EXTENSION OF FINITE PROJECTIVE PLANES
I. UNIFORM HJELMSLEV PLANES

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1. Introduction. In his paper **(3)** on finite H-planes, Kleinfeld has defined invariants s and t for an H-plane π as follows: let P and k be any point and line of π such that P is incident with k, then let s be the number of non-neighbour points to P on k and let t be the number of neighbour points to P on k. He has shown that s and t are independent of the choice of P and k, t divides s, π has $s^2 + st + t^2$ points (lines), π^* has order s/t, and, if $t \neq 1$, $s \leqslant t^2$. In the case $s = t^2$, π is called uniform and has the property that each pair of neighbour lines (points) has exactly t points (lines) in common.

The neighbour relation has been shown in **(4)** to be an equivalence relation on points as well as lines. Using this, Klingenberg constructed a projective plane π^*, as above, associated with each Hjelmslev plane π. The points (lines) of π^* are the equivalence classes of neighbour points (lines) of π. Furthermore, class $\mathfrak{P}$ is incident with class $\mathfrak{L}$ in π^* if and only if there exist a P in $\mathfrak{P}$ and l in $\mathfrak{L}$ such that P is incident with l in π.

Surely, we can find the incidence matrix A^* of π^* from the incidence matrix A of π by partitioning A into blocks of neighbours and by then replacing each of these submatrices by the appropriate 0 or 1. However, it has been unknown whether or not the incidence matrix A' of a finite projective plane could be extended or "blown up" to the incidence matrix A of an H-plane with $t \neq 1$.

In § 2, a subset $\mathfrak{C}$ of the positive integers is defined and we show that $n \in \mathfrak{C}$ is a necessary and sufficient condition for a projective plane of order n to be extended to not only an H-plane but in fact a uniform H-plane. In § 3, this condition is removed as we show the equivalence of the existence question for projective planes to that of the uniform H-planes and also to the membership question for $\mathfrak{C}$.

2. Extension. The following definition was motivated by the structure of the incidence matrix of the uniform H-plane with $t = 2$, as in **(1)**.

DEFINITION 1. *$n \in \mathfrak{C}$ if and only if the positive integers $1, 2, \ldots, n^2$ can be partitioned into n n-tuples in $n + 1$ distinct ways such that each pair of distinct numbers from $1, 2, \ldots, n^2$ occurs in exactly 1 of the n-tuples. The set of the $n + 1$ partitions will be called a $\mathfrak{C}$-decomposition for n.*

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Definition 2. *A projective plane π' can be extended to an H-plane π, if π^* is isomorphic to π'.*

This definition does not seem to say that π' is necessarily isomorphic to a subspace of π.

Theorem 1. *Let π' be a projective plane of order n. Then π' can be extended to a uniform H-plane π with $t = n$, if $n \in \mathfrak{C}$.*

Proof. Let $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_N$ and $\mathfrak{L}_1, \ldots, \mathfrak{L}_N$ be the points and lines of π', where $N = n^2 + n + 1$. Let A' be the incidence matrix of π', i.e.

$$A' = [a_{ij}'],$$

where $a_{ij}' = 1$ if $\mathfrak{P}_i$ is on $\mathfrak{L}_j$, and $a_{ij}' = 0$ if $\mathfrak{P}_i$ is not on $\mathfrak{L}_j$.

Since $n \in \mathfrak{C}$, we can form a $\mathfrak{C}$-decomposition for n. For each partition in this decomposition, we form an $n^2 \times n^2$ matrix by letting the (i, j)th entry be 1 if i and j appear in the same n-tuple in that partition; otherwise let the entry be 0. After this is done for each partition we have $n + 1$ distinct (0, 1) matrices, say $M_1, \ldots, M_{n+1}$.

Now, we extend A' by replacing a zero entry by the zero square matrix of order n^2 and replace each 1 in A' by one of the matrices $M_1, \ldots$, or M_{n+1} such that no two 1's in the same row (column) of A' are replaced by the same M_i. This is readily done by writing A' as the sum of $n + 1$ permutation matrices, $A' = A_1' + \ldots + A'_{n+1}$. Then replace the 1's in A' that come from A_i' by M_i.

We now have a matrix $A = [a_{ij}]$ which can be considered as the incidence matrix for a structure π with points $P_1, \ldots, P_{Nn^2}$ and lines $l_1, \ldots, l_{Nn^2}$, where P_i is on l_j if and only if $a_{ij} = 1$. We now show that π is a uniform H-plane by checking the axioms for an H-plane as they appear in **(3)**.

I. *Two points determine at least one line.*

(a) If two points are in the same block, they have exactly n lines in common—the lines that result from the unique n-tuple they have in common.

(b) If two points are in different blocks, they have exactly one line in common—since two n-tuples in different partitions have exactly one number in common.

II. *Two lines determine at least one point.*

The dual of I holds as $M_1, \ldots, M_{n+1}$ are symmetric.

Note that two points (lines) are neighbours if and only if they belong to the same block.

III. *If $l \circ k$ and $k \not\circ m$ and l, k, m all contain P, then $m \not\circ l$.*

Surely m is in a different block than the block containing k and, hence, l.

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IV. *If* $l \circ j$ *and* $j \not\circ k$, *then* $kl \circ kj$.

Since k is in a different block, $\mathfrak{K}$, than the block $\mathfrak{L}$, containing l and j, then kl and kj have to be in the same block, namely $\mathfrak{K}\mathfrak{L}$.

V. *If* $P \circ Q$ *and* $Q \not\circ R$, *then* $RP \circ RQ$.

This is the dual of IV.

VI. *There exist points* R_1, R_2, R_3, *and* R_4 *which are pairwise non-neighbour and* $R_iR_j \not\circ R_iR_k$ *for* i, j, *and* k *all distinct as* $i, j, k = 1, 2, 3, 4$.

Pick four points from π', say $\mathfrak{P}_{t1}$, $\mathfrak{P}_{t2}$, $\mathfrak{P}_{t3}$, and $\mathfrak{P}_{t4}$, such that no three are collinear. Then pick a point of π from each of these classes, say R_1, R_2, R_3, and R_4, where $R_j \in \mathfrak{P}_{tj}$ for $j = 1, 2, 3, 4$.

These four points satisfy the axioms since:

(a) they are in different blocks and hence are non-neighbour,

(b) if $R_iR_j \circ R_iR_k$ for i, j, and k all distinct, then $\mathfrak{P}_{ti}$, $\mathfrak{P}_{tj}$, and $\mathfrak{P}_{tk}$ would be collinear—a contradiction.

Hence, π is an H-plane. Furthermore, each pair of neighbour points (lines) have $n = t$ lines (points) in common, which shows π is a uniform plane with $t = n$. Lastly, A' can be taken as the incidence matrix of π^*, i.e. π^* is isomorphic to π'. This completes the proof of Theorem 1.

THEOREM 2. *If* π *is a uniform H-plane with* $t = n$, *then* $n \in \mathfrak{C}$.

Proof. Label the points and lines of π such that they are in blocks of neighbours as in Theorem 1. Form the incidence matrix A of π and look at its $n + 1$ non-zero submatrices formed by the first row block and its incident column blocks. Each of these submatrices determines a partition of $1, \ldots, n^2$ as follows: define the partition such that i and j are in the same n-tuple if and only if P_i and P_j have a line in common in this column block. Suppose, now, that some pair i and j never occur in the same n-tuple in any of these partitions. Then, for some k, i and k appear in at least two n-tuples. Hence, P_i and P_k have at least $2n = 2t$ lines in common—a contradiction. Therefore, these partitions form a $\mathfrak{C}$-decomposition for n.

Theorems 1 and 2 combine to yield the following theorem.

THEOREM 3. *Let* n *be the order of a projective plane* π'. *Then* π' *can be extended to a uniform H-plane if and only if* $n \in \mathfrak{C}$.

Furthermore, uniform H-planes are known to exist for all prime powers t **(3)**. Therefore, we have the following corollary.

COROLLARY. *All projective planes of prime-power order can be extended to uniform H-planes.*

The extension of a projective plane to a uniform Hjelmslev plane may not

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be unique, but Theorem 2 points out the importance of the method of extension in Theorem 1.

3. Existence. The definition of $\mathfrak{C}$ is similar to the definition of orthogonal latin squares where each ordered pair must appear exactly once. Recall that there exists a complete set of orthogonal latin squares of order n if and only if there exists a projective plane of order n **(5)**. The following was a successful attempt to unite and use these concepts with the previous material.

DEFINITION 3. *Let* $\mathfrak{D}$ *be the set of all positive integers* n *such that there exists a set of* $n^2 - n$ n*-tuples of the numbers* $1, 2, \ldots, n$ *where*:

(a) *no two numbers will appear in the same respective positions in any two distinct* n*-tuples*;

(b) *they can be listed one under the other, so as to yield* $n - 1$ *latin squares, also one under the other.*

THEOREM 4. $\mathfrak{D} = \mathfrak{C}$.

Proof. Let $n \in \mathfrak{D}$. To show that this implies $n \in \mathfrak{C}$, list $n^2 - n$ n-tuples which satisfy the definition of $n \in \mathfrak{D}$. In the order they are listed, name them $P_{n+1}, \ldots, P_{n^2}$.

Now start to construct a $\mathfrak{C}$-decomposition for n in the standard way. That is, construct

$$\begin{array}{lll}
(1, 2, \ldots, n) & (n+1, \ldots, 2n) \ldots & (\ldots, n^2) \\
(1, \qquad\quad) & (2, \qquad\qquad) \ldots & (n, \qquad) \\
(1, \qquad\quad) & (2, \qquad\qquad) \ldots & (n, \qquad) \\
\quad\vdots & \quad\vdots & \quad\vdots \\
(1, \qquad\quad) & (2, \qquad\qquad) \ldots & (n, \qquad)
\end{array}$$

Let $P_i = (a_{i1}, \ldots, a_{in})$ for $i = n + 1, \ldots, n^2$. Then, for each i, put i in the a_{ij}th "n-tuple" of the $(j + 1)$st partition.

Since $a_{ij} \neq a_{ik}$ for $j \neq k$, every number will appear once, and, therefore, only once, in each partition. Furthermore, since the P_i's form $n - 1$ latin squares, each "n-tuple" will receive $n - 1$ new elements. Therefore, each partition will consist of n n-tuples of the numbers $1, 2, \ldots, n^2$.

It remains to show that each pair of distinct numbers occurs in exactly one of the n-tuples.

If this is not the case, then some pair of distinct numbers, i and j, will appear in the same n-tuple in at least two different partitions. Surely neither i nor j can be less than $n + 1$. So we can assume that $i, j \geqslant n + 1$. Suppose they appear in the same n-tuple in both the kth and lth partitions, $k \neq l$. Then, by the construction, we have

$$a_{ik} = a_{jk} \quad \text{and} \quad a_{il} = a_{jl}.$$

However, this gives

$$\begin{array}{l} P_i = (\ldots, a_{ik}, \ldots, a_{il}, \ldots), \\ \qquad\qquad\; \| \qquad\quad \| \\ P_j = (\ldots, a_{jk}, \ldots, a_{jl} \ldots), \end{array}$$

which contradicts part (a) of the definition of $n \in \mathfrak{D}$. Hence, this is a $\mathfrak{C}$-decomposition for n. Therefore, $\mathfrak{D} \subseteq \mathfrak{C}$.

To show that $\mathfrak{C} \subseteq \mathfrak{D}$ reverse the previous steps.

THEOREM 5. *There exists a complete set of orthogonal latin squares of order n if and only if $n \in \mathfrak{D}$.*

Proof. Assume we have a complete set of orthogonal latin squares, say $A^1, \ldots, A^{n-1}$ that are in normal form, i.e. the first row is $1\ 2\ 3 \ldots n$.

Form the following $(n-1)$-tuple for each position (i, j), $i \neq 1$, of the set of latin squares:

$$(a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^{n-1}),$$

where $a^k{}_{ij}$ is the element in the ith row and jth column of the square A^k.

These give a set of $n^2 - n$ $(n-1)$-tuples with entries from $1, \ldots, n$ such that no two members appear in the same respective positions in any two of the $(n-1)$-tuples, since the squares are mutually orthogonal.

Now, extend these $(n-1)$-tuples to n-tuples by extending

$$(a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^{n-1})$$

to

$$(j, a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^{n-1}).$$

Suppose (a) of the definition for $\mathfrak{D}$ was now not true. Then we would have the following setup: for some i, j, k, and l, $i \neq l$,

$$\begin{array}{l} (j, a_{ij}^1, \ldots, a_{ij}^k, \ldots), \\ (j, a_{lj}^1, \ldots, a_{lj}^k, \ldots), \end{array}$$

where $a^k{}_{ij} = a^k{}_{lj}$. This, however, implies that the jth column of A^k has two entries the same—a contradiction. Therefore, (a) holds.

Now, list these n-tuples one under the other by the positions they came from in the following order:

$$(2, 1), (2, 2), \ldots, (2, n), (3, 1), \ldots, (3, n), \ldots, (n, n).$$

The top n n-tuples form a latin square since

(i) the rows clearly have no repetitions;

(ii) if the same number appeared twice in the same column, say the ith column, then the second row of A^i would have a repetition—a contradiction. Therefore, $n \in \mathfrak{D}$.

To prove the converse, reverse the previous argument.

We can now replace Theorem 3 by our main result.

THEOREM 6. *Every finite projective plane can be extended to a uniform H-plane.*

Proof. Let π be a projective plane of order n. Then $n \in \mathfrak{C}$ by applying first the remark at the beginning of this section and then Theorems 5 and 4.

Lastly, we can combine all of these results in the following Theorem.

THEOREM 7. *The following statements are equivalent:*
(a) $n \in \mathfrak{C}$,
(b) *there exists a projective plane of order* n,
(c) *there exists a uniform H-plane with* $t = n$.

Proof.
(a) $\Leftrightarrow$ (b) as in proof of Theorem 6.
(b) $\Rightarrow$ (c) by Theorem 6.
(c) $\Rightarrow$ (b) the associated projective plane satisfies this.

One of the most important unanswered questions dealing with projective planes is: For what n do projective planes of order n exist? The only known projective planes have prime-power order and there is a projective plane for each of these prime-power orders **(5)**. Moreover, Bruck and Ryser **(2)** have shown necessary conditions for n to be the order of a projective plane. However, there is a gap between the two as, for example, nothing is known for $n = 10$.

Theorem 7 itself does not add anything to the final solution of this question. However, it suggests new methods of attack that are worthy of consideration.

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