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POSITIVE SOLUTIONS FOR NON-RESONANT SINGULAR BOUNDARY-VALUE PROBLEMS WITH A LINEAR TERM

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Abstract This paper presents new existence results for the singular boundary-value problem

$$-u'' + p(t)u = f(t,u), \quad t \in (0,1),$$
$$u(0) = 0 = u(1).$$

In particular, our nonlinearity f may be singular at $t = 0, 1$ and $u = 0$.

Keywords: non-resonant singular boundary-value problems; positive solution; upper and lower solution

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1. Introduction

The singular boundary-value problem (BVP) of the form

$$\left.\begin{aligned} -u'' = f(t,u), \quad t \in (0,1), \\ u(0) = 0 = u(1), \end{aligned}\right\} \tag{1.1}$$

occurs in several problems in applied mathematics [**1**–**4**]. In this paper we investigate a more general non-resonant singular Dirichlet BVP, namely

$$\left.\begin{aligned} -u'' + p(t)u = f(t,u), \quad t \in (0,1), \\ u(0) = 0 = u(1). \end{aligned}\right\} \tag{1.2}$$

where $p \in C[0,1]$, $p(t) > 0$ for $t \in (0,1)$, and $f : (0,1) \times (0,\infty) \to R$ is continuous. Notice that f may be singular at $t = 0, 1$ and $u = 0$. We obtain the existence of $C[0,1] \cap C^2(0,1)$ non-negative solutions. Of course, by a solution u of the BVP (1.2) we mean $u : [0,1] \to R$, which satisfies the differential equation in (1.2) on $(0,1)$ and the stated boundary data.

We will let $C[0,1]$ denote the class of maps u which are continuous on $[0,1]$, with norm $|u|_\infty = \max_{t\in[0,1]} |u(t)|$. Let

$$M = \left\{ h \in C(0,1) : \int_0^1 s(1-s)|h(s)|\,\mathrm{d}s < \infty, \right.$$
$$\lim_{t\to 0^+} t^2(1-t)|h(t)| = 0 \text{ if } \int_0^1 (1-s)|h(s)|\,\mathrm{d}s = \infty$$
$$\left. \text{and } \lim_{t\to 1^-} t(1-t)^2|h(t)| = 0 \text{ if } \int_0^1 s|h(s)|\,\mathrm{d}s = \infty \right\}. \quad (1.3)$$

The main results of the paper are as follows.

Theorem 1.1. *Suppose the following conditions hold.*

(H1) *There exists a constant $L > 0$ such that, for any compact set $K \subset (0,1)$, there is $\varepsilon = \varepsilon_K > 0$ with*

$$f(t,x) \geqslant L \quad \text{for all } t \in K,\ x \in (0,\varepsilon].$$

(H2) *For any $\delta > 0$ there exist $h_\delta \in M$, $h_\delta(t) > 0$ for $t \in (0,1)$ such that*

$$|f(t,x)| \leqslant h_\delta(t) \quad \text{for all } t \in (0,1),\ x \geqslant \delta.$$

Then problem (1.2) has at least one positive solution $u \in C[0,1] \cap C^2(0,1)$. If, moreover, $f(t,\cdot)$ is non-increasing, for each $t \in (0,1)$, then the solution is unique.

Theorem 1.2. *Suppose that (H1) holds. Moreover, suppose the following conditions also hold.*

(H3) *$f(t,x) = q(t)m(t,x)$ with $q > 0$ on $(0,1)$, $q \in M$ and $m : [0,1] \times (0,\infty) \to R$ is continuous with*

$$|m(t,x)| \leqslant g(x) + h(x) \text{ on } [0,1] \times (0,\infty),$$
$$g > 0 \text{ continuous and non-increasing on } (0,\infty),$$
$$h \geqslant 0 \text{ continuous on } [0,\infty)$$
$$h/g \text{ non-decreasing on } (0,\infty).$$

(H4) *For any $R > 0$, $1/g$ is differentiable on $(0,R]$ with $g' < 0$ a.e. on $(0,R]$ and $g'/g^2 \in L^1[0,R]$. In addition, suppose that there exists $C > 0$ with*

$$\left[1 + \frac{h(C)}{g(C)}\right]^{-1} \int_0^C \frac{\mathrm{d}u}{g(u)} > b_0$$

holding; here

$$b_0 = 2\max\left\{ \int_0^{1/2} t(1-t)q(t)\,\mathrm{d}t,\ \int_{1/2}^1 t(1-t)q(t)\,\mathrm{d}t \right\}.$$

Then problem (1.2) has at least one positive solution $u \in C[0,1] \cap C^2(0,1)$.

Remark 1.3. In [**3**], the authors consider the BVP (1.2) with $p(t) \equiv 0$ for $t \in [0,1]$ under conditions (H1) and (H2).

Remark 1.4. In [**1**, p. 186], the authors consider the BVP (1.2) with $p(t) \equiv 0$ for $t \in [0,1]$ under conditions (H1), (H3) and (H4).

Remark 1.5. If $p \in C[0,1]$, $p(t) > 0$ for $t \in (0,1)$, then note that

$$-u'' + p(t)u = 0, \quad t \in (0,1),$$
$$u(0) = 0 = u(1),$$

has only the trivial solution.

Corollary 1.6. *Suppose (H1) and (H2) (or (H1), (H3) and (H4)) hold. Then, for every fixed $\lambda > 0$, the problem*

$$-u'' + \lambda u = f(t,u), \quad t \in (0,1),$$
$$u(0) = 0 = u(1),$$

has at least one positive solution $u \in C[0,1] \cap C^2(0,1)$.

To conclude this section we look at an example. Consider the BVP

$$\left.\begin{aligned} -u''(t) + \lambda u &= \frac{1}{u^{\alpha}} \quad \text{for } t \in (0,1), \\ u(0) &= u(1) = 0, \end{aligned}\right\} \tag{1.4}$$

where $\lambda \geqslant 0$ and $\alpha > 0$.

For this example we cannot apply [**3**, Theorem 2]. Also it is difficult to demonstrate the conditions (for example $\lambda = 2$, $\alpha = 20$) [**1**, Theorem 2.7.7]. However Corollary 1.6 immediately guarantees that (1.4) at least has a solution $u \in C[0,1] \cap C^2(0,1)$ with $u(t) > 0$ for $t \in (0,1)$ for every fixed $\lambda \geqslant 0$, $\alpha > 0$.

2. The proof of Theorem 1.1

From [**1**, Theorem 1.11.1], we know that

$$-u'' + p(t)u = 0, \quad t \in (0,1),$$
$$u(0) = 0, \quad u'(0) = 1,$$

has only one increasing positive solution $e_1(t) = tb_1(t) \in C[0,1] \cap C^1[0,1)$, where $b_1 \in C[0,1]$ satisfies

$$b_1(t) = 1 + \frac{1}{t}\int_0^t \int_0^{\eta} \tau p(\tau) b_1(\tau)\,\mathrm{d}\tau\,\mathrm{d}\eta.$$

Also,

$$-u'' + p(t)u = 0, \quad t \in (0,1),$$
$$u(1) = 0, \quad u'(1) = -1$$

has only one decreasing positive solution $e_2(t) = (1-t)b_2(t) \in C[0,1] \cap C^1(0,1]$, where $b_2 \in C[0,1]$ satisfies

$$b_2(t) = 1 + \frac{1}{1-t}\int_t^1 \int_\eta^1 (1-\tau)p(\tau)b_2(\tau)\,\mathrm{d}\tau\,\mathrm{d}\eta.$$

Let

$$G(t,s) = \frac{1}{\omega}\begin{cases} e_2(t)e_1(s), & 0 \leqslant s \leqslant t \leqslant 1, \\ e_2(s)e_1(t), & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{2.1}$$

where

$$\omega = \begin{vmatrix} e_2(t) & e_2'(t) \\ e_1(t) & e_1'(t) \end{vmatrix} = \text{const.} > 0.$$

It is easy to see that

$$0 \leqslant G(t,s) \leqslant G(s,s), \quad 0 \leqslant s,t \leqslant 1. \tag{2.2}$$

Consider the two-point BVP

$$\left.\begin{aligned} -u'' + p(t)u = v(t,u), \quad t \in (0,1), \\ u(0) = a = u(1), \end{aligned}\right\} \tag{2.3}$$

where $v : D \to R$ is a continuous function and $D \subset (0,1) \times R$. By a solution $u(\cdot)$ of (2.3) we mean a function $u \in C[0,1] \cap C^2(0,1)$ such that $(t,u(t)) \in D$ for all $t \in (0,1)$ and $-u'' + p(t)u = v(t,u)$ for all $t \in (0,1)$ with $u(0) = a = u(1)$.

Let $\alpha \in C[0,1] \cap C^2(0,1)$ satisfy the following conditions: $(t,\alpha(t)) \in D$ for all $t \in (0,1)$ and

$$-\alpha'' + p(t)\alpha \leqslant v(t,\alpha), \quad t \in (0,1),$$
$$\alpha(0) \leqslant a, \quad \alpha(1) \leqslant a.$$

In this case, we say that $\alpha(\cdot)$ is a lower solution of problem (2.3). The definition of an upper solution $\beta(\cdot)$ of problem (2.3) is given in a completely similar way, just by reversing the above inequalities. Also, if $\alpha, \beta \in C[0,1]$ are such that $\alpha(t) \leqslant \beta(t)$ for all $t \in [0,1]$, we define the set

$$D_\alpha^\beta := \{(t,x) \in (0,1) \times R : \alpha(t) \leqslant x \leqslant \beta(t)\}.$$

We then have the following result.

Theorem 2.1. *Let α and β be, respectively, a lower solution and an upper solution of problem (2.3) such that*

(a1) *$\alpha(t) \leqslant \beta(t)$ for all $t \in [0,1]$, and*

(a2) *$D_\alpha^\beta \subset D$.*

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Assume also that there is a function $h \in M$, $h(t) > 0$, for $t \in (0,1)$, such that

(a3) $|v(t,x)| \leqslant h(t)$ *for all* $(t,x) \in D_\alpha^\beta$.

Then problem (2.3) has at least one solution $\tilde{u}(\cdot)$ *such that*

$$\alpha(t) \leqslant \tilde{u}(t) \leqslant \beta(t) \quad \text{for all } t \in (0,1).$$

Proof of Theorem 2.1. The proof follows the argument in [**3**]. For convenience, we sketch it here.

First of all we define an auxiliary function

$$v^*(t,x) := \begin{cases} v(t,\alpha(t)), & x < \alpha(t), \\ v(t,x), & \alpha(t) \leqslant x \leqslant \beta(t), \\ v(t,\beta(t)), & x > \beta(t). \end{cases}$$

By (a2) and the definition of v^* it can easily be checked that $v^* : (0,1) \times R \to R$ is continuous. From (a3) we have

$$|v^*(t,x)| \leqslant h(t) \quad \text{for } (t,x) \in (0,1) \times R. \tag{2.4}$$

Consider now the problem

$$\left.\begin{aligned} -u'' + p(t)u = v^*(t,u) \quad \text{for } t \in (0,1), \\ u(0) = a = u(1). \end{aligned}\right\} \tag{2.5}$$

It can easily be verified that the Green function of the problem

$$-u'' + p(t)u = v^*(t,u) \quad \text{for } t \in (0,1),$$
$$u(0) = 0 = u(1)$$

is the function $G : [0,1] \times [0,1] \to [0,\infty)$ given by (2.1). Define the operator T by

$$(Tu)(t) := a + \int_0^1 G(t,s) v^*(s,u(s))\, \mathrm{d}s.$$

From (2.4) and the definition of v^* it follows that

$$T : X = C[0,1] \to X$$

is defined, continuous and that $T(X)$ is a bounded set. Moreover, $u \in X$ is a solution of (2.5) if and only if $u = Tu$.

The existence of a fixed point for the operator T will now follow from the Schauder fixed-point theorem if we show that $T(X)$ is relatively compact.

Let $t \in (0,1)$. Then, using (2.4), we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} T(u)(t)\right| \leqslant \frac{C_1}{\omega}\left[\int_t^1 e_2(s)h(s)\,\mathrm{d}s + \int_0^t e_1(s)h(s)\,\mathrm{d}s\right],$$

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where

$$C_1 = \max\left\{\left(1 + \int_0^1 \tau p(\tau) b_1(\tau)\,\mathrm{d}\tau\right), \left(1 + \int_0^1 (1-\tau) p(\tau) b_2(\tau)\,\mathrm{d}\tau\right)\right\}.$$

Letting

$$\tau(t) = \int_t^1 e_2(s) h(s)\,\mathrm{d}s + \int_0^t e_1(s) h(s)\,\mathrm{d}s,$$

we obtain

$$\int_0^1 |\tau(t)|\,\mathrm{d}t \leqslant 2\omega \int_0^1 G(s,s) h(s)\,\mathrm{d}s < \infty.$$

This is sufficient to ensure the relative compactness of the image $T(X)$ via the Ascoli–Arzelà theorem.

As a result, (2.5) has a solution $u \in C[0,1]$. We claim that

$$\alpha(t) \leqslant u(t) \leqslant \beta(t) \quad \text{for all } t \in [0,1]. \tag{2.6}$$

Suppose that, without loss of generality, the first inequality is not true. Then there exists a $t^* \in (0,1)$ with $u(t^*) < \alpha(t^*)$. By continuity, we can find a maximal open interval $(t_1, t_2) \subset (0,1)$ such that $t^* \in (t_1, t_2)$ and

$$u(t_1) = \alpha(t_1), \quad u(t_2) = \alpha(t_2), \quad u(t) < \alpha(t) \quad \text{for all } t \in (t_1, t_2). \tag{2.7}$$

For $t \in (t_1, t_2)$, we have $v^*(t, u(t)) = v(t, \alpha(t))$ and, therefore,

$$-u'' + p(t) u = v(t, \alpha(t)) \quad \text{for all } t \in (t_1, t_2).$$

On the other hand, as α is a lower solution of (2.3), we also have

$$-\alpha'' + p(t)\alpha \leqslant v(t, \alpha(t)) \quad \text{for all } t \in (t_1, t_2).$$

Then, setting

$$z(t) := \alpha(t) - u(t) \quad \text{for } t \in [t_1, t_2],$$

we obtain

$$-z'' + p(t) z \leqslant 0 \quad \text{for } t \in (t_1, t_2), \tag{2.8}$$

with $z(t) > 0$ for $t \in (t_1, t_2)$ and $z(t_1) = 0 = z(t_2)$. Multiplying (2.8) by

$$G_0(t,s) = \frac{1}{t_2 - t_1}\begin{cases} (s - t_1)(t_2 - t) & \text{for } t_1 \leqslant s \leqslant t \leqslant t_2, \\ (t - t_1)(t_2 - s) & \text{for } t_1 \leqslant t \leqslant s \leqslant t_2, \end{cases}$$

and integrating both sides from t_1 to t_2 we have

$$-\int_{t_1}^{t_2} G_0(t,s) z''(s)\,\mathrm{d}s + \int_{t_1}^{t_2} G_0(t,s) p(s) z(s)\,\mathrm{d}s \leqslant 0.$$

Using

$$-\int_{t_1}^{t_2} G_0(t,s)z''(s)\,\mathrm{d}s = z(t),$$

we have

$$z(t) + v(t) \leqslant 0 \quad \text{for } t \in [t_1, t_2], \tag{2.9}$$

where

$$w(t) = \int_{t_1}^{t_2} G_0(t,s)p(s)z(s)\,\mathrm{d}s.$$

Now, since $z(t) > 0$ for $t \in (t_1, t_2)$, we have

$$w'' = -p(t)z(t) < 0 \quad \text{for } t \in (t_3, t_4)$$

and $w(t_1) = w(t_2) = 0$. Thus, $w(t) \geqslant 0$ for $t \in (t_1, t_2)$, so $z(t) + w(t) > 0$ for $t \in (t_1, t_2)$. This contradicts (2.9). □

The proof of Theorem 1.1 follows closely the ideas in [**3**]. For completeness we briefly sketch the proof.

Proof of Theorem 1.1. For any $n \in N$, $n \geqslant 1$, let e_n be the compact subinterval of $(0,1)$ defined by

$$e_n := \left[\frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+1}}\right].$$

From assumption (H1), there exists an $\varepsilon_n > 0$ such that

$$f(t,u) > L \text{ for } (t,u) \in e_n \times (0, \varepsilon_n] \quad \text{and} \quad \varepsilon_n \leqslant \frac{L}{\max_{t\in[0,1]} p(t)}.$$

Without loss of generality (taking, if we need to, a smaller ε_n), we can assume that $\{\varepsilon_n\}$ is a decreasing sequence and $\lim_{n\to+\infty} \varepsilon_n = 0$.

We can choose a function $\alpha \in C[0,1] \cap C^2(0,1)$ (see [**3**, p. 692]) such that

$$\left.\begin{aligned} &\alpha(0) = 0, \qquad \alpha(1) = 0, \\ &\alpha(t) > 0 \quad \text{for } t \in (0,1), \\ \alpha(t) \leqslant &\begin{cases} \varepsilon_1 & \text{for } t \in e_1, \\ \varepsilon_n & \text{for } t \in e_n \setminus e_{n-1},\ n \geqslant 2. \end{cases} \end{aligned}\right\} \tag{2.10}$$

Note that

$$f(t,u) \geqslant L, \quad \forall (t,u) \in (0,1) \times \{u \in (0,\infty) : 0 < u \leqslant \alpha(t)\}. \tag{2.11}$$

Set

$$k_0 := \min\left\{1, \frac{L}{|\alpha''|_\infty + |p\alpha|_\infty + 1}\right\}.$$

Now we make some claims that yield the proof of the theorem.

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Claim 1. *Let $h(t,u) \geqslant f(t,u)$ for $(t,u) \in (0,1) \times (0,\infty)$ with $h : (0,1) \times (0,\infty) \to (0,\infty)$ a continuous function and let $v \in C[0,1] \cap C^2(0,1)$, $v(t) > 0$ for $t \in (0,1)$ be any solution of*

$$-v'' + p(t)v = h(t,v),$$
$$v(0) \geqslant 0, \quad v(1) \geqslant 0.$$

Then

$$v(t) \geqslant k_0 \alpha(t) \quad \text{for } t \in [0,1]. \tag{2.12}$$

*The proof is similar to the proof of [**1**, Theorem 2] and that of (2.6) in this paper. We omit it here.*

We define now, for each $n \in N$, $n \geqslant 1$,

$$\eta_n(t) := \max\left\{\frac{1}{2^{n+1}}, \min\left\{t, 1 - \frac{1}{2^{n+1}}\right\}\right\} \quad \text{for } t \in (0,1)$$

and set

$$\tilde{f}_n(t,u) := \max\{f(\eta_n(t),u), f(t,u)\}.$$

We find that, for each index n, $\tilde{f}_n : (0,1) \times (0,\infty) \to (-\infty,\infty)$ is continuous and

$$\tilde{f}_n(t,u) \geqslant f(t,u) \quad \text{for } (t,u) \in (0,1) \times (0,\infty),$$
$$\tilde{f}_n(t,u) = f(t,u) \quad \text{for } (t,u) \in e_n \times (0,\infty).$$

Hence, the sequence of function $\{\tilde{f}_n\}$ converges to f uniformly on any set of the form $K \times (0,\infty)$, where K is an arbitrary compact subset of $(0,1)$.

Next we define, by induction,

$$f_1(t,u) := \tilde{f}_1(t,u),$$
$$f_2(t,u) := \min\{f_1(t,u), \tilde{f}_2(t,u)\},$$
$$\vdots$$
$$f_{n+1}(t,u) := \min\{f_n(t,u), \tilde{f}_{n+1}(t,u)\},$$
$$\vdots$$

Each of the f_i is a continuous function defined on $(0,1) \times (0,\infty)$. Moreover,

$$f_1(t,u) \geqslant f_2(t,u) \geqslant \cdots \geqslant f_n(t,u) \geqslant f_{n+1}(t,u) \geqslant \cdots \geqslant f(t,u) \tag{2.13}$$

and the sequence $\{f_n\}$ converges to f uniformly on compact subsets of $(0,1) \times (0,\infty)$. We also note that

$$f_n(t,u) = f(t,u) \quad \text{for } (t,u) \in e_n \times (0,\infty).$$

Consider the sequence of BVPs

$$\left.\begin{aligned} -u'' + p(t)u = f_n(t,u) \quad \text{in } (0,1), \\ u(0) = u(1) = \varepsilon_n. \end{aligned}\right\} \qquad (2.14)_n$$

Claim 2. *For any $c \in (0, \varepsilon_n]$, the constant function $\alpha_n(\cdot) \equiv c$ is a lower solution of problem (2.14)$_n$.*

It is easy to prove (i.e. it is clear once we prove (use induction), for each $t \in (0, 1)$, that $cp(t) \leqslant f_n(t, c)$ for $t \in (0, \varepsilon_n]$), so we leave the details to the reader.

Claim 3. *Any solution $u_n(\cdot)$ of (2.14)$_n$ is an upper solution of (2.14)$_{n+1}$.*

Proof of Claim 3. From (2.13) we have

$$-u_n'' + p(t)u_n = f_n(t, u_n) \geqslant f_{n+1}(t, u_n) \quad \text{for } t \in (0, 1).$$

Moreover, $u_n(0) = u(1) = \varepsilon_n > \varepsilon_{n+1}$ and the conclusion follows.

Claim 4. *Problem (2.14)$_1$ has at least one solution.*

Proof of Claim 4. We fix a constant $c_1 > \varepsilon_1$. From (H2) we can find a function $h_{c_1} \in M$ such that

$$|f(t, u)| \leqslant h_{c_1}(t) \quad \text{for } (t, u) \in (0, 1) \times (c_1, \infty).$$

Moreover,

$$|f(\eta_1(t), u)| \leqslant h_{c_1}(\eta_1(t)) \leqslant R \quad \text{for } (t, u) \in (0, 1) \times (c_1, \infty),$$

where $R > c_1 \max_{t\in[0,1]} p(t)$ is a suitable constant. Setting $q(t) := h_{c_1}(t) + R$, we have $q \in M$ with

$$|f_1(t, u)| \leqslant q(t) \quad \text{for } (t, u) \in (0, 1) \times (c_1, \infty). \tag{2.15}$$

Let $\beta \in C[0, 1] \cap C^2(0, 1)$ be the solution of the BVP

$$-u'' + p(t)u = q(t),$$
$$u(0) = u(1) = c_1.$$

It is easy to check that such a solution exists. We can prove (see the proof of (2.6)) that

$$\beta(t) \geqslant c_1 \quad \text{for } t \in [0, 1].$$

From (2.15), we have

$$-\beta'' + p(t)\beta = q(t) \geqslant f_1(t, \beta),$$

and so β is an upper solution of problem (2.14)$_1$.

If we now take $\alpha_1 \equiv \varepsilon_1$ and recall Claim 2, we find that α_1 and $\beta_1 := \beta$ are a lower solution and an upper solution, respectively, of problem (2.14)$_1$ with $\alpha_1(t) \leqslant \beta_1(t)$ for $t \in (0, 1)$. Then, by Theorem 2.1 we know that there is a solution $u_1(\cdot)$ of (2.14)$_1$ such that $\varepsilon_1 = \alpha_1(t) \leqslant u_1(t) \leqslant \beta_1(t)$ for $t \in (0, 1)$. Claim 4 is thus proved. □

By Claim 2 and proceeding by induction using Claim 3, we obtain (via Theorem 2.1) a sequence $\{u_n(\cdot)\}$ of solutions to $(2.14)_n$ such that

$$\begin{aligned} \varepsilon_n \leqslant u_n(t) \leqslant u_{n-1}(t) \qquad & \text{for } t \in [0,1], \\ k_0\alpha(t) \leqslant u_n(t) \qquad & \text{for } t \in [0,1], \\ u_n(0) = \varepsilon_n, \quad u_n(1) = \varepsilon_n. & \end{aligned}$$

We see that the series of functions $\{u_j(t)\}_{j=1}^{\infty}$ converges pointwise on $[0,1]$. Let

$$u(t) = \lim_{n\to\infty} u_n(t).$$

It is clear that, for any $n \geqslant 1$,

$$k_0\alpha(t) \leqslant u(t) \leqslant u_n(t) \quad \text{for } t \in [0,1]. \tag{2.16}$$

Now let $K \subset (0,1)$ be a compact interval.

There is an index $n^* = n^*(K)$ such that $K \subset K_n$ for all $n \geqslant n^*$ and, therefore, for these $n \geqslant n^*$,

$$-u_n'' + p(t)u_n = f_n(t, u_n(t)) = f(t, u_n(t)) \quad \text{for } t \in K.$$

Hence, the function u_n is a solution of equation (1.2) for all $t \in K$ and $n \geqslant n^*$. Moreover,

$$\sup\{|f(t,x)| + p(t)x : t \in K,\ k_0\alpha(t) \leqslant x \leqslant u_{n^*}(t)\} < \infty.$$

Thus, by the Ascoli–Arzelà theorem one can conclude that u is a solution of (1.2) on interval K. Since K was arbitrary, we find that

$$-u'' + p(t)u = f(t,u) \quad \text{for } t \in (0,1).$$

Moreover, $u(0) = u(1) = \lim_{n\to\infty} \varepsilon_n = 0$. One can easily prove (see [**3**, p. 697]) that u is continuous at $t = 0, 1$.

Using the method in the proof of (2.6) we can easily make the following claim.

Claim 5. *Suppose that, for each $t \in (0,1)$, $f(t\cdot)$ is non-increasing. Then (1.2) has at most one solution.*

□

3. The proof of Theorem 1.2

Let

$$f^*(t,x) = \begin{cases} f(t,x), & x \leqslant C, \\ f(t,C), & x > C, \end{cases}$$

and

$$m^*(t,x) = \begin{cases} m(t,x), & x \leqslant C, \\ m(t,C), & x > C. \end{cases}$$

Consider the BVP

$$\left.\begin{aligned} -u'' + p(t)u = f^*(t,u), \quad t \in (0,1), \\ u(0) = 0 = u(1). \end{aligned}\right\} \tag{3.1}$$

Theorem 1.1 guarantees that problem (3.1) has a positive solution $u^* \in C[0,1] \cap C^2(0,1)$.

Next we show that

$$u^*(t) \leqslant C \quad \text{for } t \in [0,1]. \tag{3.2}$$

Suppose that (3.2) is false. Now, since $u^*(0) = u^*(1) = 0$, there exists either

(i) $t_1, t_2 \in (0,1)$, $t_2 < t_1$ with $0 < u^*(t) \leqslant C$ for $t \in [0,t_2)$, $u^*(t) = C$ and $u^*(t) > C$ on (t_2,t_1) with $u^{*\prime}(t_1) = 0$, or

(ii) $t_3, t_4 \in (0,1)$, $t_4 < t_3$ with $0 < u^*(t) \leqslant C$ for $t \in (t_3,1]$, $u^*(t_3) = C$ and $u^*(t) > C$ on (t_4,t_3) with $u^{*\prime}(t_4) = 0$.

We can assume without loss of generality that either $t_1 \leqslant \frac{1}{2}$ or $t_4 \geqslant \frac{1}{2}$. Suppose $t_1 \leqslant \frac{1}{2}$. Notice that for $t \in (t_2,t_1)$ we have

$$\begin{aligned} -u^{*\prime\prime} &\leqslant -u^{*\prime\prime} + p(t)u^* \\ &= q(t)m^*(t,u^*) \\ &= q(t)m(t,C) \\ &\leqslant q(t)[g(C) + h(C)]. \end{aligned} \tag{3.3}$$

Integrate (3.3) from t_2 to t_1 to obtain

$$u^{*\prime}(t_2) \leqslant [g(C) + h(C)] \int_{t_2}^{t_1} q(s)\,\mathrm{d}s,$$

and this, together with $u^*(t_2) = C$, yields

$$\frac{u^{*\prime}(t_2)}{g(u^*(t_2))} \leqslant \left[1 + \frac{h(C)}{g(C)}\right] \int_{t_2}^{t_1} q(s)\,\mathrm{d}s. \tag{3.4}$$

Also, for $t \in (0,t_2)$ we have

$$\begin{aligned} -u^{*\prime\prime} &\leqslant -u^{*\prime\prime} + p(t)u^* = q(t)m(t,u^*) \\ &\leqslant q(t)[g(u^*(t)) + h(u^*(t))], \end{aligned}$$

and so

$$\begin{aligned} \frac{-u^{*\prime\prime}(t)}{g(u^*(t))} &\leqslant q(t)\left[1 + \frac{h(u^*(t))}{g(u^*(t))}\right] \\ &\leqslant q(t)\left[1 + \frac{h(C)}{g(C)}\right] \quad \text{for } t \in (0,t_2). \end{aligned}$$

Integrate from $t \in (0,t_2)$ to t_2 to obtain

$$\frac{-u^{*\prime}(t_2)}{g(u^*(t_2))} + \frac{u^{*\prime}(t)}{g(u^*(t))} + \int_t^{t_2} \left\{\frac{-g'(u^*(t))}{g^2(u^*(t))}\right\}[u^{*\prime}(t)]^2\,\mathrm{d}t \leqslant \left[1 + \frac{h(C)}{g(C)}\right] \int_t^{t_2} q(s)\,\mathrm{d}s \tag{3.5}$$

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and this, together with (3.4) and (3.5), yields

$$\frac{u^{*\prime}(t)}{g(u^*(t))} \leqslant \left[1+\frac{h(C)}{g(C)}\right]\int_t^{t_1} q(s)\,\mathrm{d}s \quad \text{for } t\in(0,t_2).$$

Integrate from 0 to t_2 to find

$$\int_0^C \frac{\mathrm{d}v}{g(v)} \leqslant \left[1+\frac{h(C)}{g(C)}\right]\frac{1}{1-t_1}\int_0^{t_1} s(1-s)q(s)\,\mathrm{d}s,$$

i.e.

$$\int_0^C \frac{\mathrm{d}v}{g(v)} \leqslant 2\left[1+\frac{h(C)}{g(C)}\right]\int_0^{1/2} s(1-s)q(s)\,\mathrm{d}s$$
$$\leqslant b_0\left[1+\frac{h(C)}{g(C)}\right].$$

This is a contradiction, so (3.2) holds (a similar argument yields a contradiction if $t_4 \geqslant \frac{1}{2}$). Thus, we have

$$0 < u^*(t) \leqslant C \quad \text{for } t\in(0,1), \qquad u^*(0)=u^*(1)=0,$$

so $u^* \in C[0,1]\cap C^2(0,1)$ is a positive solution of problem (1.2).

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