DOI:10.1017/S0013091506264939

Schilling, R. *Measures, integrals and martingales* (Cambridge University Press, 2005), xi + 352 pp., 0 521 61525 9 (paperback), £24.99, 0 521 85015 0 (hardback), £50.

Whereas measure theory and integration are of central importance in modern analysis, no single treatment of this topic has become standard in the undergraduate curriculum. The outstanding issues are whether to emphasize Lebesgue or Riemann integration, whether or not to teach integration alongside probability, and whether to define integrals from measures or vice versa. In each case, Schilling advocates the first option.

In this attractive textbook, Schilling presents the theory in a style consistent with the historical development due to Lebesgue, Kolmogorov and others. The mathematical level of this book is appropriate for undergraduates at levels 4 or 3, and these students could attempt the accompanying exercises, for which solutions appear on the author's web page. Generally, the author strikes the right balance when stating results, making them short enough to be memorable, while sufficiently detailed as to be precise.

The author wisely starts with elementary material on counting and sets, before considering measures as countably additive set functions on σ-algebras. Carathéodory's extension theorem leads directly to the existence of Lebesgue measure on $\mathbb{R}^n$ and thence to the integrals of simple functions; the full definition of the Lebesgue integral and the convergence theorems follow.

In a style consistent with multivariable calculus, Jacobi's transformation formula has a rigorous treatment which follows naturally from the discussion of product measures and induced measures. The chapters on Hilbert space and L^p have an elegant conciseness, benefiting most those students who have some previous familiarity with these topics.

After a detailed discussion of uniform integrability, martingales emerge midway through the book, and the martingale convergence theorem is used to prove Kolmogorov's strong law of large numbers and the Borel–Cantelli lemmas. Martingales also find employment in harmonic analysis, as in the Calderón–Zygmund decomposition theorem. The presentation of the Hardy–Littlewood maximal theorem on $\mathbb{R}^n$ features a universal constant and hence seems elaborate; some lecturers might prefer to present first the one-dimensional case, where Riesz's sunrise lemma [**4**] gives a simpler proof. There is one exercise on rearrangements, a topic which is central to Hardy and Littlewood's presentation of maximal functions, as in their example in [**2**]: 'a batsman's total satisfaction for the season is a maximum, for a given stock of innings, when the innings are played in decreasing order'.

Bourbaki's [**1**] approach to integration starts with the space $C(\Omega)$ of continuous bounded functions on a locally compact Hausdorff space and emphasizes Riesz's representation theorem for bounded linear functionals on $C(\Omega)$, a result which is virtually suppressed in the book under review. For probabilists, a significant drawback of Bourbaki's approach to integration is the unsatisfactory treatment of conditional probability when Ω is not necessarily a Polish space. Overcoming such problems, Schilling introduces conditional expectations via orthogonal projections onto subspaces of L^2 associated with σ-algebras, a natural approach which admits further development into Markov processes and non-commutative integration. The coverage of independence here is conceptually similar, although less systematic, than that by Itô [**3**]; in particular, there is no zero–one law.

To illustrate the theory, the author provides examples including Friedrichs mollifiers, Bernoulli random variables and the Haar wavelet. In the discussion of classical function systems such as Laguerre polynomials, the author could have mentioned that their orthogonality follows easily from differential equations. Brownian motion provides the most advanced application of the theory, and here it is introduced by Haar orthogonal series with Gaussian coefficients; one can pursue this route and introduce the stochastic integral in L^2 relatively easily.

The appendices feature a concise summary of point-set topology and a construction of sets that are not Lebesgue measurable. For the benefit of those still interested, there is a summary of Riemann integration. There are good indices of notation and subjects, and ample references.

The author and Cambridge University Press have produced an elegant and self-contained introduction to the modern theory of integration which leads naturally into postgraduate study—this reviewer strongly recommends the book to students of analysis and probability.

References

1. N. Bourbaki, *Éléments de mathematique, Livre VI, Intégration*, 2nd edn (Hermann, Paris, 1965).
2. G. H. Hardy and J. E. Littlewood, A maximal theorem with function-theoretic applications, *Acta Math.* **54** (1930), 81–116.
3. K. Itô, *Introduction to probability theory* (Cambridge University Press, 1984).
4. F. Riesz, Sur un théorème de maximum de MM. Hardy et Littlewood, *J. Lond. Math. Soc.* **7** (1932), 10–13.

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