

Motivic zeta functions of hyperplane arrangements

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Abstract

For each central essential hyperplane arrangement \mathcal{A} over an algebraically closed field, let $Z_{\mathcal{A}}^{\hat{\mu}}(T)$ denote the Denef–Loeser motivic zeta function of \mathcal{A} . We prove a formula expressing $Z_{\mathcal{A}}^{\hat{\mu}}(T)$ in terms of the Milnor fibers of related hyperplane arrangements. This formula shows that, in a precise sense, the degree to which $Z_{\mathcal{A}}^{\hat{\mu}}(T)$ fails to be a combinatorial invariant is completely controlled by these Milnor fibers. As one application, we use this formula to show that the map taking each complex arrangement \mathcal{A} to the Hodge–Deligne specialization of $Z_{\mathcal{A}}^{\hat{\mu}}(T)$ is locally constant on the realization space of any loop-free matroid. We also prove a combinatorial formula expressing the motivic Igusa zeta function of \mathcal{A} in terms of the characteristic polynomials of related arrangements.

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1. Introduction

We study hyperplane arrangements and the motivic zeta functions of Denef and Loeser. Let k be an algebraically closed field, and let H_1, \dots, H_n be a central essential arrangement of hyperplanes in \mathbb{A}_k^d . If f_1, \dots, f_n are linear forms defining H_1, \dots, H_n , respectively, then we can consider the Denef–Loeser motivic zeta function $Z_f^{\hat{\mu}}(T)$ of $f = f_1 \cdots f_n$ and the motivic Igusa zeta function $Z_f^{\text{naive}}(T)$ of f .

Inspired by Kontsevich’s theory of motivic integration [Kon95], Denef and Loeser defined zeta functions [DL98, DL01, DL02] that are power series with coefficients in a Grothendieck ring of varieties. These zeta functions are related to multiple well-known invariants in singularity theory and birational geometry, and they have implications for Igusa’s monodromy conjecture, a longstanding conjecture concerning the poles of Igusa’s local zeta function. There has been interest in understanding these motivic zeta functions, and the closely related topological zeta function, in the case of polynomials defining hyperplane arrangements [BSY11, BMT11, vdV18].

In this paper, we prove Theorem 1.3, which provides a formula for $Z_f^{\hat{\mu}}(T)$ in terms of the classes of Milnor fibers of certain related hyperplane arrangements. We use Theorem 1.3 and a result in [KU22] to show Corollary 1.7, which states that certain specialisations of $Z_f^{\hat{\mu}}(T)$, including the Hodge–Deligne specialisation, remain constant as we vary the arrangement H_1, \dots, H_n within the same connected component of a matroid’s realization space. Thus when $k = \mathbb{C}$, the Hodge–Deligne specialisation of the Denef–Loeser motivic zeta function is a lattice-isotopy invariant, in the sense of e.g. [Ran89, Ran97].

There has been much interest in understanding how invariants of hyperplane arrangements, particularly those invariants arising in singularity theory, vary as the arrangements vary with fixed combinatorial type. For example, a major open conjecture predicts that when $k = \mathbb{C}$, the Betti numbers of a hyperplane arrangement’s Milnor fiber depend only on combinatorial type, i.e., they depend only on the matroid. Budur and Saito proved that a related invariant, the Hodge spectrum, depends only on the combinatorial type [BS10]. Randell proved that the diffeomorphism type, and thus Betti numbers, of the Milnor fiber is a lattice-isotopy invariant [Ran97]. See [Suc17] for a survey on such questions. Our perspective on Corollary 1.7 is in the context of that literature, and we hope it illustrates the utility of the formula in Theorem 1.3 as a tool for answering related questions.

Our methods also provide a combinatorial formula for $Z_f^{\text{naive}}(T)$ in terms of the characteristic polynomials of certain related matroids. See Theorem 1.10 below.

Remark 1.1. We note that the right-hand side of the formulas in Theorem 1.10 make sense even for non-realisable matroids. In a paper with Jensen [JKU21], we use this to define motivic zeta functions for not-necessarily-realisable matroids. In that paper, we show how Poincaré duality for Chow rings of matroids, in the sense of Adiprasito–Huh–Katz [AHK18], implies that these matroid zeta functions satisfy a certain functional equation.

1.1. Statements of main results

Throughout this paper, k will be an algebraically closed field. Before we state our results, we set some notation.

For each $n \in \mathbb{Z}_{>0}$, let $\mu_n \subset k^\times$ be the group of n -th roots of unity, let $K_0^{\mu_n}(\mathbf{Var}_k)$ be the μ_n -equivariant Grothendieck ring of k -varieties, let $\mathbb{L} \in K_0^{\mu_n}(\mathbf{Var}_k)$ be the class of \mathbb{A}_k^1 with the trivial μ_n -action, and let $\mathcal{M}_k^{\mu_n} = K_0^{\mu_n}(\mathbf{Var}_k)[\mathbb{L}^{-1}]$. Let $\hat{\mathcal{M}}_k = \varinjlim_n \mathcal{M}_k^{\mu_n}$, and let $\mathbb{L} \in \hat{\mathcal{M}}_k$ be the image of $\mathbb{L} \in \mathcal{M}_k^{\mu_n}$ for any n .

Let $d, n \in \mathbb{Z}_{>0}$, and let $\text{Gr}_{d,n}$ be the Grassmannian of d -dimensional linear subspaces in $\mathbb{A}_k^n = \text{Spec}(k[x_1, \dots, x_n])$. For each $\mathcal{A} \in \text{Gr}_{d,n}(k)$, let $X_{\mathcal{A}} \subset \mathbb{A}_k^d$ denote the corresponding linear subspace, let $F_{\mathcal{A}}$ be the scheme theoretic intersection of $X_{\mathcal{A}}$ with the closed subscheme of \mathbb{A}_k^n defined by $(x_1 \cdots x_n - 1)$, and endow $F_{\mathcal{A}}$ with the restriction of the μ_n -action on \mathbb{A}_k^n where each $\xi \in \mu_n$ acts by scalar multiplication. Let $Z_{\mathcal{A},k}^{\hat{\mu}}(T) \in \hat{\mathcal{M}}_k[[T]]$ be the Denef–Loeser motivic zeta function of $(x_1 \cdots x_n)|_{X_{\mathcal{A}}}$, and let $Z_{\mathcal{A},0}^{\hat{\mu}}(T) \in \hat{\mathcal{M}}_k[[T]]$ be the Denef–Loeser motivic zeta function of $(x_1 \cdots x_n)|_{X_{\mathcal{A}}}$ at the origin of \mathbb{A}_k^n .

If $X_{\mathcal{A}}$ is not contained in a coordinate hyperplane of \mathbb{A}_k^n , then the restrictions of the coordinates x_i define a central essential hyperplane arrangement in $X_{\mathcal{A}}$. The Milnor fiber of that hyperplane arrangement is $F_{\mathcal{A}}$, the μ_n -action on $F_{\mathcal{A}}$ is the monodromy action, and $Z_{\mathcal{A},k}^{\hat{\mu}}(T)$ and $Z_{\mathcal{A},0}^{\hat{\mu}}(T)$ are the Denef–Loeser motivic zeta functions associated to that arrangement. Note that we are using a definition of the Milnor fiber that takes advantage of the fact

that a hyperplane arrangement is defined by a homogeneous polynomial. This definition is common in the hyperplane arrangement literature, and it allows us to consider the Milnor fiber $F_{\mathcal{A}}$ as a variety.

Remark 1.2. If H_1, \dots, H_n is a central essential hyperplane arrangement in \mathbb{A}_k^d , then any choice of linear forms defining H_1, \dots, H_n gives a linear embedding of \mathbb{A}_k^d into \mathbb{A}_k^n , and H_1, \dots, H_n is the arrangement associated to the resulting subspace of \mathbb{A}_k^n . Therefore, we lose no generality by considering the arrangements associated to d -dimensional linear subspaces in \mathbb{A}_k^n .

Let \mathcal{M} be a rank d loop-free matroid on $\{1, \dots, n\}$, let $\text{Trop}(\mathcal{M}) \subset \mathbb{R}^n$ be the Bergman fan of \mathcal{M} , and let $\text{Gr}_{\mathcal{M}} \subset \text{Gr}_{d,n}$ be the locus parametrising linear subspaces whose associated hyperplane arrangements have combinatorial type \mathcal{M} . For any $w \in \text{Trop}(\mathcal{M})$, there exists a rank d loop-free matroid \mathcal{M}_w on $\{1, \dots, n\}$ such that for all $\mathcal{A} \in \text{Gr}_{\mathcal{M}}(k)$, the initial degeneration $\text{in}_w(X_{\mathcal{A}} \cap \mathbb{G}_{m,k}^n)$ is equal to $X_{\mathcal{A}_w} \cap \mathbb{G}_{m,k}^n$ for some unique $\mathcal{A}_w \in \text{Gr}_{\mathcal{M}_w}(k)$. We refer to Section 2.4 for the definition of \mathcal{M}_w . Let $\mathcal{B}(\mathcal{M})$ be the set of bases in \mathcal{M} , and set

$$\text{wt}_{\mathcal{M}} : \mathbb{R}^n \longrightarrow \mathbb{R} : (w_1, \dots, w_n) \mapsto \max_{B \in \mathcal{B}(\mathcal{M})} \sum_{i \in B} w_i.$$

In this paper, we will prove the following formulas that express the motivic zeta functions $Z_{\mathcal{A},k}^{\hat{\mu}}(T)$ and $Z_{\mathcal{A},0}^{\hat{\mu}}(T)$ in terms of classes of the Milnor fibers $F_{\mathcal{A}_w}$.

THEOREM 1.3. *Let $\mathcal{A} \in \text{Gr}_{\mathcal{M}}(k)$. Then*

$$Z_{\mathcal{A},k}^{\hat{\mu}}(T) = \sum_{w \in \text{Trop}(\mathcal{M}) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\})} [F_{\mathcal{A}_w}, \hat{\mu}] \mathbb{L}^{-d - \text{wt}_{\mathcal{M}}(w)}(T, \dots, T)^w \in \mathcal{M}_k^{\hat{\mu}}[[T]],$$

and

$$Z_{\mathcal{A},0}^{\hat{\mu}}(T) = \sum_{w \in \text{Trop}(\mathcal{M}) \cap \mathbb{Z}_{>0}^n} [F_{\mathcal{A}_w}, \hat{\mu}] \mathbb{L}^{-d - \text{wt}_{\mathcal{M}}(w)}(T, \dots, T)^w \in \mathcal{M}_k^{\hat{\mu}}[[T]].$$

Remark 1.4. Theorem 1.3 tells us that, in a precise sense, all of the “non-combinatorial” information seen by $Z_{\mathcal{A},k}^{\hat{\mu}}$ and $Z_{\mathcal{A},0}^{\hat{\mu}}$ is determined by Milnor fibers of the arrangements \mathcal{A}_w . See Corollary 1.7 and Remark 1.9 below for more on the utility of this perspective.

In the course of proving Theorem 1.3, we prove Theorem 5.2 and Corollary 5.4, which give formulas for motivic zeta functions when certain tropical hypotheses are satisfied. We think of Theorem 5.2 and Corollary 5.4 as being in the spirit of the formulas for zeta functions of so-called Newton non-degenerate hypersurfaces [DH01, Gui02, BV16, BN16]. To prove Theorem 5.2 and Corollary 5.4, we use certain $k[[\pi]]$ -schemes whose special fibers are the initial degenerations that arise in tropical geometry. These $k[[\pi]]$ -schemes have played an essential role in much of tropical geometry. See for example [Gub13]. We also use Sebag’s [Seb04] theory of motivic integration for Greenberg schemes, which are non-constant coefficient versions of arc schemes. For our proofs to account for the $\hat{\mu}$ -action, we use Hartmann’s [Har15] equivariant version of Sebag’s motivic integration.

Theorem 1.3 allows us to use results about additive invariants of the Milnor fibers $F_{\mathcal{A}_w}$ to obtain results about specializations of the Denef–Loeser motivic zeta functions. To state

such an application, we first define some terminology that can apply to additive invariants. Let $\mathbb{Z}[\mathbb{L}]$ be the polynomial ring over the symbol \mathbb{L} , and endow $\mathcal{M}_k^{\hat{\mu}}$ with the $\mathbb{Z}[\mathbb{L}]$ -algebra structure given by $\mathbb{L} \mapsto \mathbb{L}$.

Definition 1.5. Let P be a $\mathbb{Z}[\mathbb{L}]$ -module, and let $\nu : \mathcal{M}_k^{\hat{\mu}} \rightarrow P$ be a $\mathbb{Z}[\mathbb{L}]$ -module morphism. We say that ν is *constant on smooth projective families with μ_n -action* if the following always holds.

If S is a connected separated finite type k -scheme with trivial μ_n -action and $X \rightarrow S$ is a μ_n -equivariant smooth projective morphism from a scheme X with μ_n -action, then the map $S(k) \rightarrow P : s \mapsto \nu[X_s, \hat{\mu}]$ is constant, where X_s denotes the fiber of $X \rightarrow S$ over s .

Remark 1.6. If $k = \mathbb{C}$ and $\text{HD} : \mathcal{M}_k^{\hat{\mu}} \rightarrow \mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$ is the morphism that sends the class of each variety to its Hodge–Deligne polynomial, then HD is constant on smooth projective families with μ_n -action.

Note that if $w \in \text{Trop}(\mathcal{M})$ and $\mathcal{A}_1, \mathcal{A}_2 \in \text{Gr}_{\mathcal{M}}(k)$ are in the same connected component of $\text{Gr}_{\mathcal{M}}$, then $(\mathcal{A}_1)_w, (\mathcal{A}_2)_w \in \text{Gr}_{\mathcal{M}_w}(k)$ are in the same connected component of $\text{Gr}_{\mathcal{M}_w}$. See for example [KU22, lemma 2.4]. Therefore the following corollary is a direct consequence of Theorem 1.3 and [KU22, theorem 1.4].

COROLLARY 1.7. *Let P be a torsion-free $\mathbb{Z}[\mathbb{L}]$ -module, let $\nu : \mathcal{M}_k^{\hat{\mu}} \rightarrow P$ be a $\mathbb{Z}[\mathbb{L}]$ -module morphism that is constant on smooth projective families with μ_n -action, and assume that the characteristic of k does not divide n .*

If $\mathcal{A}_1, \mathcal{A}_2 \in \text{Gr}_{\mathcal{M}}(k)$ are in the same connected component of $\text{Gr}_{\mathcal{M}}$, then

$$\nu\left(Z_{\mathcal{A}_1, k}^{\hat{\mu}}(T)\right) = \nu\left(Z_{\mathcal{A}_2, k}^{\hat{\mu}}(T)\right) \in P[[T]],$$

and

$$\nu\left(Z_{\mathcal{A}_1, 0}^{\hat{\mu}}(T)\right) = \nu\left(Z_{\mathcal{A}_2, 0}^{\hat{\mu}}(T)\right) \in P[[T]].$$

Remark 1.8. In the statement of Corollary 1.7, by ν applied to a power series, we mean the power series obtained by applying ν to each coefficient.

In particular, Corollary 1.7 implies that the Hodge–Deligne specialisation of the Denef–Loeser motivic zeta function remains constant as we vary the linear subspace within the same connected component of $\text{Gr}_{\mathcal{M}}$.

Remark 1.9. It is natural to ask if the Hodge–Deligne specialisation of the Denef–Loeser motivic zeta function is furthermore a combinatorial invariant. Theorem 1.3 tells us that, to verify this, it would be sufficient to show that the Hodge–Deligne polynomial of the Milnor fiber is a combinatorial invariant. More generally, Theorem 1.3 shows that for any combinatorial invariant obtained by applying an additive invariant ν (that respects multiplication by \mathbb{L}) to the class $[F_{\mathcal{A}}, \hat{\mu}]$, we automatically obtain another combinatorial invariant

by applying ν to $Z_{\mathcal{A},k}^{\hat{\mu}}(T)$ or $Z_{\mathcal{A},0}^{\hat{\mu}}(T)$. For example, when χ^{top} is the topological Euler characteristic, we obtain that $\chi^{\text{top}}(Z_{\mathcal{A},k}^{\hat{\mu}}(T))$ and $\chi^{\text{top}}(Z_{\mathcal{A},0}^{\hat{\mu}}(T))$ are combinatorial invariants. Note that $\chi^{\text{top}}(Z_{\mathcal{A},k}^{\hat{\mu}}(T))$ and $\chi^{\text{top}}(Z_{\mathcal{A},0}^{\hat{\mu}}(T))$ can also be seen to be combinatorial by their relationship with the so-called monodromy zeta functions (see [DL02, theorem 1.1] for this relationship, and, e.g., the proof of Lemma 2.3 in [BMT11] for the fact that the monodromy zeta function is a combinatorial invariant).

Our final main result consists of combinatorial formulas for the motivic Igusa zeta functions of a hyperplane arrangement. It is well known that the motivic Igusa zeta functions are combinatorial invariants. For example, one can see this by using De Concini and Procesi’s wonderful models [DCP95] and Denef and Loeser’s formula for the motivic Igusa zeta function in terms of a log resolution [DL01, corollary 3.3.2]. Regardless, it is worth stating Theorem 1.10 below, as it follows from the methods of this paper with little extra effort, and because we are not aware of these particular formulas having appeared in the literature. The formulas in Theorem 1.10 also provide the inspiration for an upcoming paper with Jensen, in which we define motivic zeta functions for matroids.

Let $K_0(\mathbf{Var}_k)$ be the Grothendieck ring of k -varieties, let $\mathbb{L} \in K_0(\mathbf{Var}_k)$ be the class of \mathbb{A}_k^1 , and let $\mathcal{M}_k = K_0(\mathbf{Var}_k)[\mathbb{L}^{-1}]$. For each $\mathcal{A} \in \text{Gr}_{d,n}(k)$, let $Z_{\mathcal{A},k}^{\text{naive}}(T) \in \mathcal{M}_k[[T]]$ be the motivic Igusa zeta function of $(x_1 \cdots x_n)|_{\mathcal{X}_{\mathcal{A}}}$, and let $Z_{\mathcal{A},0}^{\text{naive}}(T) \in \mathcal{M}_k[[T]]$ be the motivic Igusa zeta function of $(x_1 \cdots x_n)|_{\mathcal{X}_{\mathcal{A}}}$ at the origin of \mathbb{A}_k^n .

THEOREM 1.10. *Let $\mathcal{A} \in \text{Gr}_{\mathcal{M}}(k)$. Then*

$$Z_{\mathcal{A},k}^{\text{naive}}(T) = \sum_{w \in \text{Trop}(\mathcal{M}) \cap \mathbb{Z}_{\geq 0}^n} \chi_{\mathcal{M}_w}(\mathbb{L}) \mathbb{L}^{-d - \text{wt}_{\mathcal{M}}(w)}(T, \dots, T)^w \in \mathcal{M}_k[[T]]$$

and

$$Z_{\mathcal{A},0}^{\text{naive}}(T) = \sum_{w \in \text{Trop}(\mathcal{M}) \cap \mathbb{Z}_{> 0}^n} \chi_{\mathcal{M}_w}(\mathbb{L}) \mathbb{L}^{-d - \text{wt}_{\mathcal{M}}(w)}(T, \dots, T)^w \in \mathcal{M}_k[[T]],$$

where $\chi_{\mathcal{M}_w}(\mathbb{L}) \in \mathcal{M}_k$ is the characteristic polynomial of \mathcal{M}_w evaluated at \mathbb{L} .

We now outline the structure of this paper.

- (i) In Section 2, we set notation and recall some facts that will be used throughout this paper.
- (ii) In Section 3, we describe one way of expressing the Denef–Loeser motivic zeta function in terms of Hartmann’s equivariant motivic integration. The results in this section are unsurprising and likely well-known, but as we are unaware of these results having appeared in the literature, we include this section for completeness and readers’ convenience.
- (iii) In Section 4, we consider certain group actions on subvarieties of an algebraic torus. In the course of proving Theorem 1.3, the results in this section are used to manipulate classes in the equivariant Grothendieck ring of varieties.

- (iv) In Section 5, we prove formulas for motivic zeta functions in a setting involving varieties with smooth initial degenerations. We do this by computing the motivic volumes of the fibers of a certain tropicalisation map.
- (v) We end the paper with Section 6, where we use the results of previous sections to prove our main results.

2. Preliminaries

In this section, we set some notation and recall facts about the equivariant Grothendieck ring of varieties, the motivic zeta functions of Denef and Loeser, Hartmann’s equivariant motivic integration, and linear subspaces and matroids.

2.1. The equivariant Grothendieck ring of varieties

Suppose X is a separated finite type scheme over k . We let $K_0(\mathbf{Var}_X)$ denote the Grothendieck ring of varieties over X , we let $\mathbb{L} \in K_0(\mathbf{Var}_X)$ denote the class of $\mathbb{A}_k^1 \times_k X$, and for each separated finite type X -scheme Y , we let $[Y/X] \in K_0(\mathbf{Var}_X)$ denote the class of Y . We will let \mathcal{M}_X denote the ring obtained by inverting \mathbb{L} in $K_0(\mathbf{Var}_X)$, and by slight abuse of notation, we write $\mathbb{L}, [Y/X] \in \mathcal{M}_X$ to denote the images of $\mathbb{L}, [Y/X]$, respectively, in \mathcal{M}_X .

We will let $K_0(\mathbf{Var}_k)$ and \mathcal{M}_k denote $K_0(\mathbf{Var}_{\text{Spec}(k)})$ and $\mathcal{M}_{\text{Spec}(k)}$, respectively, and for each separated finite type k -scheme Y , we will write $[Y] = [Y/\text{Spec}(k)]$ in both $K_0(\mathbf{Var}_k)$ and \mathcal{M}_k .

Suppose G is a finite abelian group. An action of G on a scheme is said to be *good* if each orbit is contained in an affine open subscheme. For example, any G -action on any quasiprojective k -scheme is good. If the separated finite type k -scheme X is endowed with a good G -action, then we will let $K_0^G(\mathbf{Var}_X)$ denote the G -equivariant Grothendieck ring of varieties over X . For the precise definition of $K_0^G(\mathbf{Var}_X)$, we refer to [Har15, definition 4.1]. We will let $\mathbb{L} \in K_0^G(\mathbf{Var}_X)$ denote the class of $\mathbb{A}_k^1 \times_k X$ with the action induced by the trivial G -action on \mathbb{A}_k^1 and the given G -action on X , and for each separated finite type X -scheme Y with good G -action making the structure morphism G -equivariant, we will let $[Y/X, G] \in K_0^G(\mathbf{Var}_X)$ denote the class of Y with its given G -action. We will let \mathcal{M}_X^G denote the ring obtained by inverting \mathbb{L} in $K_0^G(\mathbf{Var}_X)$, and by slight abuse of notation, we will let $\mathbb{L}, [Y/X, G] \in \mathcal{M}_X^G$ denote the images of $\mathbb{L}, [Y/X, G]$, respectively, in \mathcal{M}_X^G .

If X is a separated finite type k -scheme with no specified G -action and we refer to $K_0^G(\mathbf{Var}_X)$ or \mathcal{M}_X^G , then we are considering X with the trivial G -action. We will let $K_0^G(\mathbf{Var}_k)$ and \mathcal{M}_k^G denote $K_0^G(\mathbf{Var}_{\text{Spec}(k)})$ and $\mathcal{M}_{\text{Spec}(k)}^G$, respectively, and for each separated finite type k -scheme Y with good G -action making the structure morphism G -equivariant, we will let $[Y, G] = [Y/\text{Spec}(k), G]$ in both $K_0^G(\mathbf{Var}_k)$ and \mathcal{M}_k^G .

For $\ell \in \mathbb{Z}_{>0}$, we will let $\mu_\ell \subset k^\times$ denote the group of ℓ th roots of unity.

Remark 2.1. We will only consider μ_ℓ as a finite group, so when the characteristic of k divides ℓ , we will not consider the non-reduced scheme structure of μ_ℓ .

For each $\ell, m \in \mathbb{Z}_{>0}$, there is a morphism $\mu_{\ell m} \rightarrow \mu_\ell : \xi \mapsto \xi^m$. Suppose that X is a separated finite type scheme over k . Then the morphism $\mu_{\ell m} \rightarrow \mu_\ell$ induces ring morphisms $K_0^{\mu_\ell}(\mathbf{Var}_X) \rightarrow K_0^{\mu_{\ell m}}(\mathbf{Var}_X)$ and $\mathcal{M}_X^{\mu_\ell} \rightarrow \mathcal{M}_X^{\mu_{\ell m}}$. We will let $K_0^{\hat{\mu}}(\mathbf{Var}_X) = \varinjlim_\ell K_0^{\mu_\ell}(\mathbf{Var}_X)$ and $\mathcal{M}_X^{\hat{\mu}} = \varinjlim_\ell \mathcal{M}_X^{\mu_\ell}$. We let $\mathbb{L} \in K_0^{\hat{\mu}}(\mathbf{Var}_X)$ denote the image of $\mathbb{L} \in K_0^{\mu_\ell}(\mathbf{Var}_X)$ for any $\ell \in \mathbb{Z}_{>0}$, and similarly we let $\mathbb{L} \in \mathcal{M}_X^{\hat{\mu}}$ denote the image of $\mathbb{L} \in \mathcal{M}_X^{\mu_\ell}$ for any $\ell \in \mathbb{Z}_{>0}$.

For each separated finite type X -scheme Y with good μ_ℓ -action making the structure morphism μ_ℓ -equivariant, we let $[Y/X, \hat{\mu}] \in K_0^{\hat{\mu}}(\mathbf{Var}_X)$ denote the image of $[Y/X, \mu_\ell] \in K_0^{\mu_\ell}(\mathbf{Var}_X)$, and we similarly let $[Y/X, \hat{\mu}] \in \mathcal{M}_X^{\hat{\mu}}$ denote the image of $[Y/X, \mu_\ell] \in \mathcal{M}_X^{\mu_\ell}$.

We will let $K_0^{\hat{\mu}}(\mathbf{Var}_k)$ and $\mathcal{M}_k^{\hat{\mu}}$ denote $K_0^{\hat{\mu}}(\mathbf{Var}_{\text{Spec}(k)})$ and $\mathcal{M}_{\text{Spec}(k)}^{\hat{\mu}}$, respectively, and for each $\ell \in \mathbb{Z}_{>0}$ and each separated finite type k -scheme Y with good μ_ℓ -action making the structure morphism μ_ℓ -equivariant, we will let $[Y, \hat{\mu}] = [Y/\text{Spec}(k), \hat{\mu}]$ in both $K_0^{\hat{\mu}}(\mathbf{Var}_k)$ and $\mathcal{M}_k^{\hat{\mu}}$.

2.2. The motivic zeta functions of Denef and Loeser

Let X be a smooth, pure dimensional, separated, finite type k -scheme. For each $\ell \in \mathbb{Z}_{\geq 0}$, we will let $\mathcal{L}_\ell(X)$ denote ℓ -th jet scheme of X , and for each $m \geq \ell$, we will let $\theta_\ell^m : \mathcal{L}_m(X) \rightarrow \mathcal{L}_\ell(X)$ denote the truncation morphism. We will let $\mathcal{L}(X) = \varinjlim_\ell \mathcal{L}_\ell(X)$ denote the arc scheme of X , and for each $\ell \in \mathbb{Z}_{\geq 0}$, we will let $\theta_\ell : \mathcal{L}(X) \rightarrow \mathcal{L}_\ell(X)$ denote the canonical morphism. The following is a special case of a theorem of Bhatt’s [Bha16, theorem 1.1].

THEOREM 2.2 (Bhatt). *The k -scheme $\mathcal{L}(X)$ represents the functor taking each k -algebra A to $\text{Hom}_k(\text{Spec}(A[[\pi]]), X)$, and under this identification, each morphism $\theta_\ell : \mathcal{L}(X) \rightarrow \mathcal{L}_\ell(X)$ is the truncation morphism.*

A subset of $\mathcal{L}(X)$ is called a *cylinder* if it is the preimage, under θ_ℓ , of a constructible subset of $\mathcal{L}_\ell(X)$ for some $\ell \in \mathbb{Z}_{\geq 0}$. We will let μ_X denote the *motivic measure* on $\mathcal{L}(X)$, which assigns a *motivic volume* in \mathcal{M}_X to each cylinder.

Suppose f is a regular function on X . If $x \in \mathcal{L}(X)$ has residue field $k(x)$, then it corresponds to a k -morphism $\psi_x : \text{Spec}(k(x)[[\pi]]) \rightarrow X$, and we will let $f(x)$ denote $f(\psi_x) \in k(x)[[\pi]]$. For each $x \in \mathcal{L}(X)$, the *order* of f at x will refer to the order of π in the power series $f(x)$, and the *angular component* of f at x will refer to the leading coefficient of the power series $f(x)$. We will let $\text{ord}_f : \mathcal{L}(X) \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ denote the function taking each $x \in \mathcal{L}(X)$ to the order of f at x . We will let $Z_f^{\text{naive}}(T) \in \mathcal{M}_X[[T]]$ denote the motivic Igusa zeta function of f . Then

$$Z_f^{\text{naive}}(T) = \sum_{\ell \in \mathbb{Z}_{\geq 0}} \mu_X(\text{ord}_f^{-1}(\ell)) T^\ell \in \mathcal{M}_X[[T]].$$

Remark 2.3. In the literature, the motivic Igusa zeta function is sometimes referred to as the naive zeta function of Denef and Loeser.

We will let $Z_f^{\hat{\mu}}(T) \in \mathcal{M}_X^{\hat{\mu}}[[T]]$ denote the Denef–Loeser motivic zeta function of f . We briefly recall the definition of $Z_f^{\hat{\mu}}(T)$. The constant term of $Z_f^{\hat{\mu}}(T)$ is equal to 0. Let $\ell \in \mathbb{Z}_{>0}$, and let $Y_{\ell,1}$ be the closed subscheme of $\mathcal{L}_\ell(X)$ where f is equal to π^ℓ . For any k -algebra A , there is a μ_ℓ -action on $A[[\pi]]$, where $\xi \in \mu_\ell$ acts by $\pi \mapsto \xi\pi$, and these actions induce a μ_ℓ -action on $\mathcal{L}_\ell(X)$ making $Y_{\ell,1}$ invariant. Note also that the truncation morphism $\theta_0^\ell : \mathcal{L}_\ell(X) \rightarrow X$ restricts to a μ_ℓ -equivariant morphism $Y_{\ell,1} \rightarrow X$. Then the coefficient of T^ℓ in $Z_f^{\hat{\mu}}(T)$ is defined to be equal to $[Y_{\ell,1}/X, \hat{\mu}]_{\mathbb{L}}^{-(\ell+1)\dim X} \in \mathcal{M}_X^{\hat{\mu}}$.

Remark 2.4. Denef and Loeser defined versions of these zeta functions with coefficients in \mathcal{M}_k and $\mathcal{M}_k^{\hat{\mu}}$ [DL98, DL02], and Looijenga introduced versions with coefficients in the relative Grothendieck rings \mathcal{M}_X and $\mathcal{M}_X^{\hat{\mu}}$ [Loo02]. See [DL01] for the definitions we are

using for $Z_f^{\text{naive}}(T)$ and $Z_f^{\hat{\mu}}(T)$, but note that compared to those definitions, ours differ by a normalisation factor of $\mathbb{L}^{-\dim X}$.

2.3. *Hartmann’s equivariant motivic integration.*

For the remainder of this paper, let $R = k[[\pi]]$, the ring of power series over k . We will set up some notation and recall facts for Greenberg schemes and Hartmann’s equivariant motivic integration [Har15], which is an equivariant version of Sebag’s motivic integration for formal schemes [Seb04]. For the non-equivariant version of this theory, we also recommend the book [CNS18].

Remark 2.5. In [Har15], Hartmann uses formal R -schemes. The analogous theory for algebraic R -schemes, as stated here, directly follows by taking π -adic completion.

Let \mathfrak{X} be a smooth, pure relative dimensional, separated, finite type R -scheme. We will let \mathfrak{X}_0 denote the special fiber of \mathfrak{X} . For each $\ell \in \mathbb{Z}_{\geq 0}$, we will let $\mathcal{G}_\ell(\mathfrak{X})$ denote the ℓ -th Greenberg scheme of \mathfrak{X} . Thus $\mathcal{G}_\ell(\mathfrak{X})$ represents the functor taking each k -algebra A to $\text{Hom}_R(\text{Spec}(A[[\pi]]/(\pi^{\ell+1})), \mathfrak{X})$. For each $m \geq \ell$, we will let $\theta_\ell^m : \mathcal{G}_m(\mathfrak{X}) \rightarrow \mathcal{G}_\ell(\mathfrak{X})$ denote the truncation morphism. We will let $\mathcal{G}(\mathfrak{X}) = \varprojlim_{\ell} \mathcal{G}_\ell(\mathfrak{X})$ denote the Greenberg scheme of \mathfrak{X} , and for each $\ell \in \mathbb{Z}_{\geq 0}$, we will let $\theta_\ell : \mathcal{G}(\mathfrak{X}) \rightarrow \mathcal{G}_\ell(\mathfrak{X})$ denote the canonical morphism. As for arc schemes, the following is a special case of [Bha16, theorem 1.1]. See for example [CNS18, chapter 4, proposition 3.1.7].

THEOREM 2.6 (Bhatt). *The k -scheme $\mathcal{G}(\mathfrak{X})$ represents the functor taking each k -algebra A to $\text{Hom}_R(\text{Spec}(A[[\pi]]), \mathfrak{X})$, and under this identification, each morphism $\theta_\ell : \mathcal{G}(\mathfrak{X}) \rightarrow \mathcal{G}_\ell(\mathfrak{X})$ is the truncation morphism.*

A subset of $\mathcal{G}(\mathfrak{X})$ is called a *cylinder* if it is the preimage, under θ_ℓ , of a constructible subset of $\mathcal{G}_\ell(\mathfrak{X})$ for some $\ell \in \mathbb{Z}_{\geq 0}$. We will let $\mu_{\mathfrak{X}}$ denote the motivic measure on $\mathcal{G}(\mathfrak{X})$, which assigns a motivic volume in $\mathcal{M}_{\mathfrak{X}_0}$ to each cylinder.

Suppose f is a regular function on \mathfrak{X} . If $x \in \mathcal{G}(\mathfrak{X})$ has residue field $k(x)$, then it corresponds to an R -morphism $\psi_x : \text{Spec}(k(x)[[\pi]]) \rightarrow \mathfrak{X}$, and we will let $f(x)$ denote $f(\psi_x) \in k(x)[[\pi]]$. As for arc schemes, this is used to define the *order* and *angular component* of f at x and the order function $\text{ord}_f : \mathcal{G}(\mathfrak{X}) \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

Now suppose G is a finite abelian group acting on R , and suppose that each element of G acts on R by a π -adically continuous k -algebra morphism. Endow \mathfrak{X} with a good G -action making the structure morphism G -equivariant, and endow \mathfrak{X}_0 with the restriction of the G -action on \mathfrak{X} . The G -action on \mathfrak{X} induces good G -actions on $\mathcal{G}(\mathfrak{X})$ and each $\mathcal{G}_\ell(\mathfrak{X})$. We refer to [Har15, section 3.2] for the construction and properties of these G -actions on the Greenberg schemes. We will let $\mu_{\mathfrak{X}}^G$ denote the G -equivariant motivic measure on $\mathcal{G}(\mathfrak{X})$, which assigns a motivic volume in $\mathcal{M}_{\mathfrak{X}_0}^G$ to each G -invariant cylinder in $\mathcal{G}(\mathfrak{X})$. We refer to [Har15, section 4.2] for the definition of $\mu_{\mathfrak{X}}^G$.

If $A \subset \mathcal{G}(\mathfrak{X})$ is a G -invariant cylinder and $\alpha : A \rightarrow \mathbb{Z}$ is a function whose fibers are G -invariant cylinders, then the integral of α is defined to be

$$\int_A \mathbb{L}^{-\alpha} d\mu_{\mathfrak{X}}^G = \sum_{\ell \in \mathbb{Z}} \mu_{\mathfrak{X}}^G(\alpha^{-1}(\ell)) \mathbb{L}^{-\ell} \in \mathcal{M}_{\mathfrak{X}_0}^G.$$

Remark 2.7. By the quasi-compactness of the constructible topology, α takes finitely many values, so the above sum is well defined. See [CNS18, chapter 6, section 1.2].

We now state the equivariant version of the motivic change of variables formula [Har15, theorem 4.18]. If $h: \mathfrak{Y} \rightarrow \mathfrak{X}$ is a morphism of R -schemes, then we let $\text{ordjac}_h: \mathcal{G}(\mathfrak{Y}) \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ denote the order function of the jacobian ideal of h .

THEOREM 2.8 (Hartmann). *Suppose $\#G$ is not divisible by the characteristic of k . Let $\mathfrak{X}, \mathfrak{Y}$ be smooth, pure relative dimensional, separated, finite type R -schemes with good G -action making the structure morphisms equivariant, and let $h: \mathfrak{Y} \rightarrow \mathfrak{X}$ be a G -equivariant morphism that induces an open immersion on generic fibers. Let A, B be G -invariant cylinders in $\mathcal{G}(\mathfrak{X}), \mathcal{G}(\mathfrak{Y})$, respectively, such that h induces a bijection $B(k') \rightarrow A(k')$ for all extensions k' of k .*

If $\alpha: A \rightarrow \mathbb{Z}$ is a function whose fibers are G -invariant cylinders, then $\alpha \circ \mathcal{G}(h) - \text{ordjac}_h: B \rightarrow \mathbb{Z}$ is a function whose fibers are G -invariant cylinders, and

$$\int_A \mathbb{L}^{-\alpha} d\mu_{\mathfrak{X}}^G = \int_B \mathbb{L}^{-(\alpha \circ \mathcal{G}(h) + \text{ordjac}_h)} d\mu_{\mathfrak{Y}}^G \in \mathcal{M}_{\mathfrak{X}_0}^G.$$

Remark 2.9. Hartmann stated the formula when $A = \mathcal{G}(\mathfrak{X})$ and $B = \mathcal{G}(\mathfrak{Y})$, but the same proof works when replacing $\mathcal{G}(\mathfrak{X})$ and $\mathcal{G}(\mathfrak{Y})$ with G -invariant cylinders. See for example the proof of the non-equivariant version in [CNS18].

We note that for all $\ell \in \mathbb{Z}_{>0}$, the characteristic of k never divides $\#\mu_\ell$.

2.4. Linear subspaces and matroids

Let $d, n \in \mathbb{Z}_{>0}$. We will let $\text{Gr}_{d,n}$ denote the Grassmannian of d -dimensional linear subspaces in $\mathbb{A}_k^n = \text{Spec}(k[x_1, \dots, x_n])$. We will let $\mathbb{G}_{m,k}^n = \text{Spec}\left(k\left[x_1^{\pm 1}, \dots, x_n^{\pm 1}\right]\right) \subset \mathbb{A}_k^n$ denote the complement of the coordinate hyperplanes, and we will let $V(x_1 \cdots x_n - 1)$ denote the closed subscheme of \mathbb{A}_k^n defined by $(x_1 \cdots x_n - 1)$. For each $\mathcal{A} \in \text{Gr}_{d,n}(k)$, we will let $X_{\mathcal{A}} \hookrightarrow \mathbb{A}_k^n$ denote the corresponding linear subspace. If $X_{\mathcal{A}}$ is not contained in a coordinate hyperplane of \mathbb{A}_k^n , then the restrictions to $X_{\mathcal{A}}$ of the coordinates x_i define a central essential hyperplane arrangement in $X_{\mathcal{A}}$. We let $U_{\mathcal{A}} = X_{\mathcal{A}} \cap \mathbb{G}_{m,k}^n$ and $F_{\mathcal{A}} = X_{\mathcal{A}} \cap V(x_1 \cdots x_n - 1)$ denote this arrangement’s complement and Milnor fiber, respectively, and we endow $F_{\mathcal{A}}$ with the restriction of the μ_n -action on \mathbb{A}_k^n where each $\xi \in \mu_n$ acts by scalar multiplication. In the context of tropical geometry, we will consider both $U_{\mathcal{A}}$ and $F_{\mathcal{A}}$ as closed subschemes of the algebraic torus $\mathbb{G}_{m,k}^n$. We will let $Z_{\mathcal{A}}^{\hat{\mu}}(T) \in \mathcal{M}_{X_{\mathcal{A}}}^{\hat{\mu}}[[T]]$ and $Z_{\mathcal{A}}^{\text{naive}}(T) \in \mathcal{M}_{X_{\mathcal{A}}}[[T]]$ denote the Denef–Loeser motivic zeta function and the motivic Igusa zeta function, respectively, of the restriction of the monomial $x_1 \cdots x_n$ to $X_{\mathcal{A}}$. We will let $Z_{\mathcal{A},k}^{\hat{\mu}}(T) \in \mathcal{M}_k^{\hat{\mu}}[[T]]$ (resp. $Z_{\mathcal{A},k}^{\text{naive}}(T) \in \mathcal{M}_k[[T]]$) denote the power series obtained by pushing forward each coefficient of $Z_{\mathcal{A}}^{\hat{\mu}}(T)$ (resp. $Z_{\mathcal{A}}^{\text{naive}}(T)$) along the structure morphism of $X_{\mathcal{A}}$. We will let $Z_{\mathcal{A},0}^{\hat{\mu}}(T) \in \mathcal{M}_k^{\hat{\mu}}[[T]]$ (resp. $Z_{\mathcal{A},0}^{\text{naive}}(T) \in \mathcal{M}_k[[T]]$) denote the power series obtained by pulling back each coefficient of $Z_{\mathcal{A}}^{\hat{\mu}}(T)$ (resp. $Z_{\mathcal{A}}^{\text{naive}}(T)$) along the inclusion of the origin into $X_{\mathcal{A}}$.

Let \mathcal{M} be a rank d loop-free matroid on $\{1, \dots, n\}$. We will let $\chi_{\mathcal{M}}(\mathbb{L}) \in \mathcal{M}_k$ denote the characteristic polynomial of \mathcal{M} evaluated at \mathbb{L} , so

$$\chi_{\mathcal{M}}(\mathbb{L}) = \sum_{I \subset \{1, \dots, n\}} (-1)^{\#I} \mathbb{L}^{d - \text{rk} I} \in \mathcal{M}_k,$$

where $\text{rk} I$ is the rank function of \mathcal{M} applied to I . We will let $\mathcal{B}(\mathcal{M})$ denote the set of bases of \mathcal{M} , and we will let $\text{wt}_{\mathcal{M}} : \mathbb{R}^n \rightarrow \mathbb{R}$ denote the function $(w_1, \dots, w_n) \mapsto \max_{B \in \mathcal{B}(\mathcal{M})} \sum_{i \in B} w_i$. For each $w = (w_1, \dots, w_n) \in \mathbb{R}^n$, we let

$$\mathcal{B}(\mathcal{M}_w) = \left\{ B \in \mathcal{B}(\mathcal{M}) \mid \sum_{i \in B} w_i = \text{wt}_{\mathcal{M}}(w) \right\}$$

be the set of all bases with maximal w -weight. The set $\mathcal{B}(\mathcal{M}_w)$ is the set of the bases of a rank d matroid on $\{1, \dots, n\}$, and we will let \mathcal{M}_w denote that matroid. We let $\text{Trop}(\mathcal{M})$ denote the Bergman fan of \mathcal{M} , so

$$\text{Trop}(\mathcal{M}) = \{w \in \mathbb{R}^n \mid \mathcal{M}_w \text{ is loop-free}\}.$$

We will let $\text{Gr}_{\mathcal{M}} \subset \text{Gr}_{d,n}$ denote the locus parametrising linear subspaces whose associated hyperplane arrangements have combinatorial type \mathcal{M} . For all $\mathcal{A} \in \text{Gr}_{\mathcal{M}}(k)$, the fact that \mathcal{M} is loop-free implies that $X_{\mathcal{A}}$ is not contained in a coordinate hyperplane. Note that if $\mathcal{A} \in \text{Gr}_{\mathcal{M}}(k)$, then $[U_{\mathcal{A}}] = \chi_{\mathcal{M}}(\mathbb{L}) \in \mathcal{M}_k$ and

$$\text{Trop}(U_{\mathcal{A}}) = \{w \in \mathbb{R}^n \mid \text{in}_w U_{\mathcal{A}} \neq \emptyset\} = \text{Trop}(\mathcal{M}).$$

For each $\mathcal{A} \in \text{Gr}_{\mathcal{M}}(k)$ and each $w \in \text{Trop}(\mathcal{M})$, we will let $\mathcal{A}_w \in \text{Gr}_{\mathcal{M}_w}(k)$ denote the unique point such that $\text{in}_w U_{\mathcal{A}} = U_{\mathcal{A}_w}$.

Before concluding the preliminaries, we recall two propositions proved in [KU22] that will be used in Section 6. If $B \in \mathcal{B}(\mathcal{M})$ and $i \in \{1, \dots, n\} \setminus B$, then we will let $C(\mathcal{M}, i, B)$ denote the fundamental circuit in \mathcal{M} of B with respect to i , so $C(\mathcal{M}, i, B)$ is the unique circuit in \mathcal{M} contained in $B \cup \{i\}$. For each circuit C in \mathcal{M} and each $\mathcal{A} \in \text{Gr}_{\mathcal{M}}(k)$, we will let $L_C^{\mathcal{A}} \in k[x_1, \dots, x_n]$ denote a linear form in the ideal defining $X_{\mathcal{A}}$ in \mathbb{A}_k^n such that the coefficient of x_i in $L_C^{\mathcal{A}}$ is nonzero if and only if $i \in C$. Such an $L_C^{\mathcal{A}}$ exists and is unique up to scaling by a unit in k . Once and for all, we fix such an $L_C^{\mathcal{A}}$ for all C and \mathcal{A} .

PROPOSITION 2.10 ([KU22, proposition 3.5]). *Let $\mathcal{A} \in \text{Gr}_{\mathcal{M}}(k)$, let $w \in \mathbb{R}^n$, and let $B \in \mathcal{B}(\mathcal{M}_w)$. Then*

$$\left\{ L_{C(\mathcal{M}, i, B)}^{\mathcal{A}} \mid i \in \{1, \dots, n\} \setminus B \right\} \subset k[x_1, \dots, x_n]$$

generates the ideal of $X_{\mathcal{A}}$ in \mathbb{A}_k^n , and

$$\left\{ \text{in}_w L_{C(\mathcal{M}, i, B)}^{\mathcal{A}} \mid i \in \{1, \dots, n\} \setminus B \right\} \subset k[x_1^{\pm}, \dots, x_n^{\pm}]$$

generates the ideal of $\text{in}_w U_{\mathcal{A}}$ in $\mathbb{G}_{m,k}^n$.

PROPOSITION 2.11 ([KU22, proposition 3.2]). *Let $w = (w_1, \dots, w_n) \in \mathbb{R}^n$, let $B \in \mathcal{B}(\mathcal{M}_w)$, and let $i \in \{1, \dots, n\} \setminus B$. Then*

$$\min_{j \in C(\mathcal{M}, i, B)} w_j = w_i.$$

For additional information on matroids and the tropical geometry of linear subspaces, we refer to [MS15, chapter 4].

3. Equivariant motivic integration and the motivic zeta function

Let $\ell \in \mathbb{Z}_{>0}$, and throughout this section, endow $R = k[[\pi]]$ with the μ_ℓ -action where each $\xi \in \mu_\ell$ acts on R by the π -adically continuous k -morphism $\pi \mapsto \xi^{-1}\pi$.

Let X be a smooth, pure dimensional, finite type, separated scheme over k . We will endow $\mathcal{L}(X)$ and each $\mathcal{L}_m(X)$ with μ_ℓ -actions that make the truncation morphisms μ_ℓ -equivariant as follows. Let $\xi \in \mu_\ell$, let A be a k -algebra, let $\xi_{A[[\pi]]} : \text{Spec}(A[[\pi]]) \rightarrow \text{Spec}(A[[\pi]])$ be the morphism whose pullback is the π -adically continuous A -algebra morphism $\pi \mapsto \xi^{-1}\pi$, and let $\xi_{A[\pi]/(\pi^{n+1})} : \text{Spec}(A[\pi]/(\pi^{n+1})) \rightarrow \text{Spec}(A[\pi]/(\pi^{n+1}))$ be the morphism whose pullback is the A -algebra morphism $\pi \mapsto \xi^{-1}\pi$.

If $x \in \mathcal{L}(X)(A)$ corresponds to a k -morphism

$$\psi_x : \text{Spec}(A[[\pi]]) \rightarrow X,$$

then let $\xi \cdot x \in \mathcal{L}(X)(A)$ correspond to the k -morphism

$$\psi_x \circ \xi_{A[[\pi]]}^{-1} : \text{Spec}(A[[\pi]]) \rightarrow X.$$

This action is clearly functorial in A , so it defines a μ_ℓ -action on $\mathcal{L}(X)$. Similarly, if $x \in \mathcal{L}_m(X)(A)$ corresponds to a k -morphism

$$\psi_x : \text{Spec}\left(A[\pi]/\left(\pi^{m+1}\right)\right) \rightarrow X,$$

then let $\xi \cdot x \in \mathcal{L}_m(X)(A)$ correspond to the k -morphism

$$\psi_x \circ \xi_{A[\pi]/(\pi^{m+1})}^{-1} : \text{Spec}\left(A[\pi]/\left(\pi^{m+1}\right)\right) \rightarrow X.$$

This action is also functorial in A , so it defines a μ_ℓ -action on $\mathcal{L}_m(X)$. We also see that these μ_ℓ -actions make the truncation morphisms μ_ℓ -equivariant.

PROPOSITION 3.1. *Let f be a regular function on X . Then f has constant order on any μ_ℓ -orbit of $\mathcal{L}(X)$. Furthermore, f has constant angular component on any μ_ℓ -orbit of $\mathcal{L}(X)$ on which f has order ℓ .*

Proof. Let $\xi \in \mu_\ell$, let $\xi_{\mathcal{L}(X)} : \mathcal{L}(X) \rightarrow \mathcal{L}(X)$ be its action on $\mathcal{L}(X)$, let $x \in \mathcal{L}(X)(k')$ for some extension k' of k , let $R' = k'[[\pi]]$, and let $\xi_{R'} : \text{Spec}(R') \rightarrow \text{Spec}(R')$ be the morphism whose pullback is the π -adically continuous k' -algebra morphism $\pi \mapsto \xi^{-1}\pi$. Then x corresponds to a k -morphism

$$\psi_x : \text{Spec}(R') \rightarrow X,$$

and $\xi_{\mathcal{L}(X)}(x) \in \mathcal{L}(X)(k')$ corresponds to the k -morphism

$$\psi_x \circ \xi_{R'}^{-1} : \text{Spec}(R') \rightarrow X.$$

Write

$$f(x) = f(\psi_x) = \sum_{i \geq 0} a_i \pi^i \in R',$$

where each $a_i \in k'$. Then

$$f(\xi_{\mathcal{L}(X)}(x)) = f\left(\psi_x \circ \xi_{R'}^{-1}\right) = \sum_{i \geq 0} a_i \xi^i \pi^i \in R'.$$

Thus the order of $f(x)$ is equal to the order of $f(\xi_{\mathcal{L}(X)}(x))$, and if $f(x)$ has order ℓ , then the fact that $\xi^\ell = 1$ implies that the angular component of $f(x)$ is equal to the angular component of $f(\xi_{\mathcal{L}(X)}(x))$. Thus we are done.

Let $\mathfrak{X} = X \times_k \text{Spec}(R)$ and endow \mathfrak{X} with the μ_ℓ -action induced by the μ_ℓ -action on R and the trivial μ_ℓ -action on X . Note that any open affine cover of X induces an open cover of \mathfrak{X} by μ_ℓ -invariant affines, so the μ_ℓ -action on \mathfrak{X} is good. Composition with the projection $\mathfrak{X} \rightarrow X$ induces isomorphisms $\mathcal{G}(\mathfrak{X}) \rightarrow \mathcal{L}(X)$ and $\mathcal{G}_m(\mathfrak{X}) \rightarrow \mathcal{L}_m(X)$ that commute with the truncation morphisms.

Remark 3.2. The motivic zeta function of X can be defined using only the arc scheme $\mathcal{L}(X)$. We pass to the constant family \mathfrak{X} and the Greenberg scheme $\mathcal{G}(\mathfrak{X})$ because our computations will utilise comparisons to R -schemes that are not constant families (see Section 5).

Because we are using Greenberg schemes, our techniques naturally generalize to the case where \mathfrak{X} is a degenerating family of linear subspaces. In this case, one would need to replace matroids with valuated matroids. For expositional simplicity, we restrict our attention to the constant coefficient case.

PROPOSITION 3.3. *The isomorphisms $\mathcal{G}(\mathfrak{X}) \rightarrow \mathcal{L}(X)$ and $\mathcal{G}_m(\mathfrak{X}) \rightarrow \mathcal{L}_m(X)$ are μ_ℓ -equivariant.*

Proof. Let $m \in \mathbb{Z}_{\geq 0}$. It will be sufficient to show that the isomorphism $\mathcal{G}_m(\mathfrak{X}) \rightarrow \mathcal{L}_m(X)$ is μ_ℓ -equivariant, as we get the remainder of the proposition by taking inverse limit.

Let $\xi \in \mu_\ell$, let $\xi_{\mathfrak{X}} : \mathfrak{X} \rightarrow \mathfrak{X}$ be its action on \mathfrak{X} , and let $\xi_{\mathcal{G}_m(\mathfrak{X})} : \mathcal{G}_m(\mathfrak{X}) \rightarrow \mathcal{G}_m(\mathfrak{X})$ be its action on $\mathcal{G}_m(\mathfrak{X})$.

Let $x \in \mathcal{G}_m(\mathfrak{X})(A)$ for some k -algebra A , and let

$$\xi_{A[\pi]/(\pi^{m+1})} : \text{Spec}\left(A[\pi]/\left(\pi^{m+1}\right)\right) \rightarrow \text{Spec}\left(A[\pi]/\left(\pi^{m+1}\right)\right)$$

be the morphism whose pullback is the A -algebra morphism $\pi \mapsto \xi^{-1}\pi$. Then x corresponds to an R -morphism

$$\psi_x : \text{Spec}\left(A[\pi]/\left(\pi^{m+1}\right)\right) \rightarrow \mathfrak{X},$$

and $\xi_{\mathcal{G}_m(\mathfrak{X})}(x) \in \mathcal{G}_m(\mathfrak{X})(A)$ corresponds to the R -morphism

$$\xi_{\mathfrak{X}} \circ \psi_x \circ \xi_{A[\pi]/(\pi^{m+1})}^{-1} : \text{Spec}\left(A[\pi]/\left(\pi^{m+1}\right)\right) \rightarrow \mathfrak{X}.$$

Because $\xi_{\mathfrak{X}}$ is trivial on the factor X , we get that the composition of the above morphism with the projection $\mathfrak{X} \rightarrow X$ is equal to the composition of

$$\psi_x \circ \xi_{A[\pi]/(\pi^{m+1})}^{-1} : \text{Spec}\left(A[\pi]/\left(\pi^{m+1}\right)\right) \rightarrow \mathfrak{X}$$

with the projection $\mathfrak{X} \rightarrow X$. Thus the proposition follows by our definition of the μ_ℓ -action on $\mathcal{L}_m(X)$.

PROPOSITION 3.4. *Let f be a regular function on \mathfrak{X} obtained by pulling back a regular function on X along the projection $\mathfrak{X} \rightarrow X$. Then f has constant order on any μ_ℓ -orbit of $\mathcal{G}(\mathfrak{X})$. Furthermore, f has constant angular component on any μ_ℓ -orbit of $\mathcal{G}(\mathfrak{X})$ on which f has order ℓ .*

Proof. Let g be a regular function on X that pulls back to f . Then the isomorphism $\mathcal{G}(\mathfrak{X}) \rightarrow \mathcal{L}(X)$ identifies the order function of g with the order function of f . By Proposition 3.3, the μ_ℓ -orbits are identified as well. Therefore the result follows from the analogous result on $\mathcal{L}(X)$, i.e., Proposition 3.1.

Let f be a regular function on X , let $Z_f^\mu(T) \in \mathcal{M}_X^\mu[[T]]$ denote the Denef–Loeser motivic zeta function of f , and let $Z_f^{\text{naive}}(T) \in \mathcal{M}_X[[T]]$ denote the motivic Igusa zeta function of f . By slight abuse of notation, we will also let f denote the regular function on \mathfrak{X} obtained by pulling back f along the projection $\mathfrak{X} \rightarrow X$.

PROPOSITION 3.5. *Let $A_{\ell,1} \subset \mathcal{G}(\mathfrak{X})$ be the subset of arcs where f has order ℓ and angular component 1. Then $A_{\ell,1}$ is a μ_ℓ -invariant cylinder, and the coefficient of T^ℓ in $Z_f^\mu(T)$ is equal to the image of $\mu_{\mathfrak{X}}^{\mu_\ell}(A_{\ell,1})$ in \mathcal{M}_X^μ .*

Proof. Let $B_{\ell,1} \subset \mathcal{L}(X)$ be the subset of arcs where f has order ℓ and angular component 1, and let $Y_{\ell,1}$ be the closed subscheme of $\mathcal{L}_\ell(X)$ consisting of jets where f is equal to π^ℓ . Then $\theta_\ell(B_{\ell,1}) = Y_{\ell,1}$. By Proposition 3.1, $B_{\ell,1}$ is a μ_ℓ -invariant subset of $\mathcal{L}(X)$, so because θ_ℓ is μ_ℓ -equivariant, we have that $Y_{\ell,1}$ is a μ_ℓ -invariant subset of $\mathcal{L}_\ell(X)$. Thus we may endow $Y_{\ell,1}$ with the μ_ℓ -action given by restriction of the μ_ℓ -action on $\mathcal{L}_\ell(X)$. By the definition of $Z_f^\mu(T)$ and the μ_ℓ -action on $Y_{\ell,1}$, the coefficient of T^ℓ in $Z_f^\mu(T)$ is equal to

$$[Y_{\ell,1}/X, \hat{\mu}]_{\mathbb{L}}^{-(\ell+1)\dim X} \in \mathcal{M}_X^\mu.$$

But by the μ_ℓ -equivariant isomorphisms $\mathcal{G}(\mathfrak{X}) \rightarrow \mathcal{L}(X)$ and $\mathcal{G}_\ell(\mathfrak{X}) \rightarrow \mathcal{L}_\ell(X)$, the fact that the image of $A_{\ell,1}$ under $\mathcal{G}(\mathfrak{X}) \rightarrow \mathcal{L}(X)$ is equal to $B_{\ell,1}$, and the fact that $\theta_\ell^{-1}(Y_{\ell,1}) = B_{\ell,1}$, we have that $A_{\ell,1}$ is a μ_ℓ -invariant cylinder and

$$\mu_{\mathfrak{X}}^{\mu_\ell}(A_{\ell,1}) = [Y_{\ell,1}/X, \mu_\ell]_{\mathbb{L}}^{-(\ell+1)\dim X} \in \mathcal{M}_X^{\mu_\ell},$$

and we are done.

PROPOSITION 3.6. *Let $A_\ell \subset \mathcal{G}(\mathfrak{X})$ be the subset of arcs where f has order order ℓ . Then A_ℓ is a cylinder and the coefficient of T^ℓ in $Z_f^{\text{naive}}(T)$ is equal to $\mu_{\mathfrak{X}}(A_\ell)$.*

Proof. This proposition follows from the definition of $Z_f^{\text{naive}}(T)$ and the fact that the isomorphism $\mathcal{G}(\mathfrak{X}) \rightarrow \mathcal{L}(X)$ is cylinder and volume preserving.

4. Actions of the roots of unity on an algebraic torus

Let T be an algebraic torus over k with character lattice M and co-character lattice $N = \text{Hom}(M, \mathbb{Z})$. For each $u \in M$, let $\chi^u \in k[M]$ denote the corresponding character on T . In this section, we establish some notation and facts regarding certain actions, by the roots of unity, on the closed subschemes of T . In particular, we prove Proposition 4.6, which will allow us

to show that the equivariant classes appearing in the statement of Theorem 1.3 involve the desired μ_ℓ -action.

Definition 4.1. Let $\ell \in \mathbb{Z}_{>0}$. Let $w \in N$, and $\mathbb{G}_{m,k} \rightarrow T$ be the corresponding co-character. Then we define the (μ_ℓ, w) -action to be the μ_ℓ -action on T induced by the group homomorphism $\mu_\ell \hookrightarrow \mathbb{G}_{m,k} \rightarrow T$.

For each closed subscheme U of T that is invariant under the (μ_ℓ, w) -action, we will let U_ℓ^w denote the scheme U endowed with the μ_ℓ -action given by restriction of the (μ_ℓ, w) -action.

Remark 4.2. Under the (μ_ℓ, w) -action, each $\xi \in \mu_\ell$ acts on T with pullback

$$\chi^u \mapsto \xi^{\langle u, w \rangle} \chi^u.$$

PROPOSITION 4.3. *Let $\ell \in \mathbb{Z}_{>0}$, let $w \in N$, and let U be a closed subscheme of T . Then the initial degeneration $\text{in}_w U$ is a closed subscheme of T that is invariant under the (μ_ℓ, w) -action.*

Proof. Let $\xi \in \mu_\ell$, and let $\xi_T : T \rightarrow T$ be its action on T . It will be sufficient to show that for all $f \in k[M]$, the pullback $\xi_T^*(\text{in}_w f)$ is contained in the ideal of $k[M]$ generated by $\text{in}_w f$.

By definition,

$$\text{supp}(\text{in}_w f) = \{u \in \text{supp}(f) \mid \langle u, w \rangle = \text{trop}(f)(w)\},$$

so by Remark 4.2

$$\xi_T^*(\text{in}_w f) = \xi^{\text{trop}(f)(w)} \text{in}_w f,$$

and we are done.

PROPOSITION 4.4. *Let $w \in N$, let $u \in M$ such that $\langle u, w \rangle > 0$, and let $V(\chi^u - 1)$ be the closed subscheme of T defined by $\chi^u - 1 \in k[M]$. Then $V(\chi^u - 1)$ is invariant under the $(\mu_{\langle u, w \rangle}, w)$ -action.*

Proof. Let $\xi \in \mu_{\langle u, w \rangle}$, and let $\xi_T : T \rightarrow T$ be its action on T . Then by Remark 4.2

$$\xi_T^*(\chi^u - 1) = \xi^{\langle u, w \rangle} (\chi^u - 1) = \chi^u - 1,$$

and we are done.

PROPOSITION 4.5. *Let U be a closed subscheme of T , let $\ell \in \mathbb{Z}_{>0}$, let $w \in N$, and let $u \in M$ be such that $\langle u, w \rangle > 0$.*

Then U is invariant under the $(\mu_{\langle u, w \rangle}, w)$ -action if and only if U is invariant under the $(\mu_{\langle u, \ell w \rangle}, \ell w)$ -action.

Furthermore, if U is invariant under the $(\mu_{\langle u, w \rangle}, w)$ -action, then

$$\left[U_{\langle u, w \rangle}^w, \hat{\mu} \right] = \left[U_{\langle u, \ell w \rangle}^{\ell w}, \hat{\mu} \right] \in K_0^{\hat{\mu}}(\mathbf{Var}_k).$$

Proof. Under the $(\mu_{\langle u, w \rangle}, w)$ -action, each $\xi \in \mu_{\langle u, w \rangle}$ acts on T with pullback

$$\chi^{u'} \mapsto \xi^{\langle u', w \rangle} \chi^{u'}.$$

The homomorphism $\mu_{\langle u, \ell w \rangle} \rightarrow \mu_{\langle u, w \rangle} : \xi \mapsto \xi^\ell$ and the $(\mu_{\langle u, w \rangle}, w)$ -action induce a $\mu_{\langle u, \ell w \rangle}$ -action on T such that each $\xi \in \mu_{\langle u, \ell w \rangle}$ acts on T with pullback

$$\chi^{u'} \mapsto \left(\xi^\ell\right)^{\langle u', w \rangle} \chi^{u'} = \xi^{\langle u', \ell w \rangle} \chi^{u'}.$$

We see that this action is equal to the $(\mu_{\langle u, \ell w \rangle}, \ell w)$ -action. Then the surjectivity of $\mu_{\langle u, \ell w \rangle} \rightarrow \mu_{\langle u, w \rangle}$ implies that U is invariant under the $(\mu_{\langle u, w \rangle}, w)$ -action if and only if it is invariant under the $(\mu_{\langle u, \ell w \rangle}, \ell w)$ -action. The remainder of the proposition follows from the definition of the map $K_0^{\mu_{\langle u, w \rangle}}(\mathbf{Var}_k) \rightarrow K_0^{\mu_{\langle u, \ell w \rangle}}(\mathbf{Var}_k)$.

We will devote the remainder of this section to proving the following proposition.

PROPOSITION 4.6. *Let U be a closed subscheme of T , let $u \in M$, let $V(\chi^u - 1)$ be the closed subscheme of T defined by $\chi^u - 1 \in k[M]$, let $w \in u^\perp \cap N$, and let $v \in N$ be such that $\ell = \langle u, v \rangle > 0$ and such that $\text{in}_w U$ is invariant under the (μ_ℓ, v) -action.*

Then $\text{in}_w U$ is invariant under the $(\mu_\ell, v - w)$ -action, and

$$\left[(V(\chi^u - 1) \cap \text{in}_w U)_\ell^v, \mu_\ell \right] = \left[(V(\chi^u - 1) \cap \text{in}_w U)_\ell^{v-w}, \mu_\ell \right] \in K_0^{\mu_\ell}(\mathbf{Var}_k).$$

Remark 4.7. In the statement of Proposition 4.6, because $\ell = \langle u, v \rangle = \langle u, v - w \rangle$, Proposition 4.4 implies that $V(\chi^u - 1)$ is invariant under the (μ_ℓ, v) -action and the $(\mu_\ell, v - w)$ -action, so the classes in the statement are well defined.

4.1. Proof of Proposition 4.6

Let U be a closed subscheme of T , let $u \in M$, let $V(\chi^u - 1)$ be the closed subscheme of T defined by $\chi^u - 1 \in k[M]$, let $w \in u^\perp \cap N$, and let $v \in N$ be such that $\ell = \langle u, v \rangle > 0$ and such that $\text{in}_w U$ is invariant under the (μ_ℓ, v) -action. Proposition 4.6 is clear when $w = 0$, so we assume that $w \neq 0$.

Let $\mathcal{O}_w = \text{Spec}(k[w^\perp \cap M])$, and let $T \rightarrow \mathcal{O}_w$ be the algebraic group homomorphism induced by the inclusion $k[w^\perp \cap M] \rightarrow k[M]$.

LEMMA 4.8. *Let $f \in k[M]$. Then there exists $u' \in M$ such that $\text{in}_w(\chi^{u'} f) \in k[w^\perp \cap M]$.*

Proof. By definition,

$$\text{supp}(\text{in}_w f) = \{u' \in \text{supp}(f) \mid \langle u', w \rangle = \text{trop}(f)(w)\}.$$

If $f = 0$, the statement is obvious. Thus we may assume that there exists $u' \in M$ such that $-u' \in \text{supp}(\text{in}_w f)$. Then we have that

$$\text{in}_w(\chi^{u'} f) = \chi^{u'} \text{in}_w f \in k[w^\perp \cap M].$$

PROPOSITION 4.9. *There exist closed subschemes Y and Z of \mathcal{O}_w such that $\text{in}_w U$ is equal to the pre-image of Y under the morphism $T \rightarrow \mathcal{O}_w$ and $V(\chi^u - 1) \cap \text{in}_w U$ is equal to the pre-image of Z under the morphism $T \rightarrow \mathcal{O}_w$.*

Proof. Let $f_1, \dots, f_m \in k[M]$ be such that $\text{in}_w f_1, \dots, \text{in}_w f_m \in k[M]$ generate the ideal defining $\text{in}_w U$ in T . By Lemma 4.8, we can assume that $\text{in}_w f_1, \dots, \text{in}_w f_m \in k[w^\perp \cap M]$. Because $w \in u^\perp$, we have that $\chi^u \in k[w^\perp \cap M]$.

Thus we may let Y be the closed subscheme of \mathcal{O}_w defined by the ideal generated by $\text{in}_w f_1, \dots, \text{in}_w f_m \in k[w^\perp \cap M]$, and we may let Z be the closed subscheme of \mathcal{O}_w defined by the ideal generated by $\text{in}_w f_1, \dots, \text{in}_w f_m, \chi^u - 1 \in k[w^\perp \cap M]$, and we are done.

LEMMA 4.10. *The composition of the co-character $\mathbb{G}_{m,k} \rightarrow T$ corresponding to v with the morphism $T \rightarrow \mathcal{O}_w$ is equal to the composition of the co-character $\mathbb{G}_{m,k} \rightarrow T$ corresponding to $v - w$ with the morphism $T \rightarrow \mathcal{O}_w$*

Proof. The composition of the co-character $\mathbb{G}_{m,k} \rightarrow T$ corresponding to v with the morphism $T \rightarrow \mathcal{O}_w$ corresponds to the map of lattices $w^\perp \cap M \hookrightarrow M \xrightarrow{\langle \cdot, v \rangle} \mathbb{Z}$, and the composition of the co-character $\mathbb{G}_{m,k} \rightarrow T$ corresponding to $v - w$ with the morphism $T \rightarrow \mathcal{O}_w$ corresponds to the map of lattices $w^\perp \cap M \hookrightarrow M \xrightarrow{\langle \cdot, v-w \rangle} \mathbb{Z}$. These are clearly the same lattice maps, so we are done.

Let $T_w = \text{Spec}(k[(\mathbb{R}w \cap N)^\vee])$. Any splitting of $0 \rightarrow \mathbb{R}w \cap N \rightarrow N \rightarrow N/(\mathbb{R}w \cap N) \rightarrow 0$ induces an isomorphism of algebraic groups $T \cong T_w \times_k \mathcal{O}_w$ such that $T \rightarrow \mathcal{O}_w$ corresponds to the projection $T_w \times_k \mathcal{O}_w \rightarrow \mathcal{O}_w$.

Let $\phi_1 : \mu_\ell \rightarrow T$ (resp. $\phi_2 : \mu_\ell \rightarrow T$) be the composition of $\mu_\ell \hookrightarrow \mathbb{G}_{m,k}$ with the co-character $\mathbb{G}_{m,k} \rightarrow T$ corresponding to v (resp. $v - w$).

Let $\varphi_1 : \mu_\ell \rightarrow \mathcal{O}_w$ (resp. $\varphi_2 : \mu_\ell \rightarrow \mathcal{O}_w$) be the composition of ϕ_1 (resp. ϕ_2) with $T \rightarrow \mathcal{O}_w$.

LEMMA 4.11 *We have that $\varphi_1 = \varphi_2$.*

Proof. This follows directly from Lemma 4.10.

Let $\psi_1 : \mu_\ell \rightarrow T_w$ (resp. $\psi_2 : \mu_\ell \rightarrow T_w$) be the composition of ϕ_1 (resp. ϕ_2) with the projection $T \cong T_w \times_k \mathcal{O}_w \rightarrow T_w$.

Remark 4.12. We see that under the identification $T \cong T_w \times_k \mathcal{O}_w$, the (μ_ℓ, v) -action (resp. $(\mu_\ell, v - w)$ -action) is the diagonal action defined by the action on \mathcal{O}_w induced by φ_1 (resp. φ_2) and the action on T_w induced by ψ_1 (resp. ψ_2).

We now prove the first part of Proposition 4.6.

PROPOSITION 4.13. *We have that $\text{in}_w U$ is invariant under the $(\mu_\ell, v - w)$ -action.*

Proof. By Proposition 4.9, there exists a closed subscheme Y of \mathcal{O}_w such that $\text{in}_w U$ is equal to the pre-image of Y under the morphism $T \rightarrow \mathcal{O}_w$. Then under the identification $T \cong T_w \times_k \mathcal{O}_w$, we have that

$$\text{in}_w U = T_w \times_k Y.$$

Because $\text{in}_w U$ is invariant under the (μ_ℓ, v) -action, Remark 4.12 implies that Y is invariant under the μ_ℓ -action on \mathcal{O}_w induced by φ_1 . By Lemma 4.11, Y is invariant under the μ_ℓ -action on \mathcal{O}_w induced by φ_2 , and by Remark 4.12, this implies that $\text{in}_w U$ is invariant under the $(\mu_\ell, v - w)$ -action.

Before we complete the proof of Proposition 4.6, we make the following observation, which follows from [KU22, lemma 7.1] and the fact that $\dim T_w = 1$.

Remark 4.14. The class in $K_0^{\mu_\ell}(\mathbf{Var}_k)$ of T_w with the μ_ℓ -action induced by ψ_1 (resp. ψ_2) is equal to $\mathbb{L} - 1$.

We now complete the proof of Proposition 4.6.

PROPOSITION 4.15. *We have that*

$$[(V(\chi^u - 1) \cap \text{in}_w U)_\ell^v, \mu_\ell] = [(V(\chi^u - 1) \cap \text{in}_w U)_\ell^{v-w}, \mu_\ell] \in K_0^{\mu_\ell}(\mathbf{Var}_k).$$

Proof. By Proposition 4.9, there exists a closed subscheme Z of \mathcal{O}_w such that $V(\chi^u - 1) \cap \text{in}_w U$ is equal to the pre-image of Z under the morphism $T \rightarrow \mathcal{O}_w$. Then under the identification $T \cong T_w \times_k \mathcal{O}_w$, we have that

$$V(\chi^u - 1) \cap \text{in}_w U = T_w \times_k Z.$$

Because $V(\chi^u - 1) \cap \text{in}_w U$ is invariant under the (μ_ℓ, ν) -action, Remark 4.12 implies that Z is invariant under the μ_ℓ -action on \mathcal{O}_w induced by φ_1 .

Now endow Z with the μ_ℓ -action given by restriction of the μ_ℓ -action on \mathcal{O}_w induced by φ_1 , which by Lemma 4.11 is the same as the μ_ℓ -action given by restriction of the μ_ℓ -action on \mathcal{O}_w induced by φ_2 . Then by Remarks 4.12 and 4.14,

$$\begin{aligned} [(V(\chi^u - 1) \cap \text{in}_w U)_\ell^v, \mu_\ell] &= (\mathbb{L} - 1)[Z, \mu_\ell] \\ &= [(V(\chi^u - 1) \cap \text{in}_w U)_\ell^{v-w}, \mu_\ell]. \end{aligned}$$

5. Motivic zeta functions and smooth initial degenerations

The purpose of this section is to prove Corollary 5.4, which provides formulas for the motivic zeta functions that will specialise, in the case where X is a linear space, to Theorem 1.3. Because the proof is long and technical, we provide here a brief outline of our approach.

We begin, in subsection 5.1, by showing that Corollary 5.4 follows from Proposition 5.1 and Theorem 5.2. In subsection 5.2, we identify certain motivic volumes, the computation of which will give Theorem 5.2. The remaining subsections are devoted to computing these volumes via the change of variables formula. Specifically, in subsection 5.3, we construct the maps to which we will apply the change of variables formula. The main result of subsection 5.4 is a technical computation of a jacobian ideal. Finally, in subsection 5.5, we assemble these results to prove Proposition 5.1 and Theorem 5.2.

Let $n \in \mathbb{Z}_{>0}$, let $\mathbb{A}_k^n = \text{Spec}(k[x_1, \dots, x_n])$, and let $\mathbb{G}_{m,k}^n = \text{Spec}\left(k\left[x_1^{\pm 1}, \dots, x_n^{\pm 1}\right]\right)$. Let X be a smooth pure dimension d closed subscheme of \mathbb{A}_k^n such that $U = X \cap \mathbb{G}_{m,k}^n$ is nonempty and such that for all $w \in \text{Trop}(U) \cap \mathbb{Z}_{\geq 0}^n$, the initial degeneration $\text{in}_w U$ is smooth and there exist $f_1, \dots, f_{n-d} \in k[x_1, \dots, x_n]$ that generate the ideal of X in \mathbb{A}_k^n such that $\text{in}_w f_1, \dots, \text{in}_w f_{n-d} \in k\left[x_1^{\pm 1}, \dots, x_n^{\pm 1}\right]$ generate the ideal of $\text{in}_w U$ in $\mathbb{G}_{m,k}^n$.

Let $u \in \mathbb{Z}_{>0}^n$, let $Z_{X,u}^\mu(T) \in \mathcal{M}_X^\mu[[T]]$ be the Denef–Loeser motivic zeta function of the restriction $(x_1, \dots, x_n)^u|_X$, and let $Z_{X,u}^{\text{naive}}(T) \in \mathcal{M}_X[[T]]$ be the motivic Igusa zeta function of $(x_1, \dots, x_n)^u|_X$.

PROPOSITION 5.1. *Let $w = (w_1, \dots, w_n) \in \text{Trop}(U) \cap \mathbb{Z}_{\geq 0}^n$, and let $\varphi : \mathbb{G}_{m,k}^n \rightarrow \mathbb{A}_k^n$ be the morphism whose pullback is given by $x_i \mapsto 0^{w_i} x_i$. Then the restriction of φ to $\text{in}_w U$ factors*

through X , and if $w \neq 0$, the induced map $(\text{in}_w U)_{u \cdot w}^w \rightarrow X$ is $\mu_{u \cdot w}$ -equivariant with respect to the trivial $\mu_{u \cdot w}$ -action on X .

Proposition 5.1 allows us to state the following theorem.

THEOREM 5.2. *Let V_u be the subscheme of $\mathbb{G}_{m,k}^n$ defined by $(x_1, \dots, x_n)^u - 1$. For any $w \in \text{Trop}(U) \cap \mathbb{Z}_{\geq 0}^n$, endow the initial degeneration $\text{in}_w U$ and the intersection $V_u \cap \text{in}_w U$ with the X -scheme structure given by Proposition 5.1.*

Then there exists a function $\text{ordjac} : \text{Trop}(U) \cap \mathbb{Z}_{\geq 0}^n \rightarrow \mathbb{Z}$ that satisfies the following.

- (a) *If $w = (w_1, \dots, w_n) \in \text{Trop}(U) \cap \mathbb{Z}_{\geq 0}^n$ and $f_1, \dots, f_{n-d} \in k[x_1, \dots, x_n]$ are a generating set for the ideal of X such that $\text{in}_w f_1, \dots, \text{in}_w f_{n-d} \in k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ generate the ideal of $\text{in}_w U$, then*

$$\text{ordjac}(w) = w_1 + \dots + w_n - (\text{trop}(f_1)(w) + \dots + \text{trop}(f_{n-d})(w)) \in \mathbb{Z}.$$

- (b) *We have that*

$$Z_{X,u}^{\hat{\mu}}(T) = \sum_{w \in \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\})} [(V_u \cap \text{in}_w U)_{u \cdot w}^w / X, \hat{\mu}] \mathbb{L}^{-d - \text{ordjac}(w)} T^{u \cdot w},$$

and

$$Z_{X,u}^{\text{naive}}(T) = \sum_{w \in \text{Trop}(U) \cap \mathbb{Z}_{\geq 0}^n} [\text{in}_w U / X] \mathbb{L}^{-d - \text{ordjac}(w)} T^{u \cdot w}.$$

Remark 5.3. The classes above are well defined by Propositions 4.3, 4.4 and 5.1.

Let $Z_{X,u,k}^{\hat{\mu}}(T) \in \mathcal{M}_k^{\hat{\mu}}[[T]]$ be the power series obtained by pushing forward each coefficient of $Z_{X,u}^{\hat{\mu}}(T)$ along the structure morphism of X , and if the origin of \mathbb{A}_k^n is contained in X , let $Z_{X,u,0}^{\hat{\mu}}(T) \in \mathcal{M}_k^{\hat{\mu}}[[T]]$ be the power series obtained by pulling back each coefficient of $Z_{X,u}^{\hat{\mu}}(T)$ along the inclusion of the origin into X .

COROLLARY 5.4. *Again let V_u be the subscheme of $\mathbb{G}_{m,k}^n$ defined by $(x_1, \dots, x_n)^u - 1$. Suppose there exists $v \in \mathbb{Z}^n$ such that $u \cdot v > 0$ and such that for all $w \in \mathbb{Z}^n$,*

$$\text{in}_w U = \text{in}_{w+v} U.$$

Then for all $w \in \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\})$, we have that $V_u \cap \text{in}_w U$ is invariant under the $(\mu_{u \cdot v}, v)$ -action, and there exists a function $\text{ordjac} : \text{Trop}(U) \cap \mathbb{Z}_{\geq 0}^n \rightarrow \mathbb{Z}$ that satisfies the following.

- (a) *If $w = (w_1, \dots, w_n) \in \text{Trop}(U) \cap \mathbb{Z}_{\geq 0}^n$ and $f_1, \dots, f_{n-d} \in k[x_1, \dots, x_n]$ are a generating set for the ideal of X such that $\text{in}_w f_1, \dots, \text{in}_w f_{n-d} \in k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ generate the ideal of $\text{in}_w U$, then*

$$\text{ordjac}(w) = w_1 + \dots + w_n - (\text{trop}(f_1)(w) + \dots + \text{trop}(f_{n-d})(w)) \in \mathbb{Z}.$$

(b) We have that

$$Z_{X,u,k}^{\hat{\mu}}(T) = \sum_{w \in \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\})} [(V_u \cap \text{in}_w U)_{u \cdot v}^v, \hat{\mu}] \mathbb{L}^{-d - \text{ordjac}(w)} T^{u \cdot w}.$$

(c) If the origin of \mathbb{A}_k^n is contained in X , then

$$Z_{X,u,0}^{\hat{\mu}}(T) = \sum_{w \in \text{Trop}(U) \cap \mathbb{Z}_{> 0}^n} [(V_u \cap \text{in}_w U)_{u \cdot v}^v, \hat{\mu}] \mathbb{L}^{-d - \text{ordjac}(w)} T^{u \cdot w}.$$

5.1. Proof of Corollary 5.4

The following proposition will be used for Corollary 5.4(c). Its purpose is to show that $Z_{X,u,0}^{\hat{\mu}}$ is computed by summing over lattice points in $\text{Trop}(U)$ with strictly positive entries.

PROPOSITION 5.5. Let $w \in \text{Trop}(U) \cap \mathbb{Z}_{> 0}^n$, suppose that the origin of \mathbb{A}_k^n is contained in X , and endow $\text{in}_w U$ with the X -scheme structure given by Proposition 5.1. Then:

- (a) if $w \in \mathbb{Z}_{> 0}^n$, the fiber of $\text{in}_w U$ over the origin of \mathbb{A}_k^n is equal to $\text{in}_w U$;
- (b) and if $w \notin \mathbb{Z}_{> 0}^n$, the fiber of $\text{in}_w U$ over the origin of \mathbb{A}_k^n is empty.

Proof. This is a direct consequence of the X -scheme structure of $\text{in}_w U$.

Now, using the notation in the theorem's statement, Theorem 5.2 implies

$$Z_{X,u,k}^{\hat{\mu}}(T) = \sum_{w \in \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\})} [(V_u \cap \text{in}_w U)_{u \cdot w}^w, \hat{\mu}] \mathbb{L}^{-d - \text{ordjac}(w)} T^{u \cdot w},$$

and if in addition, the origin of \mathbb{A}_k^n is contained in X , Proposition 5.5 and Theorem 5.2 imply

$$Z_{X,u,0}^{\hat{\mu}}(T) = \sum_{w \in \text{Trop}(U) \cap \mathbb{Z}_{> 0}^n} [(V_u \cap \text{in}_w U)_{u \cdot w}^w, \hat{\mu}] \mathbb{L}^{-d - \text{ordjac}(w)} T^{u \cdot w}.$$

Thus Corollary 5.4 follows from Theorem 5.2 and the following proposition.

PROPOSITION 5.6. Suppose there exists $v \in \mathbb{Z}^n$ such that $u \cdot v > 0$ and such that for all $w \in \mathbb{Z}^n$,

$$\text{in}_w U = \text{in}_{w+v} U.$$

Let $w \in \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\})$, and let V_u be the subscheme of $\mathbb{G}_{m,k}^n$ defined by $(x_1, \dots, x_n)^u - 1$. Then $V_u \cap \text{in}_w U$ is invariant under the $(\mu_{u \cdot v}, \nu)$ -action and

$$[(V_u \cap \text{in}_w U)_{u \cdot w}^w, \hat{\mu}] = [(V_u \cap \text{in}_w U)_{u \cdot v}^v, \hat{\mu}] \in K_0^{\hat{\mu}}(\mathbf{Var}_k).$$

Proof. Because $u \cdot w > 0$, there exist $\ell, \ell' \in \mathbb{Z}_{> 0}$ and $w' \in \mathbb{Z}^n$ such that $u \cdot w' = 0$ and

$$\ell w = \ell' v + w'.$$

By Proposition 4.5, $V_u \cap \text{in}_w U$ is invariant under the $(\mu_{u \cdot \ell w}, \ell w)$ -action and

$$[(V_u \cap \text{in}_w U)_{u \cdot w}^w, \hat{\mu}] = [(V_u \cap \text{in}_w U)_{u \cdot \ell w}^{\ell w}, \hat{\mu}] \in K_0^{\hat{\mu}}(\mathbf{Var}_k).$$

By the hypotheses on v , we have that

$$\text{in}_{w'} U = \text{in}_{\ell w} U,$$

so by Proposition 4.3, $\text{in}_{w'} U$ is invariant under the $(\mu_{u \cdot \ell w}, \ell w)$ -action. Then by Proposition 4.6, $\text{in}_{w'} U$ is invariant under the $(\mu_{u \cdot \ell' v}, \ell' v)$ -action, and noting that $u \cdot \ell w = u \cdot \ell' v$,

$$[(V_u \cap \text{in}_{w'} U)_{u \cdot \ell w}^{\ell w}, \mu_{u \cdot \ell w}] = [(V_u \cap \text{in}_{w'} U)_{u \cdot \ell' v}^{\ell' v}, \mu_{u \cdot \ell' v}] \in K_0^{\mu_{u \cdot \ell' v}}(\mathbf{Var}_k).$$

Again by Proposition 4.5, $V_u \cap \text{in}_{w'} U$ is invariant under the $(\mu_{u \cdot v}, v)$ -action and

$$[(V_u \cap \text{in}_{w'} U)_{u \cdot \ell' v}^{\ell' v}, \hat{\mu}] = [(V_u \cap \text{in}_{w'} U)_{u \cdot v}^v, \hat{\mu}] \in K_0^{\hat{\mu}}(\mathbf{Var}_k).$$

All together, noting that $\text{in}_w U = \text{in}_{\ell w} U = \text{in}_{w'} U$,

$$\begin{aligned} [(V_u \cap \text{in}_w U)_{u \cdot w}^w, \hat{\mu}] &= [(V_u \cap \text{in}_w U)_{u \cdot \ell w}^{\ell w}, \hat{\mu}] \\ &= [(V_u \cap \text{in}_{w'} U)_{u \cdot \ell w}^{\ell w}, \hat{\mu}] \\ &= [(V_u \cap \text{in}_{w'} U)_{u \cdot \ell' v}^{\ell' v}, \hat{\mu}] \\ &= [(V_u \cap \text{in}_{w'} U)_{u \cdot v}^v, \hat{\mu}] \\ &= [(V_u \cap \text{in}_w U)_{u \cdot v}^v, \hat{\mu}]. \end{aligned}$$

5.2. *Fibers of tropicalisation*

For the remainder of Section 5, fix $\ell \in \mathbb{Z}_{>0}$ and endow R with the μ_ℓ -action where each $\xi \in \mu_\ell$ acts on R by the π -adically continuous k -morphism $\pi \mapsto \xi^{-1}\pi$. Let $\mathbb{A}_R^n = \text{Spec}(R[x_1, \dots, x_n])$, let $\mathfrak{X} = X \times_k \text{Spec}(R) \subset \mathbb{A}_R^n$, and endow \mathbb{A}_R^n (resp. \mathfrak{X}) with the μ_ℓ -action induced by the μ_ℓ -action on R and the trivial μ_ℓ -action on \mathbb{A}_k^n (resp. X).

Let $A_\ell \subset \mathcal{G}(\mathfrak{X})$ be the subset of arcs where $(x_1, \dots, x_n)^\mu|_{\mathfrak{X}}$ has order ℓ , and let $A_{\ell,1} \subset \mathcal{G}(\mathfrak{X})$ be the subset of arcs where $(x_1, \dots, x_n)^\mu|_{\mathfrak{X}}$ has order ℓ and angular component 1.

Let $\text{trop} : \mathcal{G}(\mathfrak{X}) \rightarrow (\mathbb{Z}_{\geq 0} \cup \{\infty\})^n$ be the function $(\text{ord}_{x_1|_{\mathfrak{X}}}, \dots, \text{ord}_{x_n|_{\mathfrak{X}}})$. Any arc that tropicalises to a point in $\mathbb{Z}_{\geq 0}^n$ has generic point in $U \times_k \text{Spec}(R)$, so

$$\text{trop}(\mathcal{G}(\mathfrak{X})) \cap \mathbb{Z}_{\geq 0}^n \subset \text{Trop}(U).$$

Also because $u \in \mathbb{Z}_{>0}^n$ and $\ell \neq 0$,

$$\text{trop}(A_\ell) \subset \text{trop}(\mathcal{G}(\mathfrak{X})) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\}) \subset \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\}).$$

Thus

$$A_\ell = \bigcup_{\substack{w \in \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\}) \\ u \cdot w = \ell}} \text{trop}^{-1}(w).$$

This union is disjoint, and because $u \in \mathbb{Z}_{>0}^n$, it is also finite. By Proposition 3.4, for each $w \in \mathbb{Z}_{>0}^n$, we have that the fiber $\text{trop}^{-1}(w)$ and the intersection $\text{trop}^{-1}(w) \cap A_{\ell,1}$ are μ_ℓ -invariant cylinders in $\mathcal{G}(\mathfrak{X})$. We have thus proved the following.

PROPOSITION 5.7. *We have that*

$$\mu_{\mathfrak{X}}^{\mu_\ell}(A_{\ell,1}) = \sum_{\substack{w \in \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\}) \\ u \cdot w = \ell}} \mu_{\mathfrak{X}}^{\mu_\ell}(\text{trop}^{-1}(w) \cap A_{\ell,1}),$$

and

$$\mu_{\mathfrak{X}}(A_\ell) = \sum_{\substack{w \in \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\}) \\ u \cdot w = \ell}} \mu_{\mathfrak{X}}(\text{trop}^{-1}(w)).$$

5.3. Morphisms for computing volumes

Throughout subsection 5.3, we will fix some $w = (w_1, \dots, w_n) \in \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\})$ such that $u \cdot w = \ell$. We will construct a smooth, pure relative dimension d , finite type, separated R -scheme \mathfrak{X}^w with good μ_ℓ -action making the structure morphism equivariant, and we will construct a μ_ℓ -equivariant morphism $\psi_w : \mathfrak{X}^w \rightarrow \mathfrak{X}$ that will eventually be used to compute the motivic volumes of $\text{trop}^{-1}(w) \cap A_{\ell,1}$ and $\text{trop}^{-1}(w)$.

Let $\mathbb{G}_{m,R}^n = \text{Spec}(R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]) = \mathbb{G}_{m,k}^n \times_k \text{Spec}(R)$, and endow it with the μ_ℓ -action induced by the μ_ℓ -action on $\text{Spec}(R)$ and the (μ_ℓ, w) -action on $\mathbb{G}_{m,k}^n$. Let $\varphi_w : \mathbb{G}_{m,R}^n \rightarrow \mathbb{A}_R^n$ be the R -scheme morphism corresponding to the R -algebra morphism

$$\varphi_w^* : \text{Spec}(R[x_1, \dots, x_n]) \rightarrow \text{Spec}(R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]) : x_i \mapsto \pi^{w_i} x_i.$$

PROPOSITION 5.8. *The morphism $\varphi_w : \mathbb{G}_{m,R}^n \rightarrow \mathbb{A}_R^n$ is μ_ℓ -equivariant.*

Proof. Let $\xi \in \mu_\ell$, and let $\xi_1 : R[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \rightarrow R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $\xi_2 : R[x_1, \dots, x_n] \rightarrow R[x_1, \dots, x_n]$ be its actions. We will show that

$$\xi_1 \circ \varphi_w^* = \varphi_w^* \circ \xi_2.$$

Because the structure morphisms of $\mathbb{G}_{m,R}^n$ and \mathbb{A}_R^n are μ_ℓ -equivariant, it is sufficient to show that if $i \in \{1, \dots, n\}$, then

$$\xi_1(\varphi_w^*(x_i)) = \varphi_w^*(\xi_2(x_i)),$$

which holds because

$$\begin{aligned} \xi_1(\varphi_w^*(x_i)) &= \xi_1(\pi^{w_i} x_i) \\ &= (\xi^{-1} \pi)^{w_i} \xi^{w_i} x_i \\ &= \pi^{w_i} x_i \\ &= \varphi_w^*(x_i) \\ &= \varphi_w^*(\xi_2(x_i)). \end{aligned}$$

Now let \mathfrak{X}_η be the generic fiber of \mathfrak{X} , let $\varphi_{w,\eta} : \mathbb{G}_{m,K}^n \rightarrow \mathbb{A}_K^n$ be the base change of φ_w to the fraction field K of R , let $\mathfrak{X}_\eta^w \subset \mathbb{G}_{m,K}^n$ be the pre-image of \mathfrak{X}_η under $\varphi_{w,\eta}$, and let $\mathfrak{X}^w \subset \mathbb{G}_{m,R}^n$ be the unique closed subscheme of $\mathbb{G}_{m,R}^n$ that is flat over R and has generic fiber \mathfrak{X}_η^w , see for example [Gub13, section 4]. By construction, the generic fiber of \mathfrak{X}^w is isomorphic to $U \times_k \text{Spec}(K)$, and its special fiber is equal to $\text{in}_w U \subset \mathbb{G}_{m,k}^n$, which is smooth by the hypotheses on X . Thus \mathfrak{X}^w is smooth and pure relative dimension d over R . Note that by uniqueness, \mathfrak{X}^w is equal to the closed subscheme of $\varphi_w^{-1}(\mathfrak{X})$ defined by its R -torsion ideal. Thus we have a morphism $\psi_w : \mathfrak{X}^w \rightarrow \mathfrak{X}$ induced from φ_w by restriction.

Remark 5.9. Note that if $\psi_{w,\eta} : \mathfrak{X}_\eta^w \rightarrow \mathfrak{X}_\eta$ is the base change of ψ_w to K , we have that $\psi_{w,\eta}$ is isomorphic to the open immersion $U \times_k \text{Spec}(K) \rightarrow X \times_k \text{Spec}(K)$. In particular, ψ_w induces an open immersion on generic fibers.

To obtain a generating set for the ideal defining \mathfrak{X}^w in $\mathbb{G}_{m,R}^n$, we first need to prove the following lemma.

LEMMA 5.10. *Let \mathfrak{Y} be a finite type R -scheme, and let \mathfrak{Y}^b be the closed subscheme of \mathfrak{Y} defined by its R -torsion ideal. If as closed subschemes of \mathfrak{Y} , the special fiber of \mathfrak{Y}^b is equal to the special fiber of \mathfrak{Y} , then \mathfrak{Y} is a flat R -scheme.*

Proof. We may assume $\mathfrak{Y} = \text{Spec}(\mathfrak{A})$ for some finite type R -algebra \mathfrak{A} . Let $I \subset \mathfrak{A}$ be the π -torsion ideal of \mathfrak{A} . Because I is finitely generated, there exists $m \in \mathbb{Z}_{\geq 0}$ such that $\pi^m I = 0$. By the hypotheses,

$$I \subset \pi \mathfrak{A}.$$

Let $f \in I$. Then there exists $g \in \mathfrak{A}$ such that $f = \pi g$. But $\pi g \in I$ implies that $g \in I$. Thus

$$I = \pi I = \pi^m I = 0.$$

Therefore \mathfrak{A} is π -torsion free, so it is flat over R .

We can now prove the following two propositions.

PROPOSITION 5.11. *Let $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ be a generating set for the ideal defining X in \mathbb{A}_k^n such that $\text{in}_w f_1, \dots, \text{in}_w f_m \in k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ form a generating set for the ideal of $\text{in}_w U$ in $\mathbb{G}_{m,k}^n$. Then*

$$\pi^{-\text{trop}(f_1)(w)} \varphi_w^*(f_1), \dots, \pi^{-\text{trop}(f_m)(w)} \varphi_w^*(f_m) \in R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

form a generating set for the ideal defining \mathfrak{X}^w in $\mathbb{G}_{m,R}^n$.

Proof. Let \mathfrak{Y} be the closed subscheme of $\mathbb{G}_{m,R}^n$ defined by the ideal generated by

$$\pi^{-\text{trop}(f_1)(w)} \varphi_w^*(f_1), \dots, \pi^{-\text{trop}(f_m)(w)} \varphi_w^*(f_m).$$

Then by construction, the generic fiber of \mathfrak{Y} is equal to \mathfrak{X}_η^w , and \mathfrak{X}^w is equal to the closed subscheme of \mathfrak{Y} defined by its R -torsion ideal. The special fiber of \mathfrak{Y} is the closed subscheme of $\mathbb{G}_{m,k}^n$ defined by $\text{in}_w f_1, \dots, \text{in}_w f_m$ and thus is equal to $\text{in}_w U$, which is also the special fiber of \mathfrak{X}^w . Therefore by Lemma 5.10, \mathfrak{Y} is flat over R , so \mathfrak{X}^w is equal to \mathfrak{Y} .

PROPOSITION 5.12. *The closed subscheme $\mathfrak{X}^w \subset \mathbb{G}_{m,R}^n$ is μ_ℓ -invariant.*

Proof. By the hypotheses on X , we know there exist $f_1, \dots, f_{n-d} \in k[x_1, \dots, x_n]$ that generate the ideal of X such that $\text{in}_w f_1, \dots, \text{in}_w f_{n-d}$ generate the ideal of $\text{in}_w U$, so by Proposition 5.11,

$$\pi^{-\text{trop}(f_1)(w)} \varphi_w^*(f_1), \dots, \pi^{-\text{trop}(f_{n-d})(w)} \varphi_w^*(f_{n-d}) \in R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

generate the ideal defining \mathfrak{X}^w in $\mathbb{G}_{m,R}^n$.

Thus it will be sufficient to show that if $f \in k[x_1, \dots, x_n]$, $\xi \in \mu_\ell$, and $\xi_1 : R[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \rightarrow R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is its action, then $\xi_1(\pi^{-\text{trop}(f)(w)} \varphi_w^*(f))$ is in the ideal of $R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ generated by $\pi^{-\text{trop}(f)(w)} \varphi_w^*(f)$. Write

$$f = \sum_{u' \in \mathbb{Z}_{\geq 0}^n} a_{u'}(x_1, \dots, x_n)^{u'},$$

where each $a_{u'} \in k$. Then

$$\pi^{-\text{trop}(f)(w)} \varphi_w^*(f) = \sum_{u' \in \mathbb{Z}_{\geq 0}^n} \pi^{u' \cdot w - \text{trop}(f)(w)} a_{u'}(x_1, \dots, x_n)^{u'},$$

so

$$\begin{aligned} \xi_1(\pi^{-\text{trop}(f)(w)} \varphi_w^*(f)) &= \sum_{u' \in \mathbb{Z}_{\geq 0}^n} (\xi^{-1} \pi)^{u' \cdot w - \text{trop}(f)(w)} a_{u'} \xi^{u' \cdot w} (x_1, \dots, x_n)^{u'} \\ &= \xi^{\text{trop}(f)(w)} \sum_{u' \in \mathbb{Z}_{\geq 0}^n} \pi^{u' \cdot w - \text{trop}(f)(w)} a_{u'}(x_1, \dots, x_n)^{u'} \\ &= \xi^{\text{trop}(f)(w)} \pi^{-\text{trop}(f)(w)} \varphi_w^*(f). \end{aligned}$$

We now endow \mathfrak{X}^w with the restriction of the μ_ℓ -action on $\mathbb{G}_{m,R}^n$. Because \mathfrak{X}^w is affine, this μ_ℓ -action is good, and by construction, this μ_ℓ -action makes the structure morphism equivariant. By Proposition 5.8, we have that the morphism $\psi_w : \mathfrak{X}^w \rightarrow \mathfrak{X}$ is μ_ℓ -equivariant.

Remark 5.13. By construction, the special fiber of \mathfrak{X}^w with its induced μ_ℓ -action is equal to $(\text{in}_w U)_\ell^w$.

5.4. Preparing for the change of variables formula

We establish some results which will be used in the application of Theorem 2.8.

For the remainder of Section 5, let V_u be the subscheme of $\mathbb{G}_{m,k}^n$ defined by $(x_1, \dots, x_n)^u - 1$, and if $w \in \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\})$ is such that $u \cdot w = \ell$, let \mathfrak{X}^w and $\psi_w : \mathfrak{X}^w \rightarrow \mathfrak{X}$ be as constructed in subsection 5.3.

PROPOSITION 5.14. *Let $w \in \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\})$ be such that $u \cdot w = \ell$. Noting that $V_u \cap \text{in}_w U \subset \text{in}_w U = \mathfrak{X}_0^w$, the subset $\theta_0^{-1}(V_u \cap \text{in}_w U) \subset \mathcal{G}(\mathfrak{X}^w)$ is a μ_ℓ -invariant cylinder, and*

$$\mu_{\mathfrak{X}^w}^{\mu_\ell} \left(\theta_0^{-1}(V_u \cap \text{in}_w U) \right) = [(V_u \cap \text{in}_w U)_\ell^w / \mathfrak{X}_0^w, \mu_\ell] \mathbb{L}^{-d} \in \mathcal{M}_{\mathfrak{X}_0^w}^{\mu_\ell}.$$

Proof. By Proposition 4.4 and Remark 5.13, we have that $V_u \cap \text{in}_w U$ is a μ_ℓ -invariant subscheme of \mathfrak{X}_0^w , and with the restriction of this μ_ℓ -action, it is equal to $(V_u \cap \text{in}_w U)_\ell^w$. The proposition then follows from the fact that the truncation morphism $\theta_0 : \mathcal{G}(\mathfrak{X}^w) \rightarrow \mathcal{G}_0(\mathfrak{X}^w) = \mathfrak{X}_0^w$ is μ_ℓ -equivariant and the definition of the μ_ℓ -equivariant motivic measure.

LEMMA 5.15. *Let $w \in \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\})$ be such that $u \cdot w = \ell$, let $\varphi_w : \mathbb{G}_{m,R}^n \rightarrow \mathbb{A}_R^n$ be as in subsection 5.3, and let k' be an extension of k . Then φ_w induces a bijection*

$$\mathcal{G}(\mathbb{G}_{m,R}^n)(k') \rightarrow \{x \in \mathcal{G}(\mathbb{A}_R^n)(k') \mid w = (\text{ord}_{x_1}(x), \dots, \text{ord}_{x_n}(x))\}.$$

Proof. Let $R' = k'[\![\pi]\!]$, and let K' be its field of fractions. Because φ_w induces an open immersion on generic fibers, it induces an injection $\mathbb{G}_{m,R}^n(K') \rightarrow \mathbb{A}_R^n(K')$. Because $\mathbb{G}_{m,R}^n$ is separated, this implies that φ_w induces an injection $\mathcal{G}(\mathbb{G}_{m,R}^n)(k') \rightarrow \mathcal{G}(\mathbb{A}_R^n)(k')$. We thus only need to show that the image of this injection is $\{x \in \mathcal{G}(\mathbb{A}_R^n)(k') \mid w = (\text{ord}_{x_1}(x), \dots, \text{ord}_{x_n}(x))\}$.

Let $y : \text{Spec}(R') \rightarrow \mathbb{G}_{m,R}^n$. Then for each $i \in \{1, \dots, n\}$, we have that $x_i(y)$ is a unit in R' , so by construction,

$$\varphi_w(y) \in \{x \in \mathcal{G}(\mathbb{A}_R^n)(k') \mid w = (\text{ord}_{x_1}(x), \dots, \text{ord}_{x_n}(x))\}.$$

Write $w = (w_1, \dots, w_n)$, and let $x : \text{Spec}(R') \rightarrow \mathbb{A}_R^n$ be such that $\text{ord}_{x_i}(x) = w_i$ for each $i \in \{1, \dots, n\}$. Then for each $i \in \{1, \dots, n\}$, we have that $\pi^{-w_i} x_i(x)$ is a unit in R' , so we may set $y : \text{Spec}(R') \rightarrow \mathbb{G}_{m,R}^n$ to be the morphism whose pullback is given by $x_i \mapsto \pi^{-w_i} x_i(x) \in R$. By construction $\varphi_w(y) = x$, and we are done.

PROPOSITION 5.16. *Let $w \in \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\})$ be such that $u \cdot w = \ell$. Then $\psi_w : \mathfrak{X}^w \rightarrow \mathfrak{X}$ induces bijections $\mathcal{G}(\mathfrak{X}^w)(k') \rightarrow \text{trop}^{-1}(w)(k')$ and $\theta_0^{-1}(V_u \cap \text{in}_w U)(k') \rightarrow (\text{trop}^{-1}(w) \cap A_{\ell,1})(k')$ for all extensions k' of k .*

Proof. Fix an extension k' of k . Because ψ_w induces an open immersion on generic fibers and because \mathfrak{X}^w is separated, we have that φ_w induces an injection from $\mathcal{G}(\mathfrak{X}^w)(k')$ to $\mathcal{G}(\mathfrak{X})(k')$. Thus we need to show that the image of $\mathcal{G}(\mathfrak{X}^w)(k')$ is $\text{trop}^{-1}(w)(k')$ and that the image of $\theta_0^{-1}(V_u \cap \text{in}_w U)(k')$ is $(\text{trop}^{-1}(w) \cap A_{\ell,1})(k')$.

Let $y \in \mathcal{G}(\mathfrak{X}^w)(k') \subset \mathcal{G}(\mathbb{G}_{m,R}^n)(k')$. By Lemma 5.15, $\psi_w(y) \in \text{trop}^{-1}(w)(k')$. Let $x \in \text{trop}^{-1}(w)(k') \subset \{x' \in \mathcal{G}(\mathbb{A}_R^n)(k') \mid w = (\text{ord}_{x_1}(x'), \dots, \text{ord}_{x_n}(x'))\}$. By Lemma 5.15, x is in the image of φ_w , where φ_w is as in Subsection 5.3. Because \mathfrak{X}^w is the closed subscheme of $\varphi_w^{-1}(\mathfrak{X})$ defined by its R -torsion ideal, this implies that x is in the image ψ_w . Thus ψ_w induces a bijection $\mathcal{G}(\mathfrak{X}^w)(k') \rightarrow \text{trop}^{-1}(w)(k')$.

Let $y \in \mathcal{G}(\mathfrak{X}^w)(k')$. We only need to show that $\psi_w(y) \in A_{\ell,1}(k')$ if and only if $\theta_0(y) \in (V_u \cap \text{in}_w U)(k')$. Write $w = (w_1, \dots, w_n)$, and let $R' = k'[\![\pi]\!]$. Then

$$\begin{aligned} \psi_w(y) \in A_{\ell,1}(k') &\iff (x_1, \dots, x_n)^u(\psi_w(y)) = \pi^\ell(1 + \pi r) \text{ for some } r \in R' \\ &\iff (\pi^{w_1} x_1, \dots, \pi^{w_n} x_n)^u(y) = \pi^{u \cdot w}(1 + \pi r) \text{ for some } r \in R' \\ &\iff (x_1, \dots, x_n)^u(y) = 1 + \pi r \text{ for some } r \in R' \\ &\iff ((x_1, \dots, x_n)^u - 1)(\theta_0(y)) = 0 \\ &\iff \theta_0(y) \in (V_u \cap \text{in}_w U)(k'). \end{aligned}$$

PROPOSITION 5.17. Let $w \in \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\})$ be such that $u \cdot w = \ell$, and let $f_1, \dots, f_{n-d} \in k[x_1, \dots, x_n]$ be a generating set for the ideal defining X in \mathbb{A}_k^n such that $\text{in}_w f_1, \dots, \text{in}_w f_{n-d} \in k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ form a generating set for the ideal of $\text{in}_w U$ in $\mathbb{G}_{m,k}^n$. Then the jacobian ideal of ψ_w is generated by

$$\pi^{w_1 + \dots + w_n - (\text{trop}(f_1)(w) + \dots + \text{trop}(f_{n-d})(w))}.$$

Proof. Let $\varphi_w : \mathbb{G}_{m,R}^n \rightarrow \mathbb{A}_R^n$ be as in Subsection 5.3, and for any $f \in R[x_1, \dots, x_n]$, we will set

$$f^w = \pi^{-\text{trop}(f)(w)} \varphi_w^*(f) \in R[x_1^{\pm 1}, \dots, x_n^{\pm 1}].$$

Then by Proposition 5.11, the ideal defining \mathfrak{X}^w is generated by f_1^w, \dots, f_{n-d}^w . Let $\mathfrak{A}_w = R[x_1^{\pm 1}, \dots, x_n^{\pm 1}] / (f_1^w, \dots, f_{n-d}^w)$ be the coordinate ring of \mathfrak{X}^w . Then we have the diagram

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \nearrow \\
 \psi_w^* \Omega_{\mathfrak{X}/R} & \longrightarrow & \Omega_{\mathfrak{X}^w/R} & \twoheadrightarrow & \Omega_{\mathfrak{X}^w/\mathfrak{X}} & \longrightarrow & 0 \\
 \uparrow & & \uparrow & & \nearrow & & \\
 \mathfrak{A}_w^n & \longrightarrow & \mathfrak{A}_w^n & & & & \\
 \uparrow & & \uparrow & & & & \\
 \mathfrak{A}_w^n \oplus \mathfrak{A}_w^{n-d} & \longrightarrow & \mathfrak{A}_w^{n-d} & & & &
 \end{array}$$

where the right vertical sequence is the presentation for the differentials module $\Omega_{\mathfrak{X}^w/R}$ induced by our presentation for \mathfrak{A}_w , the top horizontal sequence is the standard presentation for the relative differentials module $\Omega_{\mathfrak{X}^w/\mathfrak{X}}$, and the top left vertical arrow picks out the generators of $\psi_w^* \Omega_{\mathfrak{X}/R}$ induced by the coordinates of \mathbb{A}_R^n . The diagonal sequence gives a presentation for $\Omega_{\mathfrak{X}^w/\mathfrak{X}}$ with matrix whose entries are the images in \mathfrak{A}_w of the entries in the matrix

$$\begin{pmatrix}
 \pi^{w_1} & & \partial f_1^w / \partial x_1 & & \partial f_{n-d}^w / \partial x_1 \\
 & \ddots & \vdots & \dots & \vdots \\
 & & \pi^{w_n} & \partial f_1^w / \partial x_n & \partial f_{n-d}^w / \partial x_n
 \end{pmatrix}.$$

For each $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, n-d\}$,

$$\partial \varphi_w^*(f_j) / \partial x_i = \sum_{i'=1}^n (\varphi_w^*(\partial f_j / \partial x_{i'})) (\partial \varphi_w^*(x_{i'}) / \partial x_i) = \pi^{w_i} \varphi_w^*(\partial f_j / \partial x_i),$$

so

$$\partial f_j^w / \partial x_i = \pi^{w_i - \text{trop}(f_j)(w)} \varphi_w^*(\partial f_j / \partial x_i).$$

For each $m \in \{0, 1, \dots, n - d\}$ and size m subsets $I \subset \{1, \dots, n\}$ and $J \subset \{1, \dots, n - d\}$, let Δ_I^J be the determinant of the size m minor of the matrix $(\partial f_j / \partial x_i)_{i,j}$ given by rows in I and columns in J . Then the jacobian ideal of ψ_w is generated by the images in \mathfrak{A}_w of

$$\left\{ \pi^{w_1 + \dots + w_n - \sum_{j \in J} \text{trop}(f_j)(w)} \varphi_w^* \Delta_I^J \right\}_{m,I,J}.$$

Because \mathfrak{X} is smooth and pure relative dimension d over R , the unit ideal of \mathfrak{X} is generated by the images in $R[x_1, \dots, x_n]/(f_1, \dots, f_{n-d})$ of

$$\left\{ \Delta_I^{\{1, \dots, n-d\}} \mid I \subset \{1, \dots, n\} \text{ has size } n - d \right\},$$

so the jacobian ideal of ψ_w contains

$$\pi^{w_1 + \dots + w_n - (\text{trop}(f_1)(w) + \dots + \text{trop}(f_{n-d})(w))}.$$

Because $w \in \mathbb{Z}_{\geq 0}^n$ and each $f_j \in k[x_1, \dots, x_n]$, we have that each $\text{trop}(f_j)(w) \geq 0$. Therefore the jacobian ideal of ψ_w is in fact generated by

$$\pi^{w_1 + \dots + w_n - (\text{trop}(f_1)(w) + \dots + \text{trop}(f_{n-d})(w))}.$$

5.5. *Proofs of Proposition 5.1 and Theorem 5.2*

We first prove Proposition 5.1.

Proof of Proposition 5.1. This is clear if $w = 0$, so we may assume that $w \in \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\})$ is such that $u \cdot w = \ell$. In this case, by Remark 5.13, the special fiber of \mathfrak{X}^w with its induced μ_ℓ -action is equal to $(\text{in}_w U)_{\mathbb{L}}^w$. The proposition follows by considering the special fiber of $\psi_w : \mathfrak{X}^w \rightarrow \mathfrak{X}$, where by construction ψ_w is the restriction of φ_w and the special fiber of φ_w is precisely the map φ from the statement of Proposition 5.1.

Now set $\text{ordjac} : \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\}) \rightarrow \mathbb{Z} : w \mapsto \text{ordjac}_{\psi_w}(y)$ for any $y \in \mathcal{G}(\mathfrak{X}^w)$, noting that by Proposition 5.17, this does not depend on the choice of y . Also set $\text{ordjac}(0) = 0$.

PROPOSITION 5.18. *Let $w \in \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\})$ be such that $u \cdot w = \ell$. Then*

$$\mu_{\mathfrak{X}} \left(\text{trop}^{-1}(w) \right) = [\text{in}_w U/X]_{\mathbb{L}}^{-d - \text{ordjac}(w)},$$

and

$$\mu_{\mathfrak{X}}^{\mu_\ell} \left(\text{trop}^{-1}(w) \cap A_{\ell,1} \right) = [(V_u \cap \text{in}_w U)_{u \cdot w}^w / X, \mu_\ell]_{\mathbb{L}}^{-d - \text{ordjac}(w)}.$$

Proof. By Remark 5.9 and Propositions 5.14 and 5.16, the proposition follows from the (equivariant) motivic change of variables formula applied to ψ_w .

Because $u \in \mathbb{Z}_{>0}^n$, the monomial $(x_1, \dots, x_n)^u$ vanishes on all of $X \setminus U$. Thus the constant term of $Z_{X,u}^{\text{naive}}(T)$ is equal to

$$[U/X]_{\mathbb{L}}^{-d} = [\text{in}_0 U/X]_{\mathbb{L}}^{-d - \text{ordjac}(w)}.$$

Therefore, Theorem 5.2 follows from Proposition 5.17 and the next proposition.

PROPOSITION 5.19. *The coefficient of T^ℓ in $Z_{X,u}^{\hat{\mu}}(T)$ is equal to*

$$\sum_{\substack{w \in \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\}) \\ u \cdot w = \ell}} [(V_u \cap \text{in}_w U)_{u \cdot w}^w / X, \hat{\mu}]_{\mathbb{L}}^{-d - \text{ordjac}(w)},$$

and the coefficient of T^ℓ in $Z_{X,u}^{\text{naive}}(T)$ is equal to

$$\sum_{\substack{w \in \text{Trop}(U) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\}) \\ u \cdot w = \ell}} [\text{in}_w U / X]_{\mathbb{L}}^{-d - \text{ordjac}(w)}.$$

Proof. This follows from Propositions 3.5, 3.6, 5.7 and 5.18.

6. Motivic zeta functions of hyperplane arrangements

Let $d, n \in \mathbb{Z}_{>0}$, let \mathcal{M} be a rank d loop-free matroid on $\{1, \dots, n\}$, and let $\mathcal{A} \in \text{Gr}_{\mathcal{M}}(k)$. We will prove Theorems 1.3 and 1.10. Throughout this section, we will be using the notation defined in Section 2.4.

LEMMA 6.1. *Let $w = (w_1, \dots, w_n) \in \mathbb{R}^n$. Then there exist $f_1, \dots, f_{n-d} \in k[x_1, \dots, x_n]$ that generate the ideal of $X_{\mathcal{A}}$ in \mathbb{A}_k^n such that the ideal of $\text{in}_w U_{\mathcal{A}}$ in $\mathbb{G}_{m,k}^n$ is generated by $\text{in}_w f_1, \dots, \text{in}_w f_{n-d} \in k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and such that*

$$w_1 + \dots + w_n - (\text{trop}(f_1)(w) + \dots + \text{trop}(f_{n-d})(w)) = \text{wt}_{\mathcal{M}}(w).$$

Proof. Fix some $B \in \mathcal{B}(\mathcal{M}_w)$. Then by Proposition 2.10,

$$\{L_{C(\mathcal{M},i,B)}^{\mathcal{A}} \mid i \in \{1, \dots, n\} \setminus B\} \subset k[x_1, \dots, x_n]$$

generate the ideal of $X_{\mathcal{A}}$ in \mathbb{A}_k^n and

$$\{\text{in}_w L_{C(\mathcal{M},i,B)}^{\mathcal{A}} \mid i \in \{1, \dots, n\} \setminus B\} \subset k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

generate the ideal of $\text{in}_w U_{\mathcal{A}}$ in $\mathbb{G}_{m,k}^n$. By Proposition 2.11,

$$\text{trop}(L_{C(\mathcal{M},i,B)}^{\mathcal{A}})(w) = w_i$$

for each $i \in \{1, \dots, n\} \setminus B$. Thus

$$w_1 + \dots + w_n - \sum_{i \in \{1, \dots, n\} \setminus B} \text{trop}(L_{C(\mathcal{M},i,B)}^{\mathcal{A}})(w) = \sum_{i \in B} w_i = \text{wt}_{\mathcal{M}}(w).$$

We now prove Theorem 1.3.

PROPOSITION 6.2. *We have that*

$$Z_{\mathcal{A},k}^{\hat{\mu}}(T) = \sum_{w \in \text{Trop}(\mathcal{M}) \cap (\mathbb{Z}_{\geq 0}^n \setminus \{0\})} [F_{\mathcal{A}_w}, \hat{\mu}]_{\mathbb{L}}^{-d - \text{wt}_{\mathcal{M}}(w)}(T, \dots, T)^w \in \mathcal{M}_k^{\hat{\mu}}[[T]],$$

and

$$Z_{\mathcal{A},0}^{\hat{\mu}}(T) = \sum_{w \in \text{Trop}(\mathcal{M}) \cap \mathbb{Z}_{>0}^n} [F_{\mathcal{A}_w}, \hat{\mu}]_{\mathbb{L}}^{-d - \text{wt}_{\mathcal{M}}(w)}(T, \dots, T)^w \in \mathcal{M}_k^{\hat{\mu}}[[T]].$$

Proof. By setting $X = X_{\mathcal{A}}$, $u = (1, \dots, 1)$, and $v = (1, \dots, 1)$, the proposition follows directly from Corollary 5.4 and Lemma 6.1.

We end this section by proving Theorem 1.10.

PROPOSITION 6.3. *We have that*

$$Z_{\mathcal{A},k}^{\text{naive}}(T) = \sum_{w \in \text{Trop}(\mathcal{M}) \cap \mathbb{Z}_{\geq 0}^n} \chi_{\mathcal{M}_w}(\mathbb{L})_{\mathbb{L}}^{-d - \text{wt}_{\mathcal{M}}(w)}(T, \dots, T)^w \in \mathcal{M}_k[[T]],$$

and

$$Z_{\mathcal{A},0}^{\text{naive}}(T) = \sum_{w \in \text{Trop}(\mathcal{M}) \cap \mathbb{Z}_{>0}^n} \chi_{\mathcal{M}_w}(\mathbb{L})_{\mathbb{L}}^{-d - \text{wt}_{\mathcal{M}}(w)}(T, \dots, T)^w \in \mathcal{M}_k[[T]].$$

Proof. By setting $X = X_{\mathcal{A}}$ and $u = (1, \dots, 1)$, the proposition follows from Theorem 5.2, Proposition 5.5, Lemma 6.1, and the fact that for each $w \in \text{Trop}(\mathcal{M})$, the class $[U_{\mathcal{A}_w}] \in \mathcal{M}_k$ is equal to $\chi_{\mathcal{M}_w}(\mathbb{L})$.

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