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THE ELECTROSTATIC ENERGY OF A TWO-DIMENSIONAL SYSTEM

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The use of the complex variable $z(=x+iy)$ and the complex potential $W(=U+iV)$ for two-dimensional electrostatic systems is well known and the actual system in the (x, y) plane has an image system in the (U, V) plane. It does not seem to have been noticed previously that the electrostatic energy per unit length of the actual system is simply related to the area of the image domain in the (U, V) plane.

The electrostatic energy density is $\frac{1}{2}\epsilon E^2$ (**1**) where E is the electric field and ϵ is the permittivity. The energy per unit length in a two-dimensional system, is therefore given by

$$\frac{1}{2}\int\int \epsilon E^2 dxdy \quad \text{......................................(1)}$$

where the integration is over the whole field (which will be bounded by conductors in general).

Now if $W=U+iV=f(z)=f(x+iy)$
Then (**2**)

$$|E| = \left|\frac{dW}{dz}\right|$$

and the electrostatic energy per unit length is

$$\frac{1}{2}\epsilon\int\int\left|\frac{dW}{dz}\right|^2 dxdy$$

and this is equal to (**3**)

$$\frac{1}{2}\epsilon\int\int dUdV \quad \text{......................................(2)}$$

where the integration is taken over the image domain in the (U, V) plane of the domain of the field in the (x, y) plane. Formula (2) does not seem to have been developed in this form previously, and I cannot find any mention of it in the standard treatises.

The method is best illustrated by means of an example. Consider a capacitor composed of two coaxal cylindrical conductors of radii a, $b(>a)$ whose potentials are respectively 0, V_0. Then the field between the two conductors is given by

$$W=\frac{iV_0}{\log(b/a)}\log(z/a)$$

Clearly the region between the two conductors corresponds to

$$\frac{-V_0\pi}{\log(b/a)} < U < \frac{V_0\pi}{\log(b/a)}, \quad 0 < V < V_0$$

in the (U, V) *plane and so the energy per unit length is given by*

$$\tfrac{1}{2}\epsilon\,\frac{2\pi}{\log(b/a)}\,V_0^2$$

giving a capacity per unit length of

$$2\pi\epsilon\,\{\log(b/a)\}^{-1}$$

It will be seen that the method outlined above can save a considerable amount of work in calculating the electrostatic energy of a two-dimensional system, as it is unnecessary to calculate the actual field strengths and integrate. All that is necessary is to find the image domain.

It may be remarked that the method may also be applied to two-dimensional hydrodynamical flow to find the kinetic energy of the fluid per unit length. This is given by

$$\tfrac{1}{2}\rho\int\int q^2\,dxdy \qquad \text{.........(3)}$$

where ρ is the density and q the fluid velocity. In exactly the same way, the integral (3) may be shown to reduce to

$$\tfrac{1}{2}\rho\int\int dUdV$$

where U, V are now the stream function and velocity potential respectively.

REFERENCES

(1) W. R. Smythe, *Static and Dynamic Electricity*, p. 31.
(2) *loc. cit.*, p. 75.
(3) E. G. Phillips, *Functions of a Complex Variable*, p. 39.

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