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A PROJECTIVE DESCRIPTION OF THE SPACE OF HOLOMORPHIC GERMS

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Abstract A natural topology on the set of germs of holomorphic functions on a compact subset K of a Fréchet space is the locally convex inductive limit topology of the spaces $\mathcal{O}(\Omega)$ endowed with the compact open topology; here Ω is any open subset containing K. Mujica gave a description of this space as the inductive limit of a suitable sequence of compact subsets. He used a set of intricate semi-norms for this. We give a projective characterization of this space, using simpler semi-norms, whose form is similar to the one used in the Whitney Extension Theorem for C_∞ functions. They are quite natural in a framework where extensions are involved. We also give a simple proof that this topology is strictly stronger than the topology of the projective limit of the non-quasi-analytic spaces.

Keywords: germs; projective description; holomorphic functions

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1. Introduction

Locally convex inductive limits are a very important construction in functional analysis and its applications. However, the natural definition of their continuous semi-norms is not explicit and often difficult to handle. This can be seen as a drawback for their use. Characterizations of the continuous semi-norms and projective descriptions are natural and important tasks in this framework (see, for example, [**1**]).

In the context of holomorphic functions, a natural inductive limit is the set $\mathcal{O}(S)$ of germs of holomorphic functions near an arbitrary subset S of $\mathbb{R}^n$. Of course, this limit is generally uncountable. Using abstract functional techniques, Martineau [**10**] proved that this space is the projective limit of the spaces of germs of holomorphic functions on the compact subsets of S.

On the other hand, in [**11**], Mujica gave a characterization of the topology of the space of germs of holomorphic functions on a compact subset K of a complex Fréchet space endowed with the τ_0 inductive limit topology. In general, two types of semi-norms are needed. One type is quite natural but the other one is more intricate. The semi-norms of this second type are formed to glue together several Taylor expansions at points where some ambiguity can occur. They involve unnatural sequences of points as well as estimates on limited Taylor expansions approximating the function outside the compact set K. For a locally connected compact subset K, the first type of semi-norms can be used alone.

The same semi-norms were used in [**3**]. Rusek showed in [**12**] that for L-connected compact subsets of $\mathbb{C}^n$, semi-norms of the first type generate the topology. However, Example 2.57 in [**3**], which is due to Aron, shows that the first type of semi-norms cannot be used alone for a very simple compact subset of $\mathbb{C}$. A more detailed discussion of this is given in [**3**, p. 107].

Here we introduce semi-norms of a new type which generate the topology of $\mathcal{O}(K)$ for an arbitrary compact subset K of a complex Fréchet space. They have a form analogous to the one used in the Whitney Extension Theorem for C_∞ functions in $\mathbb{R}^n$. As in this problem, they only constrain the error in the Taylor expansions between two points of K. Since the proof can be obtained easily from our constructions, we also show that in the locally connected case, the first type of semi-norms can be used alone.

The set of holomorphic functions in an open subset can also be endowed with a topology induced by the non-quasi-analytic classes. However, the projective limit topology defined in this way is strictly weaker than the previous topology. This was presented as an open question in [**4**] and solved in [**13**]. We give a simple proof of this.

2. Notation and main results

If Ω is an open subset of a complex Fréchet space E, we endow the space $\mathcal{O}(\Omega)$ of holomorphic functions in Ω with the usual compact open topology τ_0 (see, for example, [**3**]). For a compact subset K of E, we consider the inductive limit of locally convex spaces

$$\mathcal{O}(K) = \operatorname{ind}_{\Omega \supset K} \mathcal{O}(\Omega),$$

where Ω runs over all open subsets of E containing K. Denote by $\mathcal{V}(E)$ the set of all open, convex and balanced neighbourhoods of 0 in E.

If f is a holomorphic function in a neighbourhood of a compact subset K of E, $x \in K$ and $k \in \mathbb{N}$, there is a unique k-linear symmetric form $D^k f(x)$ on E such that

$$D^k f(x)(h, \dots, h) = D_t^k [f(x + th)]_{|t=0}.$$

If $j \leqslant k$, we denote by $D^k f(x) h^{(j)}$ the $(k - j)$-linear form on E defined by

$$(D^k f(x) h^{(j)})(v, \dots, v) = D^k f(x)(h, \dots, h, v, \dots, v),$$

with j copies of h and $k - j$ copies of v. If $k = j$, this is just an element of E.

If $A \subset E$ is bounded, let

$$\|D^k f(x)\|_A = \sup_{h \in A} |D^k f(x) h^{(k)}|.$$

It is well known (see [**3**, p. 5]) that if A is a bounded, convex and balanced subset of E, then

$$\sup_{h_1, \dots, h_k \in A} |D^k f(x)(h_1, \dots, h_k)| \leqslant \frac{k^k}{k!} \|D^k f(x)\|_A. \tag{2.1}$$

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If $k \geqslant 0$, we also use the notation

$$\|D^k f\|_{K,A} = \sup_{x\in K}\sup_{h\in A} |D^k f(x)h^{(k)}|.$$

Of course, if A is open and $\|D^k f\|_{K,A} \leqslant C^{1+k} k!$ for every k, then the Taylor expansion of f at any point x of K converges in $x + A/C$.

Consider the following semi-norms of Whitney type

$$\|f\|_{K,M,k} = \frac{\|D^k f\|_{K,M}}{k!} + \sup_{\substack{0\leqslant \ell<k,\\ 0<\rho<1}} \sup_{\substack{x,x+h\in K,\\ h\in\rho M}} \frac{1}{\ell!\rho^{k-\ell}} \left\| D^\ell f(x+h) - \sum_{j<k-\ell} D^{j+\ell} f(x)\frac{h^{(j)}}{j!}\right\|_M .$$

Here $k \geqslant 0$ and M is any compact subset of E. These semi-norms are simpler than the ones used in [**11**] and only constrain limited Taylor expansions between two points of K. Note that $\|f\|_{K,M,k}$ is increasing with M and that $\|f\|_{K,\lambda M,k} = \lambda^k \|f\|_{K,M,k}$ if $\lambda > 0$.

They are natural in problems where extension properties occur. Using the Cauchy inequalities, one easily sees that these semi-norms are continuous on $\mathcal{O}(K)$. More details are contained in the first part of the proof of Theorem 2.1. In a finite-dimensional space, we can fix M equal to the closed unit ball and omit it in the notation. Then we obtain the semi-norms

$$\|f\|_{K,k} = \frac{\|D^k f\|_K}{k!} + \sup_{0\leqslant \ell<k} \sup_{x,x+h\in K} \frac{1}{\ell!\|h\|^{k-\ell}} \left\| D^\ell f(x+h) - \sum_{j<k-\ell} D^{j+\ell} f(x)\frac{h^{(j)}}{j!}\right\|,$$

with

$$\|D^k f\|_K = \sup_{x\in K}\sup_{\|h\|\leqslant 1} |D^k f(x)h^{(k)}|.$$

Our main result is the following projective description of $\mathcal{O}(K)$.

Theorem 2.1. *If K is a compact subset of a Fréchet space E, then the topology of $\mathcal{O}(K)$ is defined by the semi-norms*

$$p(f) = \sup_{k\in\mathbb{N}} \epsilon_k^k \|f\|_{K,M,k},$$

where M runs over all compact subsets of E and $(\epsilon_k)_{k\geqslant 0}$ over all sequences of real numbers decreasing to 0. Moreover, if K is locally connected, then one can replace $\|f\|_{K,M,k}$ by $\|D^k f\|_{K,M}/k!$ in the definition of the semi-norms p.

An easy inspection of the proof of this theorem shows that in the definition of $\|f\|_{K,M,k}$, it is possible to replace $\ell!$ by any sequence a_ℓ satisfying $a_\ell \geqslant \ell!$. This is not a surprise since the control on the growth of the derivatives is already performed by the first part. However, the previous semi-norms are the natural ones since they correspond to the usual estimation of the error in the Taylor expansion of a holomorphic function.

A classical example (see [**3**] or [**11**]) shows that, for the set $K = \{1/k : k \in \mathbb{N}_0\} \cup \{0\}$, one cannot replace $\|f\|_{K,k}$ by $\|D^k f\|_K/k!$.

The next result shows that the set of sequences $(\epsilon_k)_{k\geqslant 0}$ cannot easily be relaxed. Let $L = (L_k)_{k\in\mathbb{N}}$ be an increasing sequence of real positive numbers. If Ω is an open subset

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of $\mathbb{R}^n$, we denote by $C^{(L)}(\Omega)$ the set of all $f \in C_\infty(\Omega)$ such that for any compact subset K of Ω, we have

$$\|D^k f\|_K \leqslant A^{1+k} L_k^k$$

for every k and some $A > 0$. This space is endowed with the semi-norms

$$p(f) = \sup_{k\in\mathbb{N}} \epsilon_k^k \frac{\|D^k f\|_K}{L_k^k},$$

where K runs over all compact subsets of Ω and $(\epsilon_k)_{k\geqslant 0}$ over all sequences of real numbers decreasing to 0.

L or $C^{(L)}$ is said to be quasi-analytic if $u \in C^{(L)}(\Omega)$ and $D^\alpha u(x) = 0$ for every $\alpha \in \mathbb{N}^n$ and some $x \in \Omega$ imply $u = 0$. A classical theorem of Denjoy–Carleman says that $C^{(L)}(\Omega)$ is non-quasi-analytic if and only if $\sum_{k=0}^{+\infty} 1/L_k < +\infty$. It is well known that, as a set,

$$\mathcal{O}(\Omega) = \bigcap_L C^{(L)}(\Omega),$$

where the intersection runs over all non-quasi-analytic sequences L.

Theorem 2.2. *If Ω is a non-void open subset of $\mathbb{R}^n$, then the topology of $\mathcal{O}(\Omega)$ is strictly stronger than the projective limit topology of the non-quasi-analytic spaces $C^{(L)}(\Omega)$.*

Note that this projective limit topology is defined by the semi-norms

$$p(f) = \sup_{k\in\mathbb{N}} \eta_k^k \|D^k f\|_K,$$

where K runs over all compact subsets of Ω and $(\eta_k)_{k\geqslant 0}$ over all sequences of real numbers decreasing to 0 and satisfying

$$\sum_{k=0}^{+\infty} \eta_k < +\infty.$$

This follows easily from the fact that, if the sequence η_k is summable, there is a sequence n_k converging to $+\infty$ such that $n_k \eta_k$ is still summable.

3. Proofs of the main results

We need a lemma proved in [**11**]. For the sake of completeness, we give a slightly different and shorter proof.

Lemma 3.1. *Let K be a locally connected compact subset of a Fréchet space E. Then for every $V \in \mathcal{V}(E)$ there is $W \in \mathcal{V}(E)$ such that every function $f \in \mathcal{O}(K)$ satisfying*

$$\|D^k f\|_{K,V} \leqslant k!$$

for every k can be extended as a holomorphic function in $K + W$.

Proof. Let $V \in \mathcal{V}(E)$. Since K is compact and locally connected, one can find some elements $z_1, \dots, z_N \in K$ and $W_1, \dots, W_N \in \mathcal{V}(E)$ satisfying the following properties:
$K \subset \bigcup_{j=1}^{N}(z_j + W_j)$

(i) , and

(ii) for any j, there is an open set $U_j \subset \frac{1}{2}V$ containing $2W_j$ such that $(z_j + U_j) \cap K$ is connected.

Let W be the intersection of the sets $\frac{1}{2}W_j$, $j = 1, \dots, N$.

Consider a function $f \in \mathcal{O}(K)$ satisfying $\|D^k f\|_{K,V} \leqslant k!$ for every k. Its Taylor expansion at any point x of K converges in $x + V$. To prove the lemma, it is enough to show that if $x, y \in K$, then the Taylor expansions f_x and f_y of f at x and y coincide in $(x + W) \cap (y + W)$.

Assume that this intersection is not empty and that f is holomorphic in an open neighbourhood U of K. Choose j such that $x \in z_j + W_j$ and denote by ω the connected component of $(x + V) \cap U$ that contains x. The set $(z_j + U_j) \cap K$ is connected, contains x and is included in $(x + V) \cap U$ since $z_j + U_j \subset x + W_j + U_j \subset x + 2U_j \subset x + V$. Hence it is included in ω. We have $y \in x + 2W \subset z_j + W_j + 2W \subset z_j + 2W_j \subset z_j + U_j$, hence $y \in \omega$. Since $f_x = f$ near x, it follows that $f_x = f = f_y$ in a neighbourhood of y. Now, f_x and f_y are holomorphic in the open and convex set $(x + 2W) \cap (y + W)$ and coincide near y. This proves the lemma. $\square$

Proof of Theorem 2.1. We first show that the semi-norms of Theorem 2.1 are continuous on the inductive limit $\mathcal{O}(K)$. Let U be an open neighbourhood of K in E and M be a compact subset of E. Replacing M by a larger compact set, we can assume that it is convex and balanced. Choose $V \in \mathcal{V}(E)$ such that $K + V \subset U$. Since M is compact, there is $\rho_0 > 0$ such that $2\rho_0 M \subset V$. Assume that $x, x + h \in K$, $h \in \rho M$ with $0 < \rho < \rho_0$ and $0 \leqslant t \leqslant 1$. Using the Cauchy inequalities, we obtain

$$\|D^k f(x + th)\|_M \leqslant k! \rho_0^{-k} \|f\|_{K + 2\rho_0 M}$$

for every $f \in \mathcal{O}(U)$. This proves that the first term in the definition of $\|f\|_{K,M,k}$ leads to a continuous semi-norm and that we can assume $0 < \rho < \rho_0$ is the second one. Using Taylor expansion and (2.1), we obtain

$$\begin{aligned} \frac{1}{\ell! \rho^{k-\ell}} \biggl\| D^\ell f(x + h) - \sum_{j < k - \ell} D^{j+\ell} f(x) \frac{h^{(j)}}{j!} \biggr\|_M & \\ \leqslant \frac{1}{\ell! \rho^{k-\ell}} \biggl\| \int_0^1 \frac{(1-t)^{k-\ell-1}}{(k-\ell-1)!} D^k f(x + th) h^{(k-\ell)} \, \mathrm{d}t \biggr\|_M & \\ \leqslant \frac{k^k}{k!} \frac{k!}{\ell!(k-\ell)!} \rho_0^{-k} \|f\|_{K + 2\rho_0 M} \leqslant \biggl(\frac{2e}{\rho_0}\biggr)^k \|f\|_{K + 2\rho_0 M}. & \end{aligned}$$

This proves the continuity of the semi-norms.

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To prove the converse, we consider a continuous semi-norm p on the inductive limit $\mathcal{O}(K)$. Let $V_j \in \mathcal{V}(E)$, $j \geqslant 1$, be a fundamental sequence of neighbourhoods of 0 in E satisfying $2V_{j+1} \subset V_j$.

(A) Let us prove that there are $\epsilon \in]0, \frac{1}{2}[$, an integer N_1 and a compact subset A_1 of V_1 such that

$$f \in \mathcal{O}(K+V_1), \quad \|f\|_{K+V_1} \leqslant 1 \quad \text{and} \quad \|D^k f\|_{K,A_1} \leqslant \epsilon k!, \quad \text{if } 0 \leqslant k < N_1$$

imply $p(f) < 1$.

Since by definition of the inductive limit the restriction of p to $\mathcal{O}(K+V_1)$ is continuous, there is $\epsilon \in]0, \frac{1}{2}[$ and a compact subset A_1 of V_1 such that

$$p(f) \leqslant \frac{1}{\epsilon}\|f\|_{K+A_1}, \quad f \in \mathcal{O}(K+V_1).$$

Since A_1 is compact, there is $\rho > 1$ such that $\rho A_1 \subset V_1$. Choose $N_1 > 0$ such that

$$\sum_{k=N_1}^{+\infty} \rho^{-k} \leqslant \epsilon/2.$$

Let $f \in \mathcal{O}(K+V_1)$ be such that $\|f\|_{K+V_1} \leqslant 1$ and

$$\|D^k f\|_{K,A_1} \leqslant \frac{\epsilon k!}{2N_1}, \quad \text{if } 0 \leqslant k < N_1.$$

By the Cauchy inequalities, we have

$$\|D^k f\|_{K,A_1} \leqslant \rho^{-k} k! \|f\|_{K+V_1} \leqslant \rho^{-k} k!$$

for any k since $\rho A_1 \subset V_1$. Using Taylor expansion, we get

$$\begin{aligned}\|f\|_{K+A_1} &= \sup_{x\in K} \sup_{h \in A_1} \left| \sum_{k=0}^{+\infty} \frac{1}{k!} D^k f(x) h^{(k)} \right| \\ &\leqslant \sum_{k=0}^{N_1-1} \frac{\epsilon}{2N_1} + \sum_{k=N_1}^{+\infty} \rho^{-k} \leqslant \epsilon.\end{aligned}$$

This proves the assertion.

(B) Let $N_0 = 0$ and $V_0 = 2V_1$. There is a strictly increasing sequence of integers $N_2, N_3, \dots,$ and a sequence of compact subsets $A_2, A_3, \dots,$ of E such that, for any $j \geqslant 1$, $A_j \subset 2V_{j-1}$ and

$$f \in \mathcal{O}(K+V_j), \quad \|f\|_{K+V_j} \leqslant j \quad \text{and} \quad \|f\|_{K,A_\ell,k} \leqslant \epsilon, \quad \text{if } N_{\ell-1} \leqslant k < N_\ell, \quad 1 \leqslant \ell \leqslant j$$

imply $p(f) < 1$.

The proof is by induction on j. If $j = 1$ this follows from (A). Assume that $N_1, \dots, N_{j-1}$ have been constructed. If no suitable N_j and A_j exist, then for every positive integer N

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and every compact subset A of V_j the set $e_{N,A}$ of the functions $f \in \mathcal{O}(K+V_j)$ such that $p(f) \geqslant 1$,

$$\|f\|_{K+V_j} \leqslant j, \quad \|f\|_{K,A_\ell,k} \leqslant \epsilon, \quad \text{if } N_{\ell-1} \leqslant k < N_\ell, \quad 1 \leqslant \ell < j,$$

and

$$\|f\|_{K,A,k} \leqslant \epsilon, \quad \text{if } N_{j-1} \leqslant k < N$$

is not empty. Since $\mathcal{O}(K+V_j)$ is a semi-Montel space and each finite intersection of such sets is also not empty, there is a function f in their intersection. We have $p(f) \geqslant 1$,

$$\|f\|_{K,A_\ell,k} \leqslant \epsilon, \quad \text{if } N_{\ell-1} \leqslant k < N_\ell, \quad 1 \leqslant \ell < j,$$

and $\|f\|_{K,A,k} \leqslant \epsilon$ for every k and every compact subset A of $2V_{j-1}$. Hence, using the same notation as before, though $2V_{j-1}$ is not compact, we get $\|f\|_{K,2V_{j-1},k} \leqslant \epsilon$ for any k.

Let us prove that f extends as a holomorphic function in $K + V_{j-1}$. Since

$$\|D^k f\|_{K,2V_{j-1}} \leqslant \epsilon k!$$

for every k, the Taylor expansion f_x of f at a point $x \in K$ converges in $x + 2V_{j-1}$. We have to prove that if $x, y \in K$, then f_x and f_y coincide in the intersection of $x + V_{j-1}$ and $y + V_{j-1}$.

Assume that this intersection is not empty and write $y = x + h$. Here $h \in 2V_{j-1}$. Let $\rho \in]0,1[$ such that $h \in 2\rho V_{j-1}$. Since $\|f\|_{K,2V_{j-1},k} \leqslant \epsilon$ for every k, we have

$$\left\| D^\ell f(x+h) - \sum_{j<k-\ell} D^{j+\ell} f(x).\frac{h^{(j)}}{j!} \right\|_{2V_{j-1}} \leqslant \epsilon \ell! \rho^{k-\ell}$$

if $0 \leqslant \ell < k$. For any ℓ, the right-hand side converges to 0 if k converges to $+\infty$. This proves that f_x and f_y coincide to infinite order at y. Hence they are equal in $(x + 2V_{j-1}) \cap (y + V_{j-1})$.

Moreover, using again Taylor expansion, we obtain

$$\begin{aligned} \|f\|_{K+V_{j-1}} &\leqslant \sup_{x\in K} \sup_{h \in V_{j-1}} \left| \sum_{k=0}^{+\infty} \frac{1}{k!} D^k f(x) h^{(k)} \right| \\ &\leqslant \sum_{k=0}^{+\infty} \|f\|_{K,V_{j-1},k} \leqslant \epsilon \sum_{k=0}^{+\infty} 2^{-k} \leqslant 2\epsilon \leqslant j-1. \end{aligned}$$

From the induction hypothesis we get $p(f) < 1$. This is an absurdity.

(C) Let $\epsilon_k = 2^{-j/2}$ if $N_{j-1} \leqslant k < N_j$. This sequence converges to 0. Define

$$M = \{0\} \cup \bigcup_{j=1}^{\infty} 2^{j/2} A_j.$$

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This is a compact subset of E since each A_j is compact,

$$A_j \subset 2V_{j-1} \subset V_{j-2} \subset \cdots \subset 2^{-[j/2]} V_{j-[j/2]-2} \subset 2^{-j/2} V_{j-[j/2]-3}$$

for j large enough and the V_j form a fundamental sequence of neighbourhoods of 0 in E. Here $[j/2]$ denotes the integer part of $j/2$.

If $f \in \mathcal{O}(K)$ and

$$\sup_{k \in \mathbb{N}} \epsilon_k^k \|f\|_{K,M,k} \leqslant \epsilon,$$

then $p(f) \leqslant 1$. Indeed, there is some j such that $f \in \mathcal{O}(K + V_j)$ and $\|f\|_{K+V_j} \leqslant j$. Moreover, if $N_{\ell-1} \leqslant k < N_\ell$ and $1 \leqslant \ell \leqslant j$ we have

$$\|f\|_{K,A_\ell,k} \leqslant 2^{-k\ell/2} \|f\|_{K,M,k} = \epsilon_k^k \|f\|_{K,M,k} \leqslant \epsilon,$$

since $A_\ell \subset 2^{-\ell/2} M$ and $\|f\|_{K,\lambda M,k} = \lambda^k \|f\|_{K,M,k}$ if $\lambda > 0$. By (B) this proves that $p(f) \leqslant 1$. The first part of the theorem is proved.

(D) If K is locally connected, we proceed along the same lines with the following modifications. Before (A), we choose the V_j in such a way that every function $f \in \mathcal{O}(K)$ satisfying

$$\|D^k f\|_{K,V_j} \leqslant k!$$

for every k can be extended as a holomorphic function in $K + V_{j+1}$. This can be done using Lemma 3.1. In (B) and (C), we replace $\|f\|_{K,A_\ell,k}$ by $\|D^k f\|_{K,A_\ell}/k!$ everywhere. Moreover, in (B) we replace the condition $A_j \subset 2V_{j-1}$ by $A_j \subset V_{j-2}$.

In (C), the extension of f as a holomorphic function in $K + V_{j-1}$ follows from the construction of the V_j and not from the argument presented there. □

Proof of Theorem 2.2. Let ϵ_k be a decreasing sequence of real numbers satisfying $\epsilon_k = 2\mathrm{e}/\ln(\ln(k))$ for $k \geqslant 3$ and let K be a non-empty compact subset of Ω. Let us show that the semi-norm

$$p(f) = \sup_{k \in \mathbb{N}} \epsilon_k^k \frac{\|D^k f\|_K}{k!}$$

is not continuous for the topology of the projective limit of the spaces $C^{(L)}(\Omega)$. Consider the functions $f_m(x) = \mathrm{e}^{\mathrm{i}mx_1}$. Choosing k as the integer part of $m/\ln(\ln(m))$, we obtain

$$\begin{aligned} p(f_m) &\geqslant \sup_{k \in \mathbb{N}} \frac{\epsilon_k^k}{k^k} m^k \\ &\geqslant \left(\frac{2\mathrm{e}\ln(\ln(m))}{\ln(\ln(m)) - \ln(\ln(\ln(m)))} \right)^{(m/\ln(\ln(m)))-1} \\ &\geqslant \mathrm{e}^{(m/\ln(\ln(m)))-1} \end{aligned}$$

if m is large enough.

If p is continuous, it follows that there are $C, N > 0$ and a decreasing summable sequence $(\eta_k)_{k \geqslant 0}$ such that

$$\mathrm{e}^{m/\ln(\ln(m))} \leqslant C \sup_{k \in \mathbb{N}} \eta_k^k m^k$$

if $m \geqslant N$. Multiplying the sequence η_k by a constant, we can assume that $C = 1$. There is a sequence k_m such that

$$\frac{m}{\ln(\ln(m))} \leqslant k_m \ln(m\eta_{k_m})$$

if m is large enough. It follows that the sequence k_m converges to ∞. Moreover, $\eta_{k_m} \geqslant 1/m$ and

$$\frac{m}{\ln(\ln(m))} \leqslant k_m \ln(m)$$

for m large. Denote by m_j the smallest integer such that $j \leqslant k_{m_j}$. The sequence m_j converges also to $+\infty$. If j is large enough, we have

$$\frac{m_j - 1}{\ln(m_j - 1)\ln(\ln(m_j - 1))} \leqslant k_{m_j - 1} < j \leqslant \frac{J}{\ln(J)\ln(\ln(J))}$$

with $J = 2j\ln(j)\ln(\ln(j))$. The function $x/(\ln(x)\ln(\ln(x)))$ is increasing if x is large enough. It follows that $m_j \leqslant 1 + 2j\ln(j)\ln(\ln(j))$. Therefore,

$$\eta_j \geqslant \eta_{k_{m_j}} \geqslant \frac{1}{m_j} \geqslant \frac{1}{1 + 2j\ln(j)\ln(\ln(j))}.$$

This is an absurdity since the sequence η_j is summable. □

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