POINT-TRANSITIVE ACTIONS BY THE UNIT INTERVAL

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1. Introduction. An action is a continuous function $\alpha: T \times X \to X$, where T is a semigroup, X is a Hausdorff space, and $\alpha(t_1, \alpha(t_2, x)) = \alpha(t_1t_2, x)$ for all $t_1, t_2 \in T$ and $x \in X$. If, for an action α, $Q(\alpha) = \{x \in X | \alpha(T \times \{x\}) = X\}$ is non-empty, then α is called a point-transitive action. Our aim in this note is to classify the point-transitive actions of the unit interval with the usual, nil, or min multiplications.

The reader is referred to [**5**; **7**; **9**] for information concerning the general theory of semigroups. All semigroups which are considered here are compact and Abelian and all spaces are compact Hausdorff. Actions by semigroups have been studied in [**1**; **3**; **8**]. We shall follow the notation and terminology of [**3**; **9**], e.g., we shall suppress the name of an action, whenever possible, and write the value of an action at (t, x) as tx, we shall write A^* for the topological closure of A, and I for a space homeomorphic to $[0, 1]$.

By an I-semigroup we mean a semigroup homeomorphic to the unit interval with the endpoints acting as a zero and an identity. A usual, I-semigroup is a semigroup iseomorphic to $[0, 1]$ under the usual multiplication. A nil I-semigroup is a semigroup iseomorphic to $[0, 1]/[0, \frac{1}{2}]$ (the Rees quotient). A min I-semigroup is a semigroup iseomorphic to $[0, 1]$ under the multiplication $xy = \min(x, y) = x \wedge y$. These types of semigroups have been studied in [**4**; **6**] and we shall use these results extensively.

Let $T = [0, 1]$ and $X = [0, 1]$. The usual-usual action is defined by giving T the usual multiplication and define $T \times X \to X$ by tx = usual product of t and x. The usual-nil action is defined by giving T the usual multiplication and defining $T \times [\frac{1}{2}, 1] \to [\frac{1}{2}, 1]$ by $tx = \max(\frac{1}{2}$, usual product of t and $x)$. We shall show that these are the only point-transitive actions of a usual I-semigroup. If T is a nil I-semigroup and $T \times X \to X$ is defined by tx = nil product of t and x, we shall call this the nil-usual action. The nil-nil action is defined by giving T the nil multiplication and define $T \times [\frac{1}{2}, 1] \to [\frac{1}{2}, 1]$ by $tx = \max(\frac{1}{2}$, nil product of t and $x)$. The nil-usual and nil-nil actions are the only point-transitive actions by a nil I-semigroup. Clearly, any semigroup acts on a one-point space. In order to avoid this case, we shall assume that if $T \times X \to X$ is an action, then card $X \geqq 2$.

2. Preliminaries. Extensive use will be made of the following theorem which is due to Aczél and Wallace [**1**].

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THEOREM 1. *Let $T \times X \to X$ be a point-transitive action. If T is Abelian and compact, if X is compact, and if $x_1 \in Q$, then there is a continuous associative multiplication $\circ$ for X such that*

(1) *x_1 is a unit for $\circ$,*

(2) *if $h: T \to X$ is defined by $h(t) = tx$, then h is a homomorphism,*

(3) *if $t, t' \in T$, $x \in X$, and if $t'x_1 = x$, then $tx = (tx_1) \circ x = tt'x_1$.*

We shall refer to $\circ$ as the CAM (continuous associative multiplication) induced by x_1. All the semigroups which we consider are Abelian and compact and all spaces are compact.

Let $\alpha_1: T_1 \times X_1 \to X_1$ and $\alpha_2: T_2 \times X_2 \to X_2$ be two actions and let $g: T_1 \to T_2$ be an iseomorphism. A continuous onto function $f: X_1 \to X_2$ is said to be *g-equivariant* with respect to α_1 and α_2 if and only if $f\alpha_1(t, x) = \alpha_2(g(t), f(x))$ for all $x \in X_1$ and $t \in T_1$. An *a-homomorphism* between α_1 and α_2 is a pair (g, f) where $g: T_1 \to T_2$ is an iseomorphism and $f: X_1 \to X_2$ is a g-equivariant map. We shall denote an a-homomorphism by $(g, f): \alpha_1 \to \alpha_2$. If $(g, f): \alpha_1 \to \alpha_2$ is an a-homomorphism and if f maps X_1 homeomorphically onto X_2, we say that $(g, f): \alpha_1 \to \alpha_2$ is an *a-isomorphism*.

Proposition 2. *Let $\alpha_1: T_1 \times X_1 \to X_1$ and $\alpha_2: T_2 \times X_2 \to X_2$ be actions.*

(1) *If $(g, f): \alpha_1 \to \alpha_2$ is an a-homomorphism, then $f(Q(\alpha_1)) \subset Q(\alpha_2)$.*

(2) *If $(g, f): \alpha_1 \to \alpha_2$ is an a-homomorphism, $x_1 \in Q(\alpha_1)$, $\circ$ is the CAM induced by x_1, and if $\cdot$ is the CAM induced by $f(x_1)$, then $f: (X_1, \circ) \to (X_2, \cdot)$ is a semigroup homomorphism.*

(3) *Suppose that $x_1 \in Q(\alpha_1)$ and $x_2 \in Q(\alpha_2)$. Define $h_i: T_i \to X_i$ by $h_i(t) = \alpha_i(t, x_i)$ for $i = 1$ or 2. If there exists an iseomorphism $g: T_1 \to T_2$ such that $h_1(t) = h_1(t')$ implies $h_2g(t) = h_2g(t')$, then α_1 is a-homomorphic to α_2. Moreover, if, in addition, $h_2g(t) = h_2g(t')$ implies $h_1(t) = h_1(t')$, then α_1 is a-isomorphic to α_2.*

(4) *Suppose that $(g, f): \alpha_1 \to \alpha_2$ is an a-homomorphism. Let $x_1 \in Q(\alpha_1)$, $x_2 = f(x_1)$, and h_i as in (3). Then $h_1(t) = h_1(t')$ implies $h_2g(t) = h_2g(t')$. Moreover, f is one-to-one if and only if $h_2g(t) = h_2g(t')$ implies $h_1(t) = h_1(t')$.*

Proof. (1) is clear.

(2) Let $h_i: T_i \to X_i$ be the homomorphism defined by $h_i(t_i) = \alpha_i(t, x_i)$ for $i = 1$ or 2, where $x_2 = f(x_1)$. Let $x, y \in X_1$ and let $t \in h_1^{-1}(x)$ and $t' \in h_1^{-1}(y)$. Thus, $f(x \circ y) = fh_1(tt') = h_2g(tt') = h_2g(t) \cdot h_2g(t') = f(x) \cdot f(y)$.

(3) Define $f: X_1 \to X_2$ by $f(x) = h_2gh_1^{-1}(x)$. It is easily verified that $(g, f): \alpha_1 \to \alpha_2$ is a homomorphism.

(4) follows from the fact that $fh_1 = h_2g$.

3. Classification theorem.

The following remark will establish some restrictions on point-transitive actions by I-semigroups.

Remark 3. Let T be an I-semigroup and let $\alpha_1: T \times X \to X$ be a point-transitive action. Then:

(1) X is homeomorphic to $[0, 1]$;

(2) card $Q(\alpha_1) = 1$ and the point in Q is an endpoint of X;

(3) If T is a usual I-semigroup and $x_0 \in Q$, then x_0 induces a usual multiplication or a nil-multiplication;

(4) If T is a nil I-semigroup and if $x_0 \in Q$, then x_0 induces the nil-multiplication;

(5) If T is a min I-semigroup and if $x_0 \in Q$, then x_0 induces the min-multiplication;

(6) If $\{x_1\} = Q(\alpha_1)$ and if x_2 is the other endpoint of X, then there exists an action α_2: $T \times X \to X$ such that $\{x_2\} = Q(\alpha_2)$ and α_2 is a-isomorphic to α_1.

Proof. (1) By the Aczél-Wallace Theorem, $(X, \circ)$, where $\circ$ is a CAM induced by a point in $Q(\alpha_1)$, is the homomorphic image of T. A result of Cohen and Krule [**2**] states that a homomorphic image of T is an I-semigroup.

(2) Let x_1 and x_2 be the two endpoints of X. Since each point in $Q(\alpha_1)$ induces an I-semigroup structure on X, $Q(\alpha_1) \subset \{x_1, x_2\}$. Suppose that $Q(\alpha_1) = \{x_1, x_2\}$. Then there are t_1, $t_2 \in T$ such that $\alpha_1(t_1, x_1) = x_2$ and $\alpha_1(t_2, x_2) = x_1$. Let $\circ$ be the CAM induced by x_2. By the previously mentioned result of Cohen and Krule [**2**], x_2 is an identity for $\circ$ and x_1 is a zero for $\circ$. But $x_1 = \alpha_1(t_2, x_2) \circ \alpha_1(t_1, x_2) = \alpha_1(t_1 t_2, x_2) = \alpha_1(t_1, \alpha_1(t_2, x_2)) = x_2$, which is a contradiction.

(3), (4), and (5) follow the results in [**2**; **4**] and the Aczél-Wallace Theorem.

(6) Let f: $X \to X$ be a homeomorphism such that $f(x_2) = x_1$ and f^2 is the identity. Define α_2: $T \times X \to X$ by $\alpha_2(t, x) = f\alpha_1(t, f(x))$. Then α_2 meets the stated requirements.

Because of the previous remark, we may restrict our attention to actions $T \times I \to I$ where $Q = \{1\}$. The following remark will give some information about the induced CAMs.

Remark 4. Let $T \times I \to I$ be an action such that $Q = \{1\}$.

(1) The CAM on I induced by 1 is an I-semigroup if and only if $K(T)1 = 0$, where $K(T)$ is the minimal ideal of T.

(2) The CAM on I induced by 1 is a usual I-semigroup if and only if $\{t \in T \mid t1 = 0\}$ is a prime ideal of T and $t^2 1 = t1$ implies $t1$ is 0 or 1.

(3) The CAM on I induced by 1 is a nil I-semigroup if and only if $\{t \in T \mid t1 = 0\}$ is an ideal of T which is not prime and $t^2 1 = t1$ implies $t1$ is 0 or 1.

(4) The CAM on I induced by 1 is a min I-semigroup if and only if $\{t \in T \mid t1 = 0\}$ is an ideal of T and $t^2 1 = t1$ for all $t \in T$.

The proof of this remark is a straightforward application of the results in [**4**].

LEMMA 5. *Let $T \times I \to I$ be an action such that $Q = \{1\}$. If T is a usual or nil I-semigroup, then $t_1 1 = t_2 1$ implies that $t_1 = t_2$ or $t_1 1 = 0 = t_2 1$.*

Proof. Suppose that $t_1 < t_2$. Then there is $t \in T$ such that $tt_1 = t_2$ and

$t < 1$. Thus, $t_2 1 = t^n t_2 1$ so that $t_1 1 = t_2 1 = 0 \cdot 1$. But by Remarks 3 and 4, $0 \cdot 1 = 0$.

THEOREM 6. *Let* $\alpha: T \times I \to I$ *be an action with* $Q = \{1\}$.

(1) *If* T *is a usual I-semigroup and if* $\{t \in T | \alpha(t, 1) = 0\} = \{0\}$, *then* α *is a-isomorphic to the usual-usual action.*

(2) *If* T *is a usual I-semigroup and if* $\{t \in T | \alpha(t, 1) = 0\} \neq \{0\}$, *then* α *is a-isomorphic to the usual-nil action.*

(3) *If* T *is a nil I-semigroup, then* α *is a-isomorphic to the nil-nil action or the nil-usual action.*

Proof. Without loss of generality, we shall assume that $T = [0, 1]$.

(1) Let $\alpha_2: T \times I \to I$ be the usual-usual action. Let $g: T \to T$ be the identity and let h_1 and h_2 be as in Proposition 2 (3). By Lemma 5, $h_1(t) = h_1(t')$ implies $h_2(t) = h_2(t')$, and the conclusion follows from Proposition 2 (3).

(2) Let $\alpha_2: T \times I \to I$ be the usual-nil action and let h_1 and h_2 be as in Proposition 2 (3). Let $g: T \to T$ be an iseomorphism such that $gh_1^{-1}(0) = h_2^{-1}(0)$. Then the conclusion follows from Lemma 5 and Proposition 2 (3).

(3) follows in a manner similar to (2).

It should be noted that the usual-usual and the usual-nil actions are not a-isomorphic because of Proposition 2.2.

We now turn our attention to action by a min I-semigroup T (assumed to be [0, 1]) acting on $I = [0, 1]$ where $Q = \{1\}$. Any monotone mapping $k: T \to I$ will determine an action by $tx = \min(k(t), x)$. The following are three examples of actions by a min I-semigroup. The first two are a-isomorphic to each other but the third is not a-isomorphic to the first two.

Examples. (1) Let $k_1: T \to I$ by

$$k_1(t) = \begin{cases} 2t & \text{if } 0 \leqq t \leqq \frac{1}{2}, \\ 1 & \text{if } t > \frac{1}{2}. \end{cases}$$

(2) Let $k_2: T \to I$ by

$$k_2(t) = \begin{cases} 3t & \text{if } 0 \leqq t \leqq \frac{1}{3}, \\ 1 & \text{if } t > \frac{1}{3}. \end{cases}$$

(3) Let $k_3: T \to I$ by

$$k_3(t) = \begin{cases} \frac{3}{2}t & \text{if } 0 \leqq t \leqq \frac{1}{3}, \\ \frac{1}{2} & \text{if } \frac{1}{3} < t < \frac{2}{3}, \\ \frac{1}{2}(3t - 1) & \text{if } \frac{2}{3} \leqq t \leqq 1. \end{cases}$$

The following remark is an application of Proposition 2 to actions by min I-semigroups.

Remark 7. Let T be a min I-semigroup and $\alpha_1, \alpha_2: T \times I \to I$ be two actions. Then α_1 is iseomorphic to α_2 if and only if there exists a homeomorphism $g: T \to T$ such that $g(Q(\alpha_1)) = (Q(\alpha_2))$ and $h_1(t) = h_1(t')$ if and only if $h_2 g(t) = h_2 g(t')$, where h_1 and h_2 are as defined in Proposition 2.

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