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ON THE SUMS OF COMPOUND NEGATIVE BINOMIAL AND GAMMA RANDOM VARIABLES

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Abstract

We study the convolution of compound negative binomial distributions with arbitrary parameters. The exact expression and also a random parameter representation are obtained. These results generalize some recent results in the literature. An application of these results to insurance mathematics is discussed. The sums of certain dependent compound Poisson variables are also studied. Using the connection between negative binomial and gamma distributions, we obtain a simple random parameter representation for the convolution of independent and weighted gamma variables with arbitrary parameters. Applications to the reliability of m-out-of-n:G systems and to the shortest path problem in graph theory are also discussed.

Keywords: Compound negative binomial distribution; convolution; random parameter representation; compound Poisson distribution; sums of gamma random variables; Poisson process

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Secondary 62E15

1. Introduction

The compound negative binomial (CNB) model arises naturally in several fields, such as insurance mathematics and actuarial sciences, and has been studied by several authors. For a recent reference, see Drekic and Willmot (2005) and the references therein. It also arises in nonactuarial applications (see Johnson *et al.* (2005, pp. 232–250) and Vellaisamy and Upadhye (2007)). Recently, Furman (2007) studied the sums of independent negative binomial random variables and obtained an interesting recurrence relation for computing its probability mass function (PMF). He also showed that the convolution of a negative binomial distribution with arbitrary parameters is a negative binomial distribution, but with a random parameter.

In Section 2 we first derive an exact expression for the distribution of sums of CNB random variables. For the negative binomial (NB) case, this expression reduces to a finite-sum expression which is numerically compared with the series expression of Furman (2007). We also obtain a simple random parameter representation for the convolution of CNB distributions, where the compounding distributions $Q_j = Q$. Theorems 2.1 and 2.2 of Furman (2007) follow as special cases. Our approach is essentially that of Furman (2007), except that we use the distribution itself rather than using its moment generating function (MGF).

If the Q_js are different then the convolution of CNB distributions is neither a CNB nor a mixture of CNBs. In such cases, a compound Poisson (CP) representation is presented. It is also shown that a sum of certain dependent CP variables is again a CP variable. It is well known

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that the weighted sums of independent gamma variables arise in several contexts in probability and statistics. The reader may refer to Diaconis and Perlman (1990) for several examples and applications. They mentioned that the distribution of such a sum is not expressible in a closed form and so discussed approximations and studied tail probabilities. Using the connection between the NB and gamma distributions (see Engel and Zijlstra (1980) or Vellaisamy and Sreehari (2008)), we obtain, in Section 3, the exact distribution of weighted sums of independent gamma random variables with arbitrary parameters.

In Section 4 we discuss several interesting examples and applications of the results obtained in Sections 2 and 3. The problems of total claim amount and the distribution of combined portfolios, which arise in insurance mathematics, are discussed as applications of convolutions of CNB variables and certain dependent CP variables, respectively. Furthermore, two important applications of gamma convolutions, namely, the reliability of m-out-of-n:G systems with dynamic failure rates and the shortest path problem in graph theory, are analyzed in detail. At the end, the main contributions of the paper are briefly outlined.

2. Convolution of CNB variables

Let $\mathbb{Z}_+ = \{0, 1, \dots\}$ be the set of nonnegative integers, let $0 < p < 1$, and let $q = (1 - p)$. Let $N \sim \mathrm{NB}(\alpha, p)$, the NB distribution with

$$\mathrm{P}(N = m) = \binom{\alpha + m - 1}{m} p^{\alpha} q^{m}, \qquad m \in \mathbb{Z}_+,\ \alpha > 0,\ 0 < p < 1. \tag{2.1}$$

Then, a real-valued random variable Y is said to follow a CNB distribution with parameters α, p, and Q, denoted by $\mathrm{CNB}(\alpha, p, Q)$, if it admits the random sum representation $Y = \sum_{i=1}^{N} W_i$, where $N \sim \mathrm{NB}(\alpha, p)$ and $\{W_i\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with distribution Q that is independent of N.

Let $m \in \mathbb{Z}_+$, let δ_m be the Dirac measure concentrated at m, and let Q^m denote the m-fold convolution of Q. Then, the distribution $\mathcal{L}(Y)$ of Y is given by

$$\begin{aligned}
\mathcal{L}(Y)(A) &= \sum_{m=0}^{\infty} \binom{\alpha + m - 1}{m} p^{\alpha} (qQ)^{m}(A) &(2.2)\\
&= p^{\alpha} \sum_{m=0}^{\infty} (-1)^m \binom{-\alpha}{m} (qQ)^{m}(A)\\
&= p^{\alpha} (\delta_0 - qQ)^{-\alpha}(A)\\
&= \left(\delta_0 - \frac{q}{p}(Q - \delta_0)\right)^{-\alpha}(A), &(2.3)
\end{aligned}$$

where A is any Borel-measurable set. Indeed, (2.3) is a formal representation for (2.2) in the sense that, when (2.3) is expanded as a power series, the powers of Q represent its convolutions.

When $Q = \delta_1$, $Q^m = \delta_1^m = \delta_m$ and, hence, from (2.2), $\mathrm{CNB}(\alpha, p, \delta_1) = \mathrm{NB}(\alpha, p)$. Also, when $\alpha = 1$, $\mathrm{CNB}(1, p, Q) := \mathrm{CG}(p, Q)$ denotes the compound geometric distribution with parameters p and Q.

First we obtain an exact representation for the convolution of CNB distributions with $Q_j = Q$.

Theorem 2.1. *Let $Y_j \sim \mathrm{CNB}(\alpha_j, p_j, Q)$ for $1 \le j \le n$, and let $S_n = \sum_{j=1}^{n} Y_j$. Then*

$$\mathrm{P}(S_n \le x) = \sum_{l=0}^{\infty}\Bigg(\sum_{m_1+\cdots+m_n=l}\prod_{j=1}^{n}\binom{\alpha_j+m_j-1}{m_j}p_j^{\alpha_j}q_j^{m_j}\Bigg)Q^l((-\infty, x]), \tag{2.4}$$

where the inside sum is over nonnegative integers m_j such that $m_1 + m_2 + \cdots + m_n = l$.

Proof. The proof is by induction. Note that, for $n = 1$, (2.4) reduces to the distribution function of Y_1. Assume that (2.4) is true for $n = k - 1$. Then

$$\begin{aligned}
\mathrm{P}(S_k \le x) &= \int_{\mathbb{R}} \mathrm{P}(S_{k-1} \le x - y)\,\mathrm{d}F_{Y_k}(y) \\
&= \int_{\mathbb{R}} \sum_{l=0}^{\infty}\Bigg(\sum_{m_1+\cdots+m_{k-1}=l}\prod_{j=1}^{k-1}\binom{\alpha_j+m_j-1}{m_j}p_j^{\alpha_j}q_j^{m_j}\Bigg)Q^l((-\infty, x-y]) \\
&\qquad \times \sum_{m_k=0}^{\infty}\binom{\alpha_k+m_k-1}{m_k}p_k^{\alpha_k}q_k^{m_k}\,\mathrm{d}Q^{m_k}(y) \\
&= \sum_{l=0}^{\infty}\sum_{m_k=0}^{\infty}\Bigg(\sum_{m_1+\cdots+m_{k-1}=l}\prod_{j=1}^{k-1}\binom{\alpha_j+m_j-1}{m_j}p_j^{\alpha_j}q_j^{m_j}\Bigg)\binom{\alpha_k+m_k-1}{m_k} \\
&\qquad \times p_k^{\alpha_k}q_k^{m_k}\int_{\mathbb{R}} Q^l((-\infty, x-y])\,\mathrm{d}Q^{m_k}(y) \\
&= \sum_{l=0}^{\infty}\sum_{m_k=0}^{\infty}\Bigg(\sum_{m_1+\cdots+m_{k-1}=l}\prod_{j=1}^{k-1}\binom{\alpha_j+m_j-1}{m_j}p_j^{\alpha_j}q_j^{m_j}\Bigg)\binom{\alpha_k+m_k-1}{m_k} \\
&\qquad \times p_k^{\alpha_k}q_k^{m_k}Q^{l+m_k}((-\infty, x]) \\
&= \sum_{r=0}^{\infty}\Bigg(\sum_{m_1+\cdots+m_k=r}\prod_{j=1}^{k}\binom{\alpha_j+m_j-1}{m_j}p_j^{\alpha_j}q_j^{m_j}\Bigg)Q^r((-\infty, x]),
\end{aligned}$$

where the last equality follows by substituting $l + m_k = r$ and then interchanging the order of summation of m_k and r. Thus, (2.4) is satisfied for $n = k$, which completes the proof.

Remark 2.1. When $Q = \delta_1$, (2.4) reduces to

$$\mathrm{P}(S_n \le x) = \sum_{l=0}^{\lfloor x\rfloor}\Bigg(\sum_{m_1+\cdots+m_n=l}\prod_{j=1}^{n}\binom{\alpha_j+m_j-1}{m_j}p_j^{\alpha_j}q_j^{m_j}\Bigg),$$

where $\lfloor x\rfloor$ denotes the integral part of x.

Also, the PMF of S_n is

$$\mathrm{P}(S_n = x) = \sum_{m_1+\cdots+m_n=x}\prod_{j=1}^{n}\binom{\alpha_j+m_j-1}{m_j}p_j^{\alpha_j}q_j^{m_j} \quad \text{for } x \in \mathbb{Z}_+. \tag{2.5}$$

An alternative form for $\mathrm{P}(S_n = x)$ is given in Furman (2007, Equation (11)), which is a series whose coefficients are recursively defined. In contrast, our expression (2.5) is compact and the exact value can be easily computed.

TABLE 1: The computation of $p_{\mathrm{VU}} = \mathrm{P}(S_n = x)$ using (2.5).

n	$x = 3$		$x = 5$		$x = 8$		$x = 10$		$x = 15$	
	p_{VU}	Time	p_{VU}	Time	p_{VU}	Time	p_{VU}	Time	p_{VU}	Time
2	0.023 20	0.0	0.034 03	0.000	0.042 83	0.000	0.044 25	0.000	0.038 56	0.000
3	0.002 73	0.0	0.007 30	0.015	0.017 24	0.000	0.024 21	0.000	0.036 07	0.000
4	0.000 20	0.0	0.000 94	0.015	0.004 08	0.000	0.007 85	0.016	0.020 99	0.015
5	0.000 01	0.0	0.000 10	0.016	0.000 76	0.000	0.001 96	0.015	0.009 20	0.063
6	0.000 00	0.0	0.000 01	0.016	0.000 14	0.031	0.000 47	0.047	0.003 65	0.234
7	0.000 00	0.0	0.000 00	0.016	0.000 03	0.063	0.000 13	0.140	0.001 54	0.844

TABLE 2: The computation of $p_{\mathrm{F}} = \mathrm{P}(S_n = x)$ using Furman's formula (11).

n	$x = 3$		$x = 5$		$x = 8$		$x = 10$		$x = 15$	
	p_{F}	Time	p_{F}	Time	p_{F}	Time	p_{F}	Time	p_{F}	Time
2	0.0232	0.000	0.0340	0.000	0.0428	0.000	0.0442	0.015	0.0385	0.015
3	0.0027	0.000	0.0073	0.063	0.0172	0.063	0.0242	0.266	0.0360	0.250
4	0.0002	0.078	0.0009	0.078	0.0040	0.328	0.0078	1.312	0.0209	10.532
5	0.0000	1.484	0.0001	11.953	0.0007	11.953	0.0019	12.156	0.0090	339.860
6	0.0000	1.844	0.0000	14.985	0.0001	14.844	0.0004	480.359	0.0034	61 324.600
7	0.0000	17.437	0.0000	557.141	0.0000	555.422	0.0000	555.921	0.0004	71 302.900

As suggested by the referee, we next compare (2.5) with Equation (11) of Furman (2007) by numerically calculating the computational time (in seconds) and $\mathrm{P}(S_n = x)$. The values of $\mathrm{P}(S_n = x)$ are calculated, using MATHEMATICA® 5.1, for some selected values of $\alpha_j = j$, $p_j = j/10$, $x = 3, 5, 8, 10, 15$, and $n = 2, \ldots, 7$, and are given in Tables 1 and 2. Since Furman's formula (11) involves recurrence relations, the order of accuracy of the values in Table 2 is restricted to 10^{-3} to bring down the computational time. A comparison of the values in Tables 1 and 2 shows that the computation of probability values using (2.5) requires much less time than that of Furman's formula (11).

2.1. Random parameter representation

In this subsection we obtain a random parameter representation for the convolution of independent CNB variables and also of certain dependent CP variables.

Let $Y_1, Y_2, \ldots, Y_n$ be independent variables, where $Y_j \sim \mathrm{CNB}(\alpha_j, p_j, Q)$ for $1 \le j \le n$. We now introduce the following notation. Let

$$p_m = \max_{1\le j\le n} p_j, \qquad q_m = 1 - p_m, \qquad s_j = \frac{q_j}{p_j}, \qquad s_l = \min_{1\le j\le n} s_j = \frac{q_m}{p_m};$$

$$\alpha = \sum_{j=1}^{n} \alpha_j, \qquad c_n = \prod_{j=1}^{n} \left(\frac{s_l}{s_j}\right)^{\alpha_j}, \quad \text{and} \quad a_i = \frac{1}{i} \sum_{j=1}^{n} \alpha_j \left(1 - \frac{s_l}{s_j}\right)^{i} \quad \text{for } i \in \mathbb{Z}_+ \setminus \{0\}.$$

Define K_n to be a $\mathbb{Z}_+$-valued random variable with probability distribution

$$\mathrm{P}(K_n = k) = c_n b_k \quad \text{for } k \in \mathbb{Z}_+, \tag{2.6}$$

where $b_0 = 1$ and $b_k = (1/k) \sum_{i=1}^{k} i a_i b_{k-i}$ for $k \ge 1$ (see Remark 2.2, below).

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We are now ready to prove the main result of this subsection. Note that our approach is essentially that of Furman (2007), except that he used the MGF, while we use the distribution itself.

Theorem 2.2. *Let* $Y_j \sim \mathrm{CNB}(\alpha_j, p_j, Q)$ *for* $1 \leq j \leq n$, *and let* $S_n = \sum_{j=1}^{n} Y_j$. *Then* $S_n \sim \mathrm{CNB}(K_n + \alpha, p_m, Q)$, *where the distribution of* K_n *is defined in (2.6).*

Proof. Using (2.3),

$$\begin{aligned}\mathcal{L}(Y_j) &= (\delta_0 - s_j(Q-\delta_0))^{-\alpha_j}\\ &= \bigg((\delta_0 - s_l(Q-\delta_0))^{\frac{s_j}{s_l}}\bigg(\delta_0 - \bigg(1-\frac{s_l}{s_j}\bigg)(\delta_0 - s_l(Q-\delta_0))^{-1}\bigg)\bigg)^{-\alpha_j}. \qquad (2.7)\end{aligned}$$

Observe that $\delta_0 - s_l(Q-\delta_0)$ is a finite signed measure and that $(\delta_0 - s_l(Q-\delta_0))^{-1} = \mathrm{CG}(p_m, Q) := G(\text{say})$. Therefore, from (2.7) we have

$$\begin{aligned}\mathcal{L}(S_n) &= \prod_{j=1}^{n} \mathcal{L}(Y_j)\\ &= (\delta_0 - s_l(Q-\delta_0))^{-\alpha} c_n \prod_{j=1}^{n}\bigg(\delta_0 - \frac{1-s_l/s_j}{\delta_0 - s_l(Q-\delta_0)}\bigg)^{-\alpha_j}\\ &= G^{\alpha} c_n \exp\bigg(\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=1}^{n}\alpha_j\bigg(1-\frac{s_l}{s_j}\bigg)^{k} G^k\bigg)\\ &= G^{\alpha} c_n \exp\bigg(\sum_{k=1}^{\infty} a_k G^k\bigg) \quad (\text{say}). \qquad (2.8)\end{aligned}$$

If we write $f(z) = \exp(\sum_{k=1}^{\infty} a_k z^k) = \sum_{k=0}^{\infty} b_k z^k$ then $b_k = f^{(k)}(0)/k!$ for $k \in \mathbb{Z}_+$, where $f^{(k)}$ denotes the kth derivative of f. Therefore, it can be seen that $b_0 = 1$, $b_1 = a_1$, and $b_2 = a_1^2/2 + a_2$, and, in general, we obtain $b_k = (1/k)\sum_{i=1}^{k} i a_i b_{k-i}$ for $k \geq 1$. Using these facts, we obtain, from (2.8),

$$\begin{aligned}\mathcal{L}(S_n) &= G^{\alpha}\sum_{k=0}^{\infty} c_n b_k G^k\\ &= G^{\alpha}\sum_{k=0}^{\infty} \mathrm{P}(K_n = k) G^k\\ &= \mathrm{CNB}(\alpha, p_m, Q)\mathrm{CNB}(K_n, p_m, Q)\\ &= \mathrm{CNB}(K_n + \alpha, p_m, Q),\end{aligned}$$

since $G^{\alpha} = \mathrm{CNB}(\alpha, p_m, Q)$. This proves the result.

Remark 2.2. Note that, from the proof of Theorem 2.2,

$$f(z) = \exp\bigg(\sum_{k=1}^{\infty} a_k z^k\bigg) = \sum_{k=0}^{\infty} b_k z^k.$$

Setting $z = 1$ in the above equation, we obtain

$$\begin{aligned}\sum_{k=0}^{\infty} b_k &= \exp\biggl(\sum_{k=1}^{\infty} a_k\biggr)\\ &= \exp\biggl(\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=1}^{n}\alpha_j\biggl(1-\frac{s_l}{s_j}\biggr)^k\biggr) \quad \text{(see (2.8))}\\ &= \exp\biggl(-\sum_{j=1}^{n}\alpha_j\ln\biggl(\frac{s_l}{s_j}\biggr)\biggr)\\ &= \prod_{j=1}^{n}\biggl(\frac{s_l}{s_j}\biggr)^{-\alpha_j}\\ &= c_n^{-1}.\end{aligned}$$

Hence, $\mathrm{P}(K_n = k) = c_n b_k,\ k \in \mathbb{Z}_+$, is a valid probability distribution.

Corollary 2.1. (Furman (2007, Theorem 2).) *Let* $Y_j \sim \mathrm{NB}(\alpha_j, p_j)$ *for* $1 \le j \le n$, *and let* $S_n = \sum_{j=1}^{n} Y_j$. *Then* $S_n \sim \mathrm{NB}(K_n + \alpha, p_m)$.

Corollary 2.2. (Furman (2007, Theorem 1).) *Let* $Y_j \sim \mathrm{NB}(\alpha_j, p_j)$ *for* $1 \le j \le n$, *and let* $S_n = \sum_{j=1}^{n} Y_j$. *Then*

$$\mathrm{P}(S_n = x) = \sum_{k=0}^{\infty} c_n b_k \binom{\alpha + k + x - 1}{x} p_m^{\alpha+k} q_m^{x}, \qquad x \in \mathbb{Z}_+. \tag{2.9}$$

Proof. The proof easily follows from Corollary 2.1 and the fact that

$$\mathrm{P}(S_n = x) = \sum_{k=0}^{\infty} \mathrm{P}(S_n = x \mid K_n = k)\,\mathrm{P}(K_n = k).$$

Next we look at the case of different Q_js. In this case, a CP representation is useful.

Theorem 2.3. *Let* $W_1, W_2, \dots, W_n$ *be independent* $\mathrm{CNB}(\alpha_j, p_j, Q_j)$ *random variables, and let* $T_n = \sum_{j=1}^{n} W_j$. *Also, let* $\lambda_j = -\alpha_j \ln p_j$, $q_j = (1 - p_j)$, *and* $G_j = \ln(\delta_0 - q_j Q_j)/\ln p_j$. *Then* $T_n \sim \mathrm{CP}(\lambda, G)$, *where* $\lambda = \sum_{j=1}^{n} \lambda_j$ *and* $G = (1/\lambda)\sum_{j=1}^{n} \lambda_j G_j$.

Proof. Note that

$$\begin{aligned}\mathcal{L}(W_j) &= \biggl(\delta_0 - \frac{q_j}{p_j}(Q_j - \delta_0)\biggr)^{-\alpha_j}\\ &= \biggl(\frac{p_j}{\delta_0 - q_j Q_j}\biggr)^{\alpha_j}\\ &= \exp(\alpha_j(\ln(1 - q_j) - \ln(\delta_0 - q_j Q_j)))\\ &= \exp\biggl(-\alpha_j\ln(1 - q_j)\biggl(\frac{\ln(\delta_0 - q_j Q_j)}{\ln(1 - q_j)} - \delta_0\biggr)\biggr)\\ &= \mathrm{CP}(\lambda_j, G_j),\end{aligned} \tag{2.10}$$

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where $\lambda_j = -\alpha_j \ln p_j$ and $G_j = \ln(\delta_0 - q_j Q_j)/\ln p_j$. Note that if $W_j \sim \mathrm{CP}(\lambda_j, G_j)$ then, by the additivity property, $T_n = \sum_{j=1}^{n} W_j \sim \mathrm{CP}(\lambda, G)$, where $\lambda = \sum_{j=1}^{n} \lambda_j$ and $G = (1/\lambda)\sum_{j=1}^{n} \lambda_j G_j$. The result now follows.

Remarks 2.3. Here we discuss the connections between CP and CNB distributions.

(i) Suppose that $Y \sim \mathrm{CP}(\lambda, F)$ so that $\mathcal{L}(Y) = \exp(\lambda(F - \delta_0))$. Equating this distribution to a $\mathrm{CNB}(\alpha, p, Q)$ distribution and then solving for p and Q, using (2.10), we obtain

$$p = \mathrm{e}^{-\lambda/\alpha} \quad \text{and} \quad Q = \frac{\delta_0 - \exp(-(\lambda/\alpha)F)}{1 - \mathrm{e}^{-\lambda/\alpha}},$$

where $\alpha > 0$ is arbitrary and Q is in general a finite signed measure.

(ii) Applying Theorem 2.3 and then using part (i), we obtain $\mathcal{L}(T_n) = \mathrm{CNB}(\alpha, p, Q)$, where

$$p = \prod_{j=1}^{n} p_j^{\alpha_j/\alpha}, \qquad Q = \frac{\delta_0 - \prod_{j=1}^{n}(\delta_0 - q_j Q_j)^{-\alpha_j/\alpha}}{1 - \prod_{j=1}^{n} p_j^{\alpha_j/\alpha}},$$

and $\alpha > 0$ is arbitrary. Since Q is in general a finite signed measure, a CP representation given in Theorem 2.3 may be useful for applications.

(iii) Let N follow a logarithmic series distribution with parameter $0 < q < 1$ so that

$$\mathrm{P}(N = k) = \frac{q^k}{k h(q)}, \qquad k = 1, 2, \ldots,$$

where $h(q) = -\ln(1 - q)$. Also, let the X_i be i.i.d. with distribution Q. Then the distribution of $\sum_{i=1}^{N} X_i$ is a compound logarithmic series distribution, denoted by $\mathrm{CL}(q, Q)$. That is,

$$\mathrm{CL}(q, Q) = \frac{\ln(\delta_0 - qQ)}{\ln(1 - q)}.$$

When $F = \mathrm{CL}(q, Q)$, we have $\mathrm{CP}(\lambda, F) = \mathrm{CNB}(-\lambda/\ln(1 - q), 1 - q, Q)$ and Q is now a probability measure.

2.2. Sums of dependent CP variables

We consider here the sums of certain dependent CP distributions, where the dependence is caused by a common mixing random variable W. Such a case arises in the distribution of combined portfolios. For example, Dhaene *et al.* (2003) considered the case of W being a gamma variable.

In the sequel, $X \stackrel{\mathcal{L}}{=} Y$ means that the distributions of X and Y are the same.

Theorem 2.4. *Let $W > 0$ be a continuous random variable, let $\{N_i(t)\}$, $1 \le i \le n$, be independent Poisson processes with rate λ_i, and let $V_i := N_i(W)$, $1 \le i \le n$. Define $S_{V_i} := \sum_{j=1}^{V_i} X_{i,j}$, where $X_{i,j} \sim Q_i$. Then $U_n = \sum_{i=1}^{n} S_{V_i} \stackrel{\mathcal{L}}{=} \sum_{i=1}^{N(W)} X_j \sim \mathrm{CP}(\lambda W, Q)$, where $\{N(t)\}$ is a Poisson process with parameter $\lambda = \sum_{i=1}^{n} \lambda_i$ and $X_j \sim Q = (1/\lambda)\sum_{i=1}^{n} \lambda_i Q_i$.*

Proof. Observe that, for any given $W > 0$, $S_{V_i} \sim \mathrm{CP}(\lambda_i W, Q_i)$ for $1 \le i \le n$. By the additivity property of CP distributions, we obtain $(U_n \mid W) \sim \mathrm{CP}(\lambda W, Q)$, and, hence, (unconditionally also) $U_n \sim \mathrm{CP}(\lambda W, Q)$, where $\lambda = \sum_{i=1}^{n} \lambda_i$ and $Q = (1/\lambda)\sum_{i=1}^{n} \lambda_i Q_i$.

3. Convolution of weighted gamma random variables

The distribution of the sum of weighted gamma random variables arises in many situations, and does not admit a closed form (see Diaconis and Perlman (1990)). As an application of Theorem 2.2, here we obtain the random parameter representation for such sums. This representation is compact and may be helpful for analytical purposes.

Theorem 3.1. *Let $Z_1, Z_2, \ldots, Z_n$ be independent random variables, where $Z_j \sim G(\beta_j, t_j)$, the gamma distribution with scale parameter β_j^{-1} and shape parameter $t_j > 0$. For $c_j > 0$ and $i \in \mathbb{Z}_+ \setminus \{0\}$, let $T_n = \sum_{j=1}^{n} c_j Z_j$, $\beta = \max_{1\le j\le n} \beta_j/(c_j + \beta_j)$, $d_n = \prod_{j=1}^{n}((1-\beta)\beta_j/(c_j\beta))^{t_j}$, and $a_i = (1/i)\sum_{j=1}^{n} t_j(1-(1-\beta)\beta_j/(c_j\beta))^i$. Then $T_n \sim G(\beta/(1-\beta), L_n + t)$, where L_n is a random variable with $\mathrm{P}(L_n = k) = d_n b_k$, $k \in \mathbb{Z}_+$, and $t = \sum_{j=1}^{n} t_j$. Here, $b_0 = 1$ and $b_k = (1/k)\sum_{i=1}^{k} i a_i b_{k-i}$ for $k \in \mathbb{Z}_+ \setminus \{0\}$.*

Proof. Note that

$$Z_j \sim G(\beta_j, t_j) \iff c_j Z_j \sim G\left(\frac{\beta_j}{c_j}, t_j\right) \iff N(c_j Z_j) \sim \mathrm{NB}\left(t_j, \frac{\beta_j}{c_j + \beta_j}\right)$$

(see Proposition 2 of Engel and Zijlstra (1980)), where $\{N(t)\}$ is a standard (parameter unity) Poisson process. Also, there exist (see Vellaisamy and Sreehari (2008)) independent standard Poisson processes $\{N_j(t)\}_{1\le j\le n}$ and $\{N(t)\}$ such that

$$N_1(c_1Z_1) + N_2(c_2Z_2) + \cdots + N_n(c_nZ_n) \stackrel{\mathcal{L}}{=} N(c_1Z_1 + c_2Z_2 + \cdots + c_nZ_n).$$

By Corollary 2.1 we have $N(c_1Z_1+c_2Z_2+\cdots+c_nZ_n) = N(T_n) \sim \mathrm{NB}(L_n+t, \beta)$, where t and β are as defined in the theorem and L_n is the discrete random variable with $\mathrm{P}(L_n = k) = d_n b_k$ for $k \in \mathbb{Z}_+$. Hence, $T_n \sim G(\beta/(1-\beta), L_n + t)$, which proves the result.

Remark 3.1. When $c_1 = c_2 = \cdots = c_n = 1$, Theorem 3.1 yields the convolution of n independent gamma variables with arbitrary parameters. It is known in the literature (see, for example, Sim (1992, p. 140)) that the density of T_n is complicated. Our Theorem 3.1 gives a simple random parameter representation for the distribution of T_n, which may be helpful for analytical or inferential purposes.

4. Examples and applications

In this section we discuss some examples and applications of the results derived in Sections 2 and 3. We start with an application of Theorem 2.2 to risk theory. The finite sums of CNB random variables naturally occur in credit risk modeling and have been studied by many authors (see, for example, Gundlach and Lehrbass (2004, pp. 32–40) and Dhaene *et al.* (2003)).

Example 4.1. (*Total claim amount.*) Let the claim sizes $X_i \sim E(\beta) = Q$, the exponential distribution with parameter β. Suppose that a company has a portfolio of n policies, and assume that the number N_i of claims of the ith policy follows $\mathrm{NB}(\alpha_i, p_i)$, which is a reasonable model, especially when $\mathrm{var}(N_i) > \mathrm{E}(N_i)$. Our interest is in the distribution of the total claim amount defined by

$$S_n = \sum_{i=1}^{n} \sum_{j=1}^{N_i} X_j.$$

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An application of Theorem 2.2 shows that

$$\begin{aligned}\mathcal{L}(S_n) &= \mathrm{CNB}(K_n+\alpha,\, p_m,\, Q)\\ &= \sum_{k=0}^{\infty} \mathrm{CNB}(k+\alpha,\, p_m,\, Q)\,\mathrm{P}(K_n=k).\end{aligned}$$

Also, the density of S_n is

$$f_{S_n}(x) = \sum_{k=0}^{\infty} c_n b_k \sum_{l=0}^{\infty} \binom{\alpha+k+l-1}{l} p_m^{\alpha+k} q_m^l f_{T_l}(x), \tag{4.1}$$

where

$$f_{T_l}(x) = \frac{\beta^l}{\Gamma(l)}\,\mathrm{e}^{-\beta x} x^{l-1} \quad \text{for } x>0$$

is the density of the gamma $G(\beta, l)$ variable, since $Q^l = G(\beta, l)$.

Panjer and Wilmot (1981) considered the case in which $n = 1$ and suggested an approximation procedure employing the methods of numerical analysis to evaluate the error in approximating S_1 to a compound binomial distribution. Our expression (4.1) gives the exact density of S_n.

The following two examples correspond to Theorem 2.4.

Example 4.2. Let $W \sim G(\beta, s)$ in Theorem 2.4 so that $V_i = N_i(W) \sim \mathrm{NB}(s, \beta/(\beta+\lambda_i))$ and $S_{V_i} \sim \mathrm{CNB}(s, \beta/(\beta+\lambda_i), Q_i)$. Then, by Theorem 2.4, $U_n = \sum_{i=1}^n S_{V_i} \sim \mathrm{CNB}(s, \beta/(\beta+\lambda), Q)$, where Q and λ are as defined in Theorem 2.4. This result is due to Dhaene *et al.* (2003).

Example 4.3. Let $W \sim \mathrm{L}(\alpha)$, the Lindley distribution with parameter α (see Johnson *et al.* (2005)), with density

$$f_W(x) = \frac{\alpha^2}{\alpha+1}(1+x)\,\mathrm{e}^{-\alpha x}, \qquad x>0,\ \alpha>0.$$

It is well known that $N_i(W) \sim \mathrm{PL}(\alpha, \lambda_i)$, the Poisson–Lindley distribution with parameters α and λ_i having distribution

$$\mathrm{P}(N_i(W)=k) = \frac{\alpha^2}{\alpha+1}\,\frac{\lambda_i^k(\alpha+\lambda_i+k+1)}{(\alpha+\lambda_i)^{k+2}}, \qquad k \in \mathbb{Z}_+.$$

Let U_n, λ, and Q be defined as in Theorem 2.4. Then U_n follows compound $\mathrm{PL}(\alpha, \lambda, Q)$.

Finally, we discuss two important applications of Theorem 3.1.

4.1. Reliability of the *m*-out-of-*n*:G system with different failure rates

Consider an m-out-of-n:G system with n i.i.d. components having exponential $E(\lambda)$ lifetimes. Initially, each component has failure rate λ_0. As the first component fails, there is an increase in the stress on the remaining $(n-1)$ components, which increases the failure rate of the components to λ_1. Generally, the failure of the ith component raises the stress on the remaining $(n-i)$ components, which increases the failure rate to λ_i. Our interest is to find the distribution of the system time to failure.

Let T_i denote the time to failure of the ith component, and let $X_i = T_i - T_{i-1}$. Then, the time to system failure $T = T_{n-m+1} = \sum_{i=1}^{n-m+1} X_i$. Observe that $X_i \sim E(\alpha_i)$, where $\alpha_i = (n-i+1)\lambda_{i-1}$, and the reliability of the system is $R(t) = \mathrm{P}(T > t)$. Using Theorem 3.1, $T \sim G(\gamma, L_{n-m+1} + (n-m+1))$, where $\gamma = \max_i (n-i+1)\lambda_{i-1}$.

Let $Z_j \sim G(\beta_j, t_j)$. Indeed, Scheuer (1988) derived the distribution of $T_n = \sum_{i=1}^n Z_j$ as

$$f_{T_n}(x) = B \sum_{k=1}^{n} \sum_{m=1}^{t_k} \frac{\Phi_{km}(-\beta_j)}{(t_k - m)!\,(m-1)!} x^{t_k - 1} \exp(-\beta_j x), \tag{4.2}$$

where $B = \prod_{j=1}^n \beta_j^{t_j}$ and

$$\Phi_{km}(x) = \frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}} \prod_{\substack{j=1 \\ j \neq k}}^{n} (\beta_j + x)^{-t_j}. \tag{4.3}$$

Using Theorem 3.1, we see that $T_n \sim G(\gamma, L_n + t)$ with

$$\begin{aligned} f_{T_n}(x) &= \sum_{k=0}^{\infty} \mathrm{P}(L_n = k) \frac{\gamma^{t+k}}{\Gamma(t+k)} \mathrm{e}^{-\gamma x} x^{t+k-1} \\ &= \sum_{k=0}^{\infty} d_n b_k \frac{\gamma^{t+k}}{\Gamma(t+k)} \mathrm{e}^{-\gamma x} x^{t+k-1}, \end{aligned} \tag{4.4}$$

where $\gamma = \beta/(1-\beta)$, $\beta = \max_{1 \leq j \leq n} \beta_j/(1+\beta_j)$, and d_n and the b_ks are defined in Theorem 3.1. Note that (4.3) involves derivatives of the mth order and, hence, (4.2) and (4.3) are difficult to compute. Equation (4.4) is much simpler and can be easily evaluated.

4.2. The shortest path problem in graph theory

The shortest path from a source node to a destination node is a path which minimizes the sum of the positive weights of its constituent links. The related shortest path tree (SPT) is the union of the shortest paths from the source node to a set of m other nodes in the graph of r nodes. If $m = r - 1$, the SPT connects all nodes and is called a spanning tree. The uniform recursive tree (URT) of size r is a random tree rooted at node A and, at each stage, a new node is attached uniformly to the existing node until all the nodes are discovered. We analyze the influence of the link weight structure on the SPT. Such problems arise in communication networks (see, for example, Mieghem (2006, pp. 347–384)).

The problem of finding the shortest path between two nodes A and B in a complete graph K_r, with link weights as $E(1)$ (exponentially distributed with mean 1) variables, can be modeled in the form of a Markov discovery process $\{X(t)\}$ with state space $S = \{1, 2, \ldots, r\}$, where $X(t)$ denotes the number of nodes discovered up to time t. Note that $X(t_0) = A$, $X(T) = B$, where t_0 is the starting time and T denotes the random time to reach B, and that the transmission rates are $\lambda_j = j(r-j)$, $j \in S$. This is because, from the first node A, $r-1$ new nodes can be reached, each with $E(1)$ link weights, and so the shortest path $Z_1 \sim E(r-1)$. Similarly, from the first two nodes, the remaining $r-2$ nodes can be reached with the shortest path $Z_2 \sim E(2(r-2))$. In general, from the $(j-1)$th node to the jth node, the shortest path $Z_j \sim E(\lambda_j)$ with $\lambda_j = j(r-j)$.

Observe that the time to reach the kth node from the source node A or the discovery time of the kth node is given by $M_k = \sum_{j=1}^k Z_j$. Using the MGF of M_k, the mean $\mathrm{E}(M_k)$ and the variance $\mathrm{var}(M_k)$ are computed (see Mieghem (2006, p. 359)), but they are rather complicated.

An application of Theorem 3.1 yields $M_k \sim G(\beta/(1-\beta), L_k + k)$, where

$$\beta = \max_{1 \le j \le n} \frac{j(r-j)}{1 + j(r+j)}.$$

Note that $\beta/(1-\beta) = k(r-k)$ if $k \le r/2$. When $k \ge r/2$, we have $\beta/(1-\beta) = r^2/4$ if r is even and $\beta/(1-\beta) = (r-1)^2/4$ if r is odd. Therefore, the density of M_k is

$$f_{M_k}(x) = \sum_{m=0}^{\infty} d_k b_m \frac{\gamma^{k+m}}{\Gamma(k+m)} \mathrm{e}^{-\gamma x} x^{k+m-1}, \tag{4.5}$$

where $\gamma = \beta/(1-\beta)$, as described above.

It is well known (see Mieghem (2006, p. 359)) that the shortest path in a complete graph with exponential $E(1)$ link weights is a URT. Now, let W_r denote the length of the shortest path in K_r, and let N denote the number of nodes, excluding the source node, discovered by the URT to reach the destination node. Then

$$W_r = \sum_{j=1}^{N} Z_j, \tag{4.6}$$

where N follows a discrete uniform distribution over $\{1, 2, \ldots, r-1\}$ and is independent of the Z_j. Also, the density of W_r is given by

$$f_{W_r}(x) = \sum_{k=1}^{r-1} \mathrm{P}(N = k) f_{M_k}(x) = \frac{1}{r-1} \sum_{k=1}^{r-1} f_{M_k}(x), \tag{4.7}$$

where $f_{M_k}(x)$ is defined in (4.5). It is mentioned in Mieghem (2006, p. 360) that the density of W_r can be obtained by using the inverse Laplace transform of the MGF of W_r. Equation (4.7) gives the exact density of W_r. Note that the moments and other characteristics of W_r can easily be computed using (4.6) or (4.7).

5. Concluding remarks

The main contributions of this paper are the derivation of an exact expression and the random parameter representation for the convolution of compound negative binomial variables. In the case of negative binomial distributions, it is numerically verified that the exact expression is computationally more efficient than the random parametric form. Some applications to insurance mathematics are also discussed. The distribution of a sum of certain dependent compound Poisson variables is obtained, which generalizes some existing results. The conditions under which a compound Poisson distribution is also a compound negative binomial distribution are analyzed. Using the connection between negative binomial and gamma distributions, the convolution of arbitrary gamma variables is derived, which is also a useful result. This result is then applied to two important practical problems which arise in reliability theory and graph theory.

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