A MINIMUM PROBLEM FOR THE EPSTEIN ZETA-FUNCTION

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1. In some recent work by D. G. Kendall and the author † on the number of points of a lattice which lie in a random circle the mean value of the variance emerged as a constant multiple of the value of the Epstein zeta-function $Z(s)$ associated with the lattice, taken at the point $s=\frac{3}{2}$. Because of the connexion with the problems of closest packing and covering it seemed likely that the minimum value of $Z(\frac{3}{2})$ would be attained for the hexagonal lattice ; it is the purpose of this paper to prove this and to extend the result to other real values of the variable s.

Let

$$h(m, n)=\alpha m^2+2\delta mn+\beta n^2 \qquad \text{.........(1)}$$

be a positive definite quadratic form of determinant $\alpha\beta-\delta^2$ equal to unity. In particular, the special forms $Q(m, n)$ and $q(m, n)$ are defined as follows :

$$Q(m, n)=\tfrac{2}{3}\sqrt{3}\, q(m, n)=\tfrac{2}{3}\sqrt{3}\,(m^2+mn+n^2). \qquad \text{.........(2)}$$

We consider the Epstein zeta-function ‡

$$Z_h(s)=\sum_{m=-\infty}^{\infty}\ \sideset{}{'}\sum_{n=-\infty}^{\infty} \{h(m, n)\}^{-s}, \qquad \text{.........(3)}$$

where the dash denotes, as always, the exclusion of the term $m=n=0$. This double series is absolutely convergent for $\Re s>1$. The function $Z_h(s)$ can be continued over the whole s-plane and is regular except for a simple pole of residue 1 at $s=1$.

It is easily shown that in the particular case $h(m, n)=Q(m, n)$

$$Z_Q(s)=6\,(\tfrac{1}{2}\sqrt{3})^s\,\zeta(s)L(s) \qquad \text{.........(4)}$$

where $\zeta(s)$ is the Riemann zeta-function and $L(s)$ is the Dirichlet L-series

$$L(s)=1-\frac{1}{2^s}+\frac{1}{4^s}-\frac{1}{5^s}+\frac{1}{7^s}-\frac{1}{8^s}+\dots .$$

We shall, throughout the paper, be concerned with real values of s in the half-plane of convergence. We prove the

THEOREM. *For all* $s\geqslant 1{\cdot}035$, $Z_h(s)\geqslant Z_Q(s)$. *Equality occurs only when* h *and* Q *are equivalent forms.*

In view of the closeness of the lower bound 1·035 to unity and the fact that all the functions $Z_h(s)$ are asymptotically equal as $s\to 1+0$ it seems very likely that the conclusion of the theorem remains valid for all $s>1$, but I have been unable to prove this. Further, I have not been able to find any single method which is applicable to the whole range $s\geqslant 1{\cdot}035$, as will be seen from the proof.

† " On the number of points of a given lattice in a random hypersphere." (To appear in the Quarterly Journal.)

‡ For the general theory of $Z_h(s)$ see Max Deuring, " Zetafunktionen quadratischer Formen " *J. reine angew. Math.* 172 (1935), 226–252.

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2. In this section we introduce some new notation and prove 10 lemmas. In the first place, as is well known, we may, by means of a unimodular transformation on the variables m and n, assume that α is the minimum of the form for all pairs of integers m, n, not both zero, and that

$$0 \leqslant 2\delta \leqslant \alpha \leqslant \beta, \quad 1 \leqslant \alpha\beta \leqslant \tfrac{4}{3}. \qquad \text{.......(5)}$$

With these restrictions on the coefficients, the Theorem states that, for $s \geqslant 1{\cdot}035$, $Z_h(s) \geqslant Z_Q(s)$ and that equality occurs only for $h = Q$.

We now introduce some symmetry by defining

$$\gamma = \alpha + \beta - 2\delta = \alpha + \beta - 2\sqrt{(\alpha\beta - 1)}, \qquad \text{.......(6)}$$

so that

$$2(\beta\gamma + \gamma\alpha + \alpha\beta) - (\alpha^2 + \beta^2 + \gamma^2) = 4, \qquad \text{.......(7)}$$

and, by (5),

$$0 < \alpha \leqslant \beta \leqslant \gamma \leqslant \alpha + \beta. \qquad \text{.......(8)}$$

Also define

$$\left.\begin{aligned} f(m, n) &= \beta m^2 + (\beta + \gamma - \alpha) mn + \gamma n^2 = h(-n, m+n) \\ g(m, n) &= \gamma m^2 + (\gamma + \alpha - \beta) mn + \alpha n^2 = h(m+n, -m) \end{aligned}\right\}. \qquad \text{.......(9)}$$

Then we find that

$$2(gh + hf + fg) - (f^2 + g^2 + h^2) = 4q^2, \qquad \text{.......(10)}$$

and deduce that

$$f = g + h \pm 2\sqrt{(gh - q^2)}, \quad g = h + f \pm 2\sqrt{(hf - q^2)}, \quad h = f + g \pm 2\sqrt{(fg - q^2)}. \qquad \text{.......(11)}$$

Lemma 1. *If* $\alpha + \beta + \gamma \leqslant 4$, *the minus sign must be taken in each of the three relations* (11).

Proof. We prove that $f + g - h$ cannot take negative values. We have

$$f + g - h = (\beta + \gamma - \alpha) m^2 + (3\gamma - \alpha - \beta) mn + (\gamma + \alpha - \beta) n^2.$$

By (8), the coefficients of m^2 and n^2 are non-negative and the determinant of the form is, by (7),

$$(\beta + \gamma - \alpha)(\gamma + \alpha - \beta) - \tfrac{1}{4}(3\gamma - \alpha - \beta)^2 = 4 - \tfrac{1}{4}(\alpha + \beta + \gamma)^2 \geqslant 0.$$

In a similar way it can be shown that $g + h - f$ and $h + f - g$ are not negative.

We now introduce new variables x, y and z defined by

$$x = \delta/\alpha, \quad y = 1/\alpha, \quad z = x + iy, \qquad \text{.......(12)}$$

so that we have

$$h(m, n) = \frac{|m + nz|^2}{y},$$

and the conditions (8) imply that z belongs to the modular region $\mathfrak{D}$ defined by

$$0 \leqslant x \leqslant \tfrac{1}{2}, \quad y > 0, \quad x^2 + y^2 \geqslant 1. \qquad \text{.......(13)}$$

Put

$$\phi(x, y;\ s) = y^s \left\{1 + \frac{1}{(x^2 + y^2)^s} + \frac{1}{\{(x-1)^2 + y^2\}^s}\right\} = \frac{1}{\alpha^s} + \frac{1}{\beta^s} + \frac{1}{\gamma^s}. \qquad \text{.......(14)}$$

We prove

Lemma 2. *If* $s > 2$ *we have*

$$\frac{1}{\alpha^s} + \frac{1}{\beta^s} + \frac{1}{\gamma^s} \geqslant 3\left(\tfrac{1}{2}\sqrt{3}\right)^s.$$

Equality holds only when $\alpha = \beta = \gamma = 2/\sqrt{3}$.

Proof. By Hölder's inequality, since $s>2$,

$$\phi(x, y;\ 2) \leqslant \{\phi(x, y;\ s)\}^{2/s}\, 3^{1-2/s},$$

and equality can only occur when $x=\frac{1}{2}$, $y=\frac{1}{2}\sqrt{3}$. Accordingly, it is enough to prove that $\phi(x, y;\ 2) \geqslant \frac{9}{4}$ in $\mathfrak{D}$ and that equality occurs only for $x=\frac{1}{2}$, $y=\frac{1}{2}\sqrt{3}$ and $x=0$, $y=1$.

Write $y^2=\eta$ and put

$$\psi(x, \eta)=\phi(x, y;\ 2)=\eta\left\{1+\frac{1}{(x^2+\eta)^2}+\frac{1}{\{(1-x)^2+\eta\}^2}\right\}.$$

Then

$$\frac{\partial\psi}{\partial x}=-4\eta\left[\frac{x}{(x^2+\eta)^3}-\frac{1-x}{\{(1-x)^2+\eta\}^3}\right],$$

and is positive if

$$\frac{x}{1-x}<\left\{\frac{x^2+\eta}{(1-x)^2+\eta}\right\}^3.$$

Since $\eta \geqslant 1-x^2$ in $\mathfrak{D}$, $\dfrac{\partial\psi}{\partial x}>0$ if

$$\frac{x}{1-x}<\frac{1}{8(1-x)^3},$$

i.e. if $0 \leqslant x < \frac{1}{4}(3-\sqrt{5})=x_0$, say.

Suppose now that $x \geqslant x_0$. We have

$$\frac{\partial\psi}{\partial\eta}=1+\frac{x^2-\eta}{(x^2+\eta)^3}+\frac{(1-x)^2-\eta}{\{(1-x)^2+\eta\}^3},$$

and

$$\frac{1}{2}\frac{\partial^2\psi}{\partial\eta^2}=\frac{\eta-2x^2}{(x^2+\eta)^4}+\frac{\eta-2(1-x)^2}{\{(1-x)^2+\eta\}^4}.$$

Now, by evaluating the derivatives at $\eta=a+b$ it is easily verified that

$$(\eta-2a)(\eta+b)^4+(\eta-2b)(\eta+a)^4$$

is not negative for $\eta \geqslant a+b$, and since $y^2 \geqslant x^2+(1-x)^2$ in $\mathfrak{D}$ we deduce that $\partial^2\psi/\partial\eta^2 \geqslant 0$ in $\mathfrak{D}$. Hence, since $\eta \geqslant 1-x^2$ in $\mathfrak{D}$,

$$\frac{1}{2y}\frac{\partial\phi}{\partial y}=\frac{\partial\psi}{\partial\eta} \geqslant 2x^2-\frac{x}{4(1-x)^2} \geqslant 0,$$

since $x_0 \leqslant x \leqslant \frac{1}{2}$.

Accordingly we deduce that $\psi(x, \eta)$ attains its minimum value in $\mathfrak{D}$ either on the line $x=0$, $y \geqslant 1$ or on the arc $x^2+y^2=1$, $0 \leqslant x \leqslant \frac{1}{2}$. Now, if $x=0$, $y \geqslant 1$ we have

$$\frac{\partial\psi}{\partial\eta}=\frac{\eta-1}{\eta^2(\eta+1)^3}\{(\eta+1)^4-\eta^2\} \geqslant 0,$$

so that $\phi(0, y;\ 2) \geqslant \phi(0, 1;\ 2)=\frac{9}{4}$.

Finally, if $x^2+y^2=1$, $0 \leqslant x \leqslant \frac{1}{2}$ we put

$$x=\frac{v-1}{v+1},\quad y^2=\frac{4v}{(v+1)^2},$$

so that $1 \leqslant v \leqslant 3$, and obtain

$$\phi(x, y;\ 2)=\tfrac{1}{4}v+\frac{8v}{(v+1)^2}=\phi(v),$$

say. Since

$$\phi'(v) = \frac{(v-3)(v^2+6v-11)}{4(v+1)^3},$$

we deduce that as v increases from 1 to 3, $\phi(v)$ increases from $\frac{9}{4}$ to a maximum at $v = 2\sqrt{5}-3$ and then decreases to a minimum value of $\frac{9}{4}$ at $v=3$. This completes the proof of Lemma 2.

We note that all that has been assumed in the proof of Lemma 2 is that $s>2$ and that α, β and γ satisfy (7) and the inequalities (8). We use this fact to prove

LEMMA 3. *If $\alpha+\beta+\gamma \leqslant 4$ and $s>2$ then, for any values of m and n, not both zero,*

$$\frac{1}{\{f(m,n)\}^s} + \frac{1}{\{g(m,n)\}^s} + \frac{1}{\{h(m,n)\}^s} \geqslant 3\left(\frac{\sqrt{3}}{2q}\right)^s.$$

Equality occurs only when $\alpha=\beta=\gamma=2/\sqrt{3}$.

Proof. Denote by a, b and c the three quantities f/q, g/q, h/q arranged in ascending order of magnitude. Then, by (10) and Lemma 1,

$$2(bc+ca+ab)-(a^2+b^2+c^2)=4,$$

and

$$0<a\leqslant b\leqslant c\leqslant a+b,$$

so that, by Lemma 2,

$$a^{-s}+b^{-s}+c^{-s} \geqslant 3(\tfrac{1}{2}\sqrt{3})^s.$$

Equality occurs only when $f=g=h=2q/\sqrt{3}$, and then, since $f+g+h=(\alpha+\beta+\gamma)q$, we have $\alpha+\beta+\gamma=2\sqrt{3}$. I.e., in terms of x and y, $(x-\frac{1}{2})^2+(y-\frac{1}{2}\sqrt{3})^2=0$, so that $x=\frac{1}{2}$, $y=\frac{1}{2}\sqrt{3}$ and therefore $\alpha=\beta=\gamma=2/\sqrt{3}$.

LEMMA 4. *If $\alpha+\beta+\gamma \geqslant 4$ and $s>2$, then*

$$\alpha^{-s}+\beta^{-s}+\gamma^{-s} \geqslant 2+2^{-s}.$$

Proof. In terms of the point z, defined by (12), we have to consider the region $0 \leqslant x \leqslant \frac{1}{2}$, $(x-\frac{1}{2})^2+(y-1)^2 \geqslant \frac{1}{4}$, which we denote by $\mathfrak{D}^*$.

Write

$$r_1=(x^2+y^2)^{\frac{1}{2}}, \quad r_2=\{(x-1)^2+y^2\}^{\frac{1}{2}},$$

so that $y=\frac{1}{2}\{2(r_1^2+r_2^2)-(r_2^2-r_1^2)^2-1\}^{\frac{1}{2}}$, and $r_2 \geqslant r_1 \geqslant 1$ by (13). Also $\mathfrak{D}^* \subseteq \mathfrak{D}^{**}$ where $\mathfrak{D}^{**}$ is the part of $\mathfrak{D}$ in which $r_2 \geqslant \sqrt{2}$. Then

$$\phi(x,y;\ s)=2^{-s}\{2(r_1^2+r_2^2)-(r_2^2-r_1^2)^2-1\}^{\frac{1}{2}s}(1+r_1^{-2s}+r_2^{-2s}),$$

and it is easily verified that, for r_2 constant,

$$\frac{\partial\phi}{\partial r_1}=2^{1-s}sr_1\{2(r_1^2+r_2^2)-(r_2^2-r_1^2)^2-1\}^{\frac{1}{2}s-1}\,\omega,$$

where

$$\omega=(r_2^2-r_1^2)(r_2^2-2)r_1^{-2s-2}+(1+r_2^2-r_1^2)(1+r_2^{-2s})-3r_1^{-2s}+r_1^{-2s-2}.$$

Now

$$\frac{1}{2r_2}\frac{\partial}{\partial r_2}(1+r_2^2-r_1^2)(1+r_2^{-2s})=r_2^{-2s-2}\{[r_2^{2s}-(s-1)]r_2^2+s(r_1^2-1)\}>0,$$

since $r_2^2 \geqslant 2$, $r_1^2 \geqslant 1$ and $2^s>s-1$. Thus, since $r_2 \geqslant r_1$ in $\mathfrak{D}^{**}$ we have

$$(1+r_2^2-r_1^2)(1+r_2^{-2s}) \geqslant 1+r_1^{-2s}$$

and therefore

$$\omega \geqslant 1+r_1^{-2s}-3r_1^{-2s}+r_1^{-2s-2} \geqslant 1-2r_1^{-2s}+r_1^{-4s} \geqslant 0.$$

Hence $\partial\phi/\partial r_1 \geqslant 0$ in $\mathfrak{D}^{**}$ and it follows that $\phi(x, y; s)$ attains its minimum in $\mathfrak{D}^*$ on the line $x=0$, $y \geqslant 1$. Since

$$\frac{\partial}{\partial y}\phi(0, y; s) = sy^{-s-1}(y^2-1)\left\{\frac{y^{2s}-1}{y^2-1} - \frac{y^{2s}}{(y^2+1)^{s+1}}\right\}$$

$$\geqslant sy^{-s-1}(y^2-1)(s-1) \geqslant 0,$$

we have

$$\phi(0, y; s) \geqslant \phi(0, 1; s) = 2 + 2^{-s},$$

which completes the proof of the Lemma.

LEMMA 5. *The real function $f(t)$ is defined for all real t and possesses the following properties:* (i) *$f(t)$, $f'(t)$ and $f''(t)$ are continuous for all t,* (ii) *the integrals $\int_{-\infty}^{\infty} f(t)\,dt$ and $\int_{-\infty}^{\infty} |f''(t)|\,dt$ converge,* (iii) *$f(t)$ and $f'(t)$ tend to zero as $t \to \pm\infty$,* (iv) *$f''(t)$ is negative for $t_1 < t < t_2$, but otherwise non-negative. Then $S = \sum_{n=-\infty}^{\infty} f(n)$ is convergent and, for some real ϑ satisfying $|\vartheta| \leqslant 1$,*

$$S = \int_{-\infty}^{\infty} f(t)\,dt + \tfrac{1}{8}\vartheta\{f'(t_1) - f'(t_2)\}.$$

Proof. We use the Euler-Maclaurin sum-formula in the form:

$$\sum_{n=-M}^{N} f(n) = \int_{-M}^{N} f(t)\,dt + \tfrac{1}{2}\{f(N) + f(-M)\} + \int_{-M}^{N} r_1(t) f''(t)\,dt,$$

where $r(t) = t - [t] - \frac{1}{2}$ and

$$r_1(t) = -\int_0^t r(u)\,du.$$

It is easily shown that $0 \leqslant r_1(t) \leqslant \frac{1}{8}$ for all t, and we deduce that the infinite series converges and that

$$S - \int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{\infty} r_1(t) f''(t)\,dt = R,$$

say. From condition (iv) it follows that

$$\tfrac{1}{8}\{f'(t_2) - f'(t_1)\} = \tfrac{1}{8}\int_{t_1}^{t_2} f''(t)\,dt \leqslant R \leqslant \tfrac{1}{8}\left(\int_{-\infty}^{t_1} + \int_{t_2}^{\infty}\right) f''(t)\,dt = \tfrac{1}{8}\{f'(t_1) - f'(t_2)\},$$

which completes the proof.

In the next two lemmas we are concerned with the function $Z_h(s)$ expressed in terms of the variables x and y, and write

$$G(x, y) = Z_h(s) = \sum_m \sum_n{}' y^s |mz + n|^{-2s}. \quad \text{.................................(15)}$$

LEMMA 6.† *For $s > 1$,*

$$\frac{1}{2s} G_y(x, y) = y^{s-1}\zeta(2s) - y^{-s}(s-1)\zeta(2s-1)\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s+1)} + R',$$

where

$$|R'| \leqslant \tfrac{1}{2} y^{-s-2}\zeta(2s+1)\left\{s\frac{(2s+1)^{s+\frac{1}{2}}}{(2s+2)^{s+1}} + 2(s+1)\frac{(2s+3)^{s+\frac{3}{2}}}{(2s+4)^{s+2}}\right\}.$$

† Deuring (loc. cit.) gives a somewhat similar formula for the function $G(x, y)$, but with a different form of remainder and without the explicit numerical constants which are essential for our purpose.

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Proof. We have, because of uniform convergence,

$$\frac{1}{2s} G_y(x, y) = \frac{1}{2s} \sum_{m=-\infty} \sum_{n=-\infty}' \frac{\partial}{\partial y} \left\{ \frac{y}{(mx+n)^2 + m^2y^2} \right\}^s$$

$$= \tfrac{1}{2} \sum_m \sum_n' y^{s-1} \frac{|mz+n|^2 - 2m^2y^2}{|mz+n|^{2s+2}}$$

$$= y^{s-1} \zeta(2s) - \sum_{m=1}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} \frac{2m^2 y^{s+1}}{|mz+n|^{2s+2}} - \sum_{n=-\infty}^{\infty} \frac{y^{s-1}}{|mz+n|^{2s}} \right\} \quad \ldots\ldots\ldots\ldots\ldots (16)$$

We now apply Lemma 5 to each of the two inner sums. Put

$$f(t) = \frac{y^\rho}{|mz+t|^{2\rho}},$$

where $\rho = s$ or $s+1$. Then

$$f'(t) = -\frac{2\rho y^\rho (mx+t)}{|mz+t|^{2\rho+2}}, \quad f''(t) = \frac{2\rho y^\rho \{(2\rho+1)(mx+t)^2 - m^2y^2\}}{|mz+t|^{2\rho+4}},$$

so that the conditions are satisfied with

$$mx + t_1 = -\frac{my}{\sqrt{(2\rho+1)}}, \quad mx + t_2 = \frac{my}{\sqrt{(2\rho+1)}}.$$

We deduce that

$$\sum_{-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} \frac{y^\rho\, dt}{|mz+t|^{2\rho}} + \frac{\vartheta\rho}{2m^{2\rho+1}y^{\rho+1}} \frac{(2\rho+1)^{\rho+\frac{1}{2}}}{(2\rho+2)^{\rho+1}},$$

and, since the integral is

$$\frac{y^{1-\rho}}{m^{2\rho-1}} \frac{\Gamma(\frac{1}{2})\Gamma(\rho-\frac{1}{2})}{\Gamma(\rho)},$$

the result follows from (16) on taking $\rho = s+1$ and s in the two parts of the sum over n.

LEMMA 7. *If* $s > 1$ *and* $y \geqslant \frac{3}{2}$ *then* $G_y(x, y) > 0$.

Proof. By Lemma 6,

$$\frac{1}{2s} G_y(x, y) \geqslant y^{s-1} \zeta(2s) - y^{-s} \phi_1(s) - y^{s-2} \phi_2(s),$$

where

$$\phi_1(s) = \frac{\Gamma(\frac{1}{2})\Gamma(s+\frac{1}{2})}{\Gamma(s+1)} \cdot \frac{2(s-1)\zeta(2s-1)}{2s-1},$$

and

$$\phi_2(s) = y^{-2s} \zeta(2s+1) \left\{ \tfrac{1}{2}s \frac{(2s+1)^{s+\frac{1}{2}}}{(2s+2)^{s+1}} + (s+1) \frac{(2s+3)^{s+\frac{3}{2}}}{(2s+4)^{s+2}} \right\} = \phi_3(s) + \phi_4(s),$$

say. Since

$$\zeta(2s-1) \leqslant 1 + \int_1^\infty u^{1-2s}\, du = \frac{2s-1}{2(s-1)},$$

and

$$\frac{\Gamma(\frac{1}{2})\Gamma(s+\frac{1}{2})}{\Gamma(s+1)} = \int_{-\infty}^{\infty} \frac{dt}{(t^2+1)^{s+1}} < \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(2)} = \tfrac{1}{2}\pi,$$

it follows that $\phi_1(s) < \frac{1}{2}\pi$.

Also, it is easily proved by differentiation that both $\phi_3(s)$ and $\phi_4(s)$ are decreasing functions of s for $y \geqslant \frac{3}{2}$. Thus we have, since $\phi_2(1) \leqslant 0{\cdot}8174\, y^{-2}$,

$$\frac{1}{2s} G_y(x, y) \geqslant y^{s-1} \{\zeta(2s) - \tfrac{1}{2}\pi\, y^{1-2s} - 0{\cdot}8174\, y^{-3}\} = y^{s-1} \phi_3(s, y),$$

say. In order to show that $\phi_3(s, y)>0$ for $y\geqslant\frac{3}{2}$ it is enough to prove that $\phi_3(s, \frac{3}{2})>0$. Now, for $1\leqslant s\leqslant 1{\cdot}3$ we have

$$\phi_3(s, \tfrac{3}{2})\geqslant\zeta(2{\cdot}6)-\tfrac{1}{2}\pi(\tfrac{3}{2})^{-1}-0{\cdot}8174(\tfrac{3}{2})^{-3}>0{\cdot}016>0,$$

for $1{\cdot}3\leqslant s\leqslant 2$ we have

$$\phi_3(s, \tfrac{3}{2})\geqslant\zeta(4)-\tfrac{1}{2}\pi(\tfrac{3}{2})^{-1\cdot 6}-0{\cdot}8174(\tfrac{3}{2})^{-3}>0{\cdot}019>0,$$

and for $s\geqslant 2$ we have

$$\phi_3(s, \tfrac{3}{2})\geqslant 1-(\tfrac{1}{2}\pi+0{\cdot}8174)(\tfrac{3}{2})^{-3}>0{\cdot}292>0.$$

This completes the proof.

LEMMA 8. *Let h and μ be fixed positive numbers and suppose that the function $H(u)$ possesses continuous derivatives $H'(u)$, $H''(u)$ for $u\geqslant u_0$ and that $H''(u)>\mu^2H(u)$ for all $u>u_0$. Then*

$$H_1(u)\equiv H(u+2h)-2\cosh\mu h\, H(u+h)+H(u)>0,$$

for all $u\geqslant u_0$.

Proof. This follows from the formula

$$H_1(u)=e^{-\mu u}\int_u^{u+h}e^{2\mu v}\,dv\int_v^{v+h}e^{-\mu w}\{H''(w)-\mu^2H(w)\}\,dw,$$

which is easily checked by integration.

For the remainder of the paper we write

$$H(u)=u^{s+\frac{1}{2}}K_{s-\frac{1}{2}}(u), \qquad\qquad (17)$$

where $K_{s-\frac{1}{2}}(u)$ is a Bessel function.

LEMMA 9. *If $0\leqslant\mu<1$ then $H''(u)>\mu^2H(u)$ provided either that* (i) *$u(1-\mu^2)\geqslant 2s$ where $s\geqslant 1$ or* (ii) *$(1-\mu^2)u\geqslant 3(1-\mu^2)+2s\mu^2$, when $1\leqslant s\leqslant 2$.*

Proof. By using the relations (1) and (5) of § 3.71 of G. N. Watson's *Bessel Functions*,† we obtain

$$H''(u)-\mu^2H(u)=(1-\mu^2)u^{s+\frac{1}{2}}K_{s-\frac{1}{2}}(u)-2su^{s-\frac{1}{2}}K_{s-\frac{3}{2}}(u), \qquad (18)$$

and since

$$K_\nu(u)=\int_0^\infty e^{-u\cosh t}\cosh \nu t\,dt,$$

and $\cosh(s-\frac{1}{2})t\geqslant\cosh(s-\frac{3}{2})t$, the first part of the Lemma follows.

Suppose now that $1\leqslant s\leqslant 2$. Then if we substitute for $K_{s-\frac{1}{2}}(u)$ in terms of $K_{s-\frac{3}{2}}(u)$ and $K_{s-\frac{5}{2}}(u)$ in (18) and use the fact that $K_\nu(z)=K_{-\nu}(z)$, we get

$$H''(u)-\mu^2H(u)=(1-\mu^2)u^{s+\frac{1}{2}}K_{\frac{5}{2}-s}(u)-\{3(1-\mu^2)+2s\mu^2\}u^{s-\frac{1}{2}}K_{\frac{3}{2}-s}(u),$$

and (ii) follows from this in a similar manner.

LEMMA 10. *We have $H''(u)>\mu^2H(u)$ for all $u\geqslant u_0$ in the following cases:* (i) $u_0=\frac{6}{5}\pi r$, $\mu=\mu_r=\dfrac{r}{u}\cosh^{-1}(1+2^{1-2s})$ $(1\leqslant s\leqslant 2;\ \ r=1, 2)$, (ii) $u_0=\frac{6}{5}\pi r$, $\mu=\mu_r=\dfrac{r}{u}\cosh^{-1}\{\zeta(2s-1)\}$ $(1{\cdot}035\leqslant s\leqslant 2;\ r=3, 4, 5, \ldots)$, (iii) $u_0=2\pi$, $\mu=\dfrac{1}{u}\cosh^{-1}\{\zeta(2s-1)\}$ $(2\leqslant s\leqslant 3)$.

Proof. (i) Put $a=\frac{6}{5}\pi$. We suppose first that $1\leqslant s\leqslant\frac{3}{2}$. By Lemma 9 (i), it suffices to show that

$$a-\frac{1}{a}\{\cosh^{-1}(1+2^{1-2s})\}^2\geqslant 2s,$$

since $u(1-\mu_r^2)$ is an increasing function of u. This is true since

$$\cosh\{a(a-2s)\}^{\frac{1}{2}}\geqslant\cosh\{a(a-3)\}^{\frac{1}{2}}>2{\cdot}838>\tfrac{3}{2}\geqslant 1+2^{1-2s}.$$

† Cambridge, 1922.

Suppose next that $\frac{3}{2} \leqslant s \leqslant 2$. Since $(1-\mu_r^2)(u-3)-2s\mu_r^2$ increases with u, it is enough, by Lemma 9 (ii), to show that

$$a^2(a-3)-(a-3+2s)\{\cosh^{-1}(1+2^{1-2s})\}^2 \geqslant 0,$$

i.e. that

$$1+2^{1-2s} \leqslant \cosh\left\{a\left(\frac{a-3}{a-3+2s}\right)^{\frac{1}{2}}\right\}.$$

This follows since $1+2^{1-2s} \leqslant \frac{5}{4} < 2{\cdot}383 < \cosh\left\{a\left(\frac{a-3}{a+1}\right)^{\frac{1}{2}}\right\} \leqslant \cosh\left\{a\left(\frac{a-3}{a-3+2s}\right)^{\frac{1}{2}}\right\}$.

(ii) By Lemma 9 (i) we have to show that

$$ar\left\{1-\frac{1}{a^2}[\cosh^{-1}(\zeta(2s-1))]^2\right\} \geqslant 2s$$

for $r=3, 4, 5, \ldots$, when $1{\cdot}035 \leqslant s \leqslant 2$, i.e. that

$$\zeta(2s-1) \leqslant \cosh\{a(a-\tfrac{2}{3}s)\}^{\frac{1}{2}}.$$

If $\frac{3}{2} \leqslant s \leqslant 2$, $\zeta(2s-1) \leqslant \zeta(2) < \cosh\{a(a-\frac{4}{3})\}^{\frac{1}{2}} \leqslant \cosh\{a(a-\frac{2}{3}s)\}^{\frac{1}{2}}$. Hence we need only prove that, if $1{\cdot}035 \leqslant s \leqslant \frac{3}{2}$ then

$$g(s) = \log\cosh\{a(a-\tfrac{2}{3}s)\}^{\frac{1}{2}} - \log\zeta(2s-1) \geqslant 0.$$

Now, if $1 < s \leqslant \frac{3}{2}$,

$$g'(s) = -\tfrac{1}{3}\left\{\frac{a}{a-\frac{2}{3}s}\right\}^{\frac{1}{2}}\tanh\{a(a-\tfrac{2}{3}s)\}^{\frac{1}{2}} - 2\frac{\zeta'}{\zeta}(2s-1)$$

$$> -\tfrac{1}{3}\left(\frac{a}{a-1}\right)^{\frac{1}{2}}\tanh\{a(a-\tfrac{2}{3})\}^{\frac{1}{2}} - 2\frac{\zeta'}{\zeta}(2) > 0{\cdot}751 > 0,$$

since $\frac{\zeta'}{\zeta}(2) = -0{\cdot}5699610$. Hence $g(s)$ is an increasing function for $1 < s \leqslant \frac{3}{2}$, and since $g(\frac{3}{2}) > 0$ it follows that there exists a unique s_0 such that $1 < s_0 < \frac{3}{2}$ and $g(s_0) = 0$. By using Gram's tables of $(s-1)\zeta(s)$ we find that $1{\cdot}03 < s_0 < 1{\cdot}035$, and this completes the proof.

(iii) By Lemma 9 (i) we have to show that

$$\zeta(2s-1) \leqslant \cosh\{4\pi(\pi-s)\}^{\frac{1}{2}}$$

for $2 \leqslant s \leqslant 3$. This is true since $\zeta(3) < \cosh\{4\pi(\pi-3)\}^{\frac{1}{2}}$.

3. We now prove the Theorem for $s \geqslant 3$.

Suppose first that $\alpha+\beta+\gamma \leqslant 4$. Then we have, by (9) and Lemma 3,

$$\begin{aligned} Z_h(s) &= \tfrac{1}{3}\{Z_f(s)+Z_g(s)+Z_h(s)\} \\ &= \tfrac{1}{3}\sum_m\sum_n{}'\{f^{-s}+g^{-s}+h^{-s}\} \\ &\geqslant (\tfrac{1}{2}\sqrt{3})^s\sum_m\sum_n{}' q^{-s} = Z_Q(s), \end{aligned}$$

equality occurring only when $h=Q$. To complete the proof we have therefore to consider the case $\alpha+\beta+\gamma \geqslant 4$. Now we have, by (14) and (15)

$$Z_h(s) = y^s\sum_m\sum_n{}'\,|mz+n|^{-2s} > 2\zeta(2s)\phi(x, y; s),$$

since the last expression is the part of the double sum corresponding to the terms

$$(m, n) = (0, \pm r), \quad (\pm r, 0), \quad (\pm r, \mp r)$$

for $r = 1, 2, 3, \ldots$.

By Lemma 4 and (4) it remains to prove that

$$F(s)=2+2^{-s}-3(\tfrac{1}{2}\sqrt{3})^s\psi(s)\geqslant 0,$$

where

$$\psi(s)=\frac{\zeta(s)L(s)}{\zeta(2s)}=(1+3^{-s})\prod_{p\equiv 1 \,(\mathrm{mod}\, 3)}\left(\frac{1+p^{-s}}{1-p^{-s}}\right).$$

Now it is easily shown by the method described in the appendix to a paper by D. G. Kendall † that $L(3)=0{\cdot}8840238$ and so $F(3)>0{\cdot}0896>0$. Also, by considering the infinite product, we see that $\psi(s)-1$ is a decreasing function of s, and since $2^{-s}-3(\frac{1}{2}\sqrt{3})^s$ is an increasing function it follows that $F(s)$ is an increasing function and is therefore positive.

4. In this section we assume that

$$1{\cdot}035\leqslant s\leqslant 3.$$

The function $G(x, y)$ of (15) is an even periodic function of x of period unity ; we express it as a Fourier series :

$$G(x,y)\sim\sum_{r=-\infty}^{\infty} a_r e^{2\pi i r x}=a_0+2\sum_{r=1}^{\infty} a_r\cos 2\pi r x.$$

We have, for $r>0$,

$$a_r=\int_{-\frac{1}{2}}^{\frac{1}{2}} G(x,y)\,e^{-2\pi i r x}\,dx=2y^s\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{e^{-2\pi i r x}\,dx}{\{(mx+n)^2+m^2y^2\}^s}.$$

In the inner sum we put $n=m\lambda+\nu$ where $0\leqslant\nu<m$ and obtain

$$\begin{aligned} a_r&=2y^s\sum_{m=1}^{\infty}\sum_{\nu=0}^{m-1}\int_{-\infty}^{\infty}\frac{e^{-2\pi i r x}\,dx}{\{(mx+\nu)^2+m^2y^2\}^s}\\ &=2y^s\int_{-\infty}^{\infty}\frac{e^{-2\pi i r t}\,dt}{(t^2+y^2)^s}\sum_{m=1}^{\infty}\frac{1}{m^{2s}}\sum_{\nu=0}^{m-1}e^{2\pi i r\nu/m}\\ &=2y^s\int_{-\infty}^{\infty}\frac{e^{-2\pi i r t}\,dt}{(t^2+y^2)^s}\sum_{m\mid r} m^{1-2s}\\ &=\frac{4\pi^s y^{\frac{1}{2}}}{\Gamma(s)}\,r^{s-\frac{1}{2}}\,\sigma_{1-2s}(r)\,K_{s-\frac{1}{2}}(2\pi r y),\end{aligned}$$

where

$$\sigma_k(n)=\sum_{d\mid n} d^k.$$

Similarly it can be shown that

$$a_0=2y^s\zeta(2s)+2y^{1-s}\zeta(2s-1)\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}.$$

The inversions of the orders of summation are all justified because of the absolute convergence of the series concerned. Also the Fourier series is uniformly convergent in x, and so we have, in fact,

$$G(x,y)=a_0+2\sum_{r=1}^{\infty} a_r\cos 2\pi r x.$$

Also, again by uniform convergence,

$$\begin{aligned} G_x(x,y)&=-4\pi\sum_{r=1}^{\infty} r a_r\sin 2\pi r x\\ &=-\frac{16\sqrt{\pi}}{2^{s+\frac{1}{2}}y^s\Gamma(s)}\sum_{r=1}^{\infty}\lambda_r\sin 2\pi r x=-A\sum_{r=1}^{\infty}\lambda_r\sin 2\pi r x,\end{aligned}$$

† "On the number of lattice points inside a random oval," *Quart. J. Math.*, Oxford Ser. 19 (1948), 1–26.

say, where

$$\lambda_r = \sigma_{1-2s}(r) H(2\pi r y), \quad \text{......................................(19)}$$

in the notation of (17). By partial summation we obtain, for $x \neq 0$,

$$G_x(x, y) = -\frac{A}{4 \sin^2 \pi x} \sum_{r=1}^{\infty} (\lambda_{r+2} - 2\lambda_{r+1} + \lambda_r)\{(r+1) \sin 2\pi x - \sin 2\pi (r+1) x\}. \quad \text{......(20)}$$

Now since

$$1 \leqslant \sigma_{1-2s}(r) \leqslant 1 + 2^{1-2s} \quad (r = 1, 2, 3),$$
$$1 \leqslant \sigma_{1-2s}(r) < \zeta(2s-1) \quad (r \geqslant 1),$$

it follows that we shall have

$$\lambda_{r+2} - 2\lambda_{r+1} + \lambda_r > 0 \quad (r = 1, 2, 3, \ldots) \quad \text{.............................(21)}$$

if

$$H\{(r+2)h\} - 2 \cosh \mu_r h \, H\{(r+1)h\} + H(rh) > 0 \quad (r = 1, 2, \ldots),$$

where $h = 2\pi y$ and μ_r is defined by

$$\cosh \mu_r h = 1 + 2^{1-2s} \quad (r = 1, 2\,;\ s \leqslant 2),$$
$$\cosh \mu_r h = \zeta(2s-1) \qquad \text{(otherwise)}.$$

By Lemmas 8 and 10 with $u = rh$ we conclude that (21) holds in the following cases: (*a*) $1{\cdot}035 \leqslant s \leqslant 2$, $y \geqslant \frac{3}{5}$, (*b*) $2 \leqslant s \leqslant 3$, $y \geqslant 1$. It follows from (20) that $G_x(x, y) < 0$ in cases (*a*) and (*b*) if $0 < x < \frac{1}{2}$.

5. We now suppose that $2 < s \leqslant 3$. By Lemma 7, the minimum of $G(x, y)$ is attained at a point z of $\mathfrak{D}$ for which $y \leqslant \frac{3}{2}$. Now z must lie in the circle $(x - \frac{1}{2})^2 + (y-1)^2 \leqslant \frac{1}{4}$ as otherwise $y \geqslant 1$, and, by case (*b*) of § 4, it would follow that $G(x, y)$ could be diminished by increasing x. Hence $(x - \frac{1}{2})^2 + (y-1)^2 \leqslant \frac{1}{4}$, i.e. $\alpha + \beta + \gamma \leqslant 4$, and the argument given at the beginning of § 3 shows that $G(x, y) \geqslant G(\frac{1}{2}, \frac{1}{2}\sqrt{3})$, equality occurring only when $x = \frac{1}{2}$, $y = \frac{1}{2}\sqrt{3}$.

6. It remains to consider the range $1{\cdot}035 \leqslant s \leqslant 2$. Again by Lemma 7 we know that $G(x, y)$ attains its minimum at a point z of $\mathfrak{D}$ for which $y \leqslant \frac{3}{2}$. Since $G_x(x, y) < 0$ in $\mathfrak{D}$ except on $x = 0$ and $x = \frac{1}{2}$ it follows that $z = \frac{1}{2} + iy$ where $\frac{1}{2}\sqrt{3} \leqslant y \leqslant \frac{3}{2}$. Now, if $\frac{1}{2}\sqrt{3} < y \leqslant \frac{3}{2}$, $G(x, y)$ takes the same value at the point

$$z' = x' + iy' = -\frac{1}{z-1} = \frac{2}{4y^2+1} + i\,\frac{4y}{4y^2+1}.$$

But $0 < x' < \frac{1}{2}$ and $y' \geqslant \frac{3}{5}$ so that $G_x(x', y') < 0$ and hence $G(x, y)$ can be decreased still further by increasing x'. This contradiction establishes that $Z_h(s)$ attains its minimum at $z = \frac{1}{2} + i\frac{1}{2}\sqrt{3}$ and at no other point of $\mathfrak{D}$; i.e. the minimum is attained only when $h(m, n) = Q(m, n)$. This completes the proof of the theorem.

We remark, in conclusion, that it is of course possible to reduce the lower bound $1{\cdot}035$ of s somewhat by more detailed arithmetical analysis, but, since the method places an upper bound on $\zeta(2s-1)$, it cannot be used without alteration to prove the Theorem for all $s > 1$.

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