{l-i} y_l x_l.$$

Then we have that

$$y_i x_i = q^2\left(1-q^2\right)^{-1}\left(z_i - q^{-1}z_{i-1}\right),\ (z_0 = 0), \qquad 1 \leqslant i \leqslant n.$$

Let C be the subalgebra of A_n generated by $y_1, \dots, y_n, z_1, \dots, z_n$ and let D be the localisation of C with respect to the multiplicative set generated by $y_1, \dots, y_n, z_1, \dots, z_n$. Then $C \subseteq A_n \subseteq D \cong kP$ since each y_i is invertible in D and so $x_i \in D$ for each i, and D has a Poisson bracket endowed from the isomorphism from D onto kP defined by

$$y_i \mapsto t_{e_i},\ z_i \mapsto t_{e_{n+i}}.$$

More precisely, D has the following Poisson bracket:

$$\begin{array}{llll} \{y_i, y_j\} = y_i y_j & (i < j) & \{y_i, z_j\} = 0 & (i \leqslant j) \\ \{y_i, z_j\} = -2y_i z_j & (i > j) & \{z_i, z_j\} = 0 & (\text{all } i, j). \end{array}$$

Thus C becomes a sub-Poisson-algebra of D and z_n is a Poisson central element of D. Moreover, A_n is also a sub-Poisson-algebra of D because we have the following formulas in D: $\widehat{q} = q^2\left(1-q^2\right)^{-1}$

$$\begin{array}{lr} \{x_i, z_j\} = \left\{\widehat{q}y_i^{-1}\left(z_i - q^{-1}z_{i-1}\right), z_j\right\} = 0 & (i \leqslant j) \\ \{x_i, z_j\} = \left\{\widehat{q}y_i^{-1}\left(z_i - q^{-1}z_{i-1}\right), z_j\right\} = 2x_i z_j & (i > j) \\ \{y_i, x_j\} = \left\{y_i, \widehat{q}y_j^{-1}\left(z_j - q^{-1}z_{j-1}\right)\right\} = -y_i x_j & (i \neq j) \\ \{y_i, x_i\} = \left\{y_i, \widehat{q}y_i^{-1}\left(z_i - q^{-1}z_{i-1}\right)\right\} = 2q^{-1}\widehat{q}z_{i-1} & (\text{all } i) \\ \{x_i, x_j\} = \left\{\widehat{q}y_i^{-1}\left(z_i - q^{-1}z_{i-1}\right), \widehat{q}y_j^{-1}\left(z_j - q^{-1}z_{j-1}\right)\right\} = -x_i x_j & (i < j). \end{array}$$

4.3. We define an admissible set of A_n as in **3.4**. We shall say that a subset T of $\wp_n = \{x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n\}$ is *admissible* if T satisfies the conditions:

(i) $y_i \in T$ or $x_i \in T$ if and only if $z_i \in T$ and $z_{i-1} \in T$, $2 \leqslant i \leqslant n$.

(ii) $y_1 \in T$ or $x_1 \in T$ if and only if $z_1 \in T$.

As in **3.7**, if T is an admissible set of A_n such that $y_i \in T$ and $x_i \in T$ for some i then the index i is said to be removable.

Lemma 4.4.

(1) *Every ideal generated by an admissible set of A_n is prime Poisson.*

(2) *For each prime Poisson ideal P of A_n, $P \cap \wp_n$ is an admissible set of A_n.*

(3) *If T is an admissible set of A_n with removable indices then there are $m < n$ and an admissible set T' of A_m such that $A_n/\langle T\rangle \cong A_m/\langle T'\rangle$ as Poisson algebras and T' has no removable indices.*

PROOF: Note that each z_i is an irreducible element of B. (1) and (2) follow immediately from **4.2**, and (3) follows from the modified proof of **3.8**. ∎

4.5. Let T be an admissible set of A_n. Define S_T, as in the proof of **3.6**, by

$$T_y = \{y_j \mid y_j \notin T, x_j \in T\}$$
$$T_x = \{x_j \mid x_j \notin T, y_j \in T\}$$
$$S_T = \{y_j, z_j \mid z_j \notin T\} \cup \{y_j \mid z_j \in T,\ y_j \notin T,\ x_j \notin T\} \cup T_y \cup T_x$$

and let C_T be the subalgebra of A_n generated by S_T. Then C_T is embedded into $A_n/\langle T\rangle$, the localisation D_T of C_T with respect to the multiplicative set generated by S_T is isomorphic to a group algebra kP' and

$$C_T \subseteq A_n/\langle T\rangle \subseteq D_T \cong kP'.$$

The commutative algebra $D_T \cong kP'$ has the Poisson bracket induced by an antisymmetric bilinear map u defined by

$$q^{u(e_i,e_j)} = \sigma^2(e_i, e_j) = \lambda_{ij},$$

where $\sigma^2(e_i, e_j) = \lambda_{ij}$ is the (i,j)-entry of the defining matrix λ_T in the twisted group algebra $P(\lambda_T)$ of **3.6**.

LEMMA. *The Poisson structures of $A_n/\langle T\rangle$ induced by that of A_n and by that of $D_T \cong kP'$ are equal and C_T is a sub-Poisson algebra.*

PROOF: Straightfoward. ∎

THEOREM 4.6. *For each admissible set T of the Poisson algebra A_n, let $Symp_T(A_n)$ be the set of all symplectic ideals Q of B with $Q \cap \wp_n = T$. Then there is a bijection between $Symp_T(A_n)$ and Max($k[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$), and $Symp(A_n) = \bigsqcup_T Symp_T(A_n)$, where $s = deg(T)$ when T is considered as an admissible set of $\mathcal{O}_q^n$. Moreover, there is a bijection between Prim $\mathcal{O}_q^n$ and symp(A_n).*

PROOF: If $n = 1$ then A_1 has trivial Poisson structure and so there is nothing to prove since symplectic ideals of Poisson algebra with trivial Poisson structure are only maximal ideals. Assume $n > 1$. By induction on n, and **4.4** (3), we may assume that T has no removable indices. By **4.5** and **2.7**, the centre of the twisted group algebra $P(\lambda_T)$ of **3.6** and the Poisson centre of D_T of **4.5** are equal, hence the conclusion follows immediately from **2.6**, **3.11** and **3.14**. ∎

THEOREM 4.7. *All symplectic ideals of the Poisson algebra A_{n+1}/I, $I = \langle y_1 - q^{1/2}(1+q)^{-1}x_1\rangle$, correspond bijectively to Prim $\mathcal{O}_q(\mathfrak{o}k^{2n+1})$.*

PROOF: By [**9, 5**], the map f from $\mathcal{O}_q^{n+1}$ into $\mathcal{O}_q(\mathfrak{o}k^{2n+1})$ given by

$$y_1 \mapsto q^{1/2}(1+q)^{-1}z_0,\ x_1 \mapsto z_0,\ y_i \mapsto y_{i-1},\ x_i \mapsto x_{i-1}\ \ (i \geqslant 2)$$

is an epimorphism with kernel $\langle y_1 - q^{1/2}(1+q)^{-1}x_1\rangle$, and the ideal $I = \langle y_1 - q^{1/2}(1+q)^{-1}x_1\rangle$ of A_{n+1} is a prime Poisson ideal and thus A_{n+1}/I is a Poisson algebra. Moreover, in **4.6**, all primitive ideals of $\mathcal{O}_q^{n+1}$ containing $\langle y_1 - q^{1/2}(1+q)^{-1}x_1\rangle$ correspond bijectively to all symplectic ideals of A_{n+1} containing I. $\square$

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https://doi.org/10.1017/S0004972700032007 Published online by Cambridge University Press