Proceedings of the Edinburgh Mathematical Society (1983) **26**, 163–167 ©

SEMI-EMBEDDINGS OF BANACH SPACES WHICH ARE HEREDITARILY c_0

by L. DREWNOWSKI

(Received 5th November 1981)

To Professor W. Orlicz on his 80th birthday

Following Lotz, Peck and Porta [**9**], a continuous linear operator from one Banach space into another is called a *semi-embedding* if it is one-to-one and maps the closed unit ball of the domain onto a closed (hence complete) set. (Below we shall allow the codomain to be an F-space, i.e., a complete metrisable topological vector space.) One of the main results established in [**9**] is that if X is a compact scattered space, then every semi-embedding of $C(X)$ into another Banach space is an isomorphism ([**9**], Main Theorem, $(a)\Rightarrow(b)$).

In this paper we extend this result, as well as ([**9**], Corollary 12), to those Banach spaces E which are hereditarily c_0, i.e., such that every closed infinite-dimensional subspace of E contains a further subspace that is isomorphic to c_0. Theorem 1 proved below shows, in particular, that every semi-embedding of such a space E into any F-space is an isomorphism. Note that the class of hereditarily c_0 Banach spaces includes all spaces isomorphic to the $C(X)$ spaces for X compact scattered ([**10**], Main Theorem, (5); cf. also [**9**], Main Theorem 11, (d)), in particular all $c_0(\Gamma)$ spaces, and the separable Banach space constructed by Hagler [**5**]. Trivially, if E is hereditarily c_0, then so is every closed subspace of E. It should be pointed out, however, that Hagler's space is not isomorphic to a subspace of $C(X)$ for X compact scattered; this follows from ([**5**], Proposition 13) and ([**10**], Main Theorem, (4)).

Our second result, Theorem 2, is, in a sense, a converse to the above mentioned special case of Theorem 1. It reveals, quite unexpectedly, that the existence of a non-hereditarily c_0 Banach space E such that, for E equipped with any of its equivalent norms, every semi-embedding of E is an isomorphism, would imply a negative solution to the following well-known and long-standing problem of Bessaga and Pełczyński ([**7**], Problem 1.d.5): Does every infinite-dimensional Banach space contain an unconditional basic sequence?

Theorem 1. *Let* $E=(E,\|\cdot\|)$ *be a Banach space and* F *its closed subspace which is hereditarily* c_0. *Suppose that*

(+) *for every closed infinite-dimensional subspace* G *of* F *there is a complemented subspace* H *of* E *such that* $\dim(G\cap H)=\infty$ *and* H *has no subspace isomorphic to* l_∞.

Then, T *is a semi-embedding of* E *into an* F*-space, implies that* $T\,|\,F$ *is an isomorphism.*

Equivalently: If $|\cdot|$ *is a weaker* F*-norm* (see [**6**]) *on* $(E,\|\cdot\|)$ *such that the closed unit ball* B *of* $(E,\|\cdot\|)$ *is complete under* $|\cdot|$, *then* $|\cdot|$ *and* $\|\cdot\|$ *are equivalent on* F.

Proof. Suppose an F-norm $|\cdot|$ satisfies the above assumptions and is strictly weaker than $\|\cdot\|$ on F. We first show that F has a norm-closed infinite-dimensional subspace G such that $|\cdot|$ is strictly weaker than $\|\cdot\|$ on every infinite-dimensional subspace of G.

Since B is $|\cdot|$-complete, and *a fortiori* $|\cdot|$-closed, the norm topology of F is polar with respect to the (strictly weaker) $|\cdot|$-topology of F. Hence, by a result of Kalton ([**6**], Theorem 3.2 or [**2**], Theorem 2.4), we may find a normalised basic sequence (u_n) in $(F,\|\cdot\|)$ such that $\sum_{n=1}^{\infty}|u_n|<\infty$. Let G be the norm-closed linear span of (u_n), and let $(f_n)\subset(G,\|\cdot\|)^*$ be the associated sequence bi-orthogonal to (u_n). Then $\sup\|f_n\|<\infty$, from which it follows easily that

$$\left|x-\sum_{i=1}^{n} f_i(x)u_i\right|\to 0 \quad \text{as} \quad n\to\infty,$$

uniformly for $x\in B\cap G$. Hence the identity map from $(G,\|\cdot\|)$ into $(E,|\cdot|)$ is compact. It follows that G is as required.

Now, by applying the hypotheses of the theorem and passing to a suitable subspace of G, we may assume that G is isomorphic to c_0 and that G is contained in a complemented subspace H of E such that $H\not\supset l_\infty$. Moreover, using a suitable isomorphism, we may identify G with c_0 so as to have $\|\cdot\|\leqq\|\cdot\|_\infty$ on G, where $\|\cdot\|_\infty$ is the usual sup-norm of c_0.

Let (e_n) be the standard basis of $c_0=G$. Since, for each n, $|\cdot|$ is strictly weaker than $\|\cdot\|$ (or $\|\cdot\|_\infty$) on $\text{lin}\{e_i: i\geqq n\}$, there is a block sequence (x_n) of (e_n) such that $\|x_n\|_\infty=1$ for all n and

$$\sum_{n=1}^{\infty}|x_n|<\infty.$$

Let $(t_n)\in l_\infty$ and $\|(t_n)\|_\infty\leqq 1$. Since the "supports" of the x_n's are pairwise disjoint, we have

$$\sum_{i=1}^{n} t_i x_i\in B \quad \text{for all } n;$$

furthermore,

$$\left|\sum_{i=m}^{n} t_i x_i\right|\leqq\sum_{i=m}^{n}|x_i|\to 0 \quad \text{as } m,n\to\infty.$$

Since B is $|\cdot|$-complete, the series $\sum_{n=1}^{\infty} t_n x_n$ is (subseries) convergent in $(E,|\cdot|)$ to a point in B.

Let P be a continuous linear projection from E onto H. We will indicate two ways of finishing the proof.

(I) From the above it follows that we may define a continuous linear operator

$R: l_\infty \to (E, \|\cdot\|)$ by

$$R(t_n) = \sum_{n=1}^{\infty} t_n x_n \qquad (|\cdot|\text{-convergence}).$$

Let $S = PR: l_\infty \to (H, \|\cdot\|)$. Since the basic sequence (x_n) is equivalent to the basis (e_n) of c_0 ([7], Proposition 2.*a*.1), $S|c_0 = R|c_0$ is an isomorphism. By a result of Rosenthal ([11], Proposition 1.2; cf. also [3] and [4]), there is an infinite subset M of $\mathbb{N} = \{1, 2, \ldots\}$ such that the restriction of S to the subspace $l_\infty(M) = \{(t_n) \in l_\infty : t_n = 0 \text{ for } n \notin M\} \approx l_\infty$ is an isomorphism. This is clearly impossible because $H \not\supset l_\infty$.

(II) From the above we see that the series $\sum_{n=1}^{\infty} x_n$ is subseries convergent in $(E, |\cdot|)$, and that the sum of each of its subseries is in B. We may therefore define a bounded finitely additive vector measure $\mu: 2^{\mathbb{N}} \to (H, \|\cdot\|)$ by

$$\mu(A) = P\left(\sum_{n \in A} x_n\right) \qquad (|\cdot|\text{-convergence}).$$

Since $H \not\supset l_\infty$, a theorem of Diestel and Faires ([1], Theorem I.4.2) implies μ is exhaustive, i.e., $\|\mu(A_n)\| \to 0$ for every infinite sequence (A_n) of pairwise disjoint subsets of $\mathbb{N}$. In particular as $\mu(\{n\}) = x_n$, we have $\|x_n\| \to 0$, contradicting the fact that (x_n) was chosen normalised in the norm $\|\cdot\|_\infty$ equivalent to $\|\cdot\|$ on G.

Remarks. (1) Since the requirements imposed in Theorem 1 on the Banach space E and its subspace F are isomorphism invariants, it is clear that the assertion remains valid for E equipped with any of its equivalent norms.

(2) The condition $(+)$ is satisfied in particular when

(*a*) $E \not\supset l_\infty$, or
(*b*) F is complemented in E, or
(*c*) every subspace of F isomorphic to c_0 contains a further subspace which is isomorphic to c_0 and complemented in E.

The simplest case when (*a*) or (*b*) holds is $E = F$.

Theorem 2. *A Banach space E is hereditarily c_0 if and only if it satisfies the following two conditions:*

(*) *For E equipped with any of its equivalent norms, every semi-embedding of E into another Banach space is an isomorphism.*
(**) *Every closed infinite-dimensional subspace of E contains an infinite unconditional basic sequence.*

Proof. In view of Theorem 1 and the above Remarks, only the "if" part needs a proof. So assume conditions (*) and (**) are satisfied, and suppose E has a closed infinite-dimensional subspace G which does not contain any isomorphic copy of c_0. Applying (**), we may assume that G has an unconditional basis. Since $G \not\supset c_0$, this basis is boundedly complete ([7], Theorem 1.c.10) and hence G is isomorphic to the conjugate space Z^* of a separable Banach space Z ([7], Proposition 1.b.4). We may equivalently renorm E so that in the new norm, $\|\cdot\|$ say, G becomes isometric to Z^*. Let B be the closed unit ball of $(E, \|\cdot\|)$. There is a norm $|\cdot|$ on G such that $|y| \leqq \|y\|$ for $y \in G$ and $B \cap G$ is $|\cdot|$-compact. (If (z_n) is a sequence dense in the unit ball of Z, then the formula $|y| = \sum_{n=1}^{\infty} 2^{-n}|y(z_n)|$ for $y \in G = Z^*$ defines a norm on G which is smaller than $\|\cdot\|$ and induces the weak*-topology on $B \cap G$.) Now the formula

$$|x|_1 = \inf\{|y| + \|x-y\| : y \in G\}$$

defines a norm on E such that

$$|x|_1 \leqq \|x\| \text{ for all } x \in E \quad \text{and} \quad |y|_1 = |y| \text{ for } y \in G.$$

Let B_1 be the $|\cdot|_1$-closure of B. We claim that B_1 is $|\cdot|_1$-complete and $B \subset B_1 \subset 3B$.

First note that on the quotient space E/G the quotient norms $\|\cdot\|^{\wedge}$ and $|\cdot|_1^{\wedge}$ corresponding to $\|\cdot\|$ and $|\cdot|_1$ are equal (and make E/G a Banach space). Let $Q: E \to E/G$ be the quotient map. Now let a sequence $(x_n) \subset B$ be $|\cdot|_1$-Cauchy. Then (Qx_n) is Cauchy in E/G and hence converges to a point $\zeta \in E/G$. Since $\|\zeta\|^{\wedge} \leqq 1$, for each $\varepsilon > 0$ there is $z \in E$ such that $\|z\| \leqq 1 + \varepsilon$ and $Qz = \zeta$. Moreover, since $\|Q(x_n - z)\|^{\wedge} \to 0$, there is a sequence $(y_n) \subset G$ such that $\|x_n - z - y_n\| \to 0$; hence also

$$|x_n - z - y_n|_1 \to 0.$$

It follows that (y_n) is $|\cdot|_1$-Cauchy. Since, for n large enough,

$$\|y_n\| \leqq \|x_n\| + \|z\| + \|x_n - z - y_n\| \leqq 2(1+\varepsilon),$$

i.e., $y_n \in 2(1+\varepsilon)(B \cap G)$, and since $B \cap G$ is $|\cdot|_1$-complete, the sequence (y_n) is $|\cdot|_1$-convergent to a point $y \in 2(1+\varepsilon)(B \cap G)$. Denoting $x = y + z$ we see that

$$|x_n - x|_1 \to 0$$

and

$$\|x\| \leqq 2(1+\varepsilon) + (1+\varepsilon) = 3(1+\varepsilon).$$

Since x does not depend on ε, we have $\|x\| \leqq 3$. This concludes the proof of the claim.

Finally, let $\|\cdot\|_1$ be the Minkowski functional of B_1. From the above proof, it is evident that $\|\cdot\|_1$ is a norm equivalent to $\|\cdot\|$, B_1 is the closed unit ball of $(E, \|\cdot\|_1)$, $|\cdot|_1 \leqq \|\cdot\|_1$, B_1 is $|\cdot|_1$-complete and $B_1 \cap G = B \cap G$ is $|\cdot|_1$-compact. Hence the identity map from $(E, \|\cdot\|_1)$ into the completion of $(E, |\cdot|_1)$ is a semi-embedding that is not an isomorphism.

Remarks. (1) Some parts of the above proof were inspired by the arguments used in the proofs of Propositions 2 and 3, and by the Remark on page 235 of [9].

https://doi.org/10.1017/S0013091500016862 Published online by Cambridge University Press

(2) Let us recall that condition (**) is certainly satisfied when E is a subspace of a Banach space with an unconditional basis ([**7**], p. 27) or, more generally, a subspace of an order continuous Banach lattice ([**8**], Theorem 1.*c*.9).

(3) We do not know whether the condition (*) of Theorem 2 is equivalent to the apparently weaker one: (*′) Every semi-embedding of E (with the given norm) into another Banach space is an isomorphism.

REFERENCES

1. J. Diestel and J. J. Uhl, Jr., *Vector Measures* (American Mathematical Society, Mathematical surveys, number 15, 1977).

2. L. Drewnowski, On minimally subspace-comparable F-spaces, *J. Funct. Analysis* **26** (1977), 315–332.

3. L. Drewnowski, An extension of a theorem of Rosenthal on operators acting from $l_\infty(\Gamma)$, *Studia Math.* **57** (1976), 209–215.

4. L. Drewnowski, Un théorème sur les opérateurs de $l_\infty(\Gamma)$, *C. R. Acad. Sci. Paris, Sér. A*, **281** (1975), 967–969.

5. J. Hagler, A counterexample to several questions about Banach spaces, *Studia Math.* **60** (1977), 289–308.

6. N. J. Kalton, Basic sequences in F-spaces and their applications, *Proc. Edinburgh Math. Soc.* **19** (1974), 151–167.

7. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I. Sequence Spaces* (Ergebnisse der Math. 92, Springer, Berlin–Heidelberg–New York, 1977).

8. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II. Function Spaces* (Ergebnisse der Math. 97, Springer, Berlin–Heidelberg–New York, 1979).

9. H. P. Lotz, N. T. Peck and H. Porta, Semi-embeddings of Banach spaces, *Proc. Edinburgh Math. Soc.* **22** (1979), 233–240.

10. A. Petczyński and Z. Semadeni, Spaces of continuous functions (III), *Studia Math.* **18** (1959), 211–222.

11. H. P. Rosenthal, On relatively disjoint families of measures, with some applications to Banach space theory, *Studia Math.* **37** (1970), 13–36.

Institute of Mathematics
A. Mickiewicz University
Poznań, Poland