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On common fixed points for a family of mappings

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The purpose of this paper is to obtain some common fixed point theorems for a family of mappings in a complete metric space. The results herein improve some of the recent theorems of Kiyoshi Iséki (*Bull. Austral. Math. Soc.* 10 (1974), 365-370).

1.

In a recent paper [1], Iséki has given some sufficient conditions for the existence of a common fixed point for a sequence of self mappings of a complete metric space. The purpose of this paper is to obtain some common fixed point theorems for a family of mappings under conditions that are considerably weaker than considered in [1]. The results herein improve the results in [1] and several other known results ([2], [3], [4], [5]).

Throughout this paper, let (X, d) be a complete metric space and R^+ the nonnegative reals. Let ψ denote a family of mappings such that each $\phi \in \psi$, $\phi : (R^+)^5 \to R^+$, and ϕ is continuous and nondecreasing in each coordinate variable.

THEOREM 1. *Let f, g be self mappings of X. Suppose there exists a $\phi \in \psi$ such that for all $x, y \in X$,*

(1) $d(fx, gy) \le \phi\big(d(x, fx), d(y, gy), d(x, gy), d(y, fx), d(x, y)\big)$,

where ϕ satisfies the condition: for any $t > 0$,

(2) $\phi(t, t, a_1t, a_2t, t) < t$, $a_i \in \{1, 2\}$ *with* $a_1 + a_2 = 2$.

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Then there exists a $u \in X$ *such that*

(a) $fu = gu = u$ *and*

(b) u *is the unique fixed point of each* f *and* g .

Proof. Define a sequence $\{x_n\}$ in X as follows. Let $x_0 \in X$, $x_1 = fx_0$, $x_2 = gx_1$, and inductively, for each $n \in I^+$ (positive integers),

$$x_{2n-1} = fx_{2n-2}, \quad x_{2n} = gx_{2n-1} .$$

Let $d_n = d(x_n, x_{n+1})$. Since $d(x_{2n-1}, x_{2n+1}) \leq d_{2n-1} + d_{2n}$, it follows by (1) that, for each $n \in I^+$,

(3) $\quad d_{2n} = d(fx_{2n}, gx_{2n-1}) \leq \phi(d_{2n}, d_{2n-1}, 0, d_{2n-1}+d_{2n}, d_{2n-1})$.

Now, if for some $n \in I^+$, $d_{2n} > d_{2n-1}$, then (3) will imply that

$$d_{2n} \leq \phi(d_{2n}, d_{2n}, 0, 2d_{2n}, d_{2n}) < d_{2n} ,$$

a contradiction. Hence $d_{2n} \leq d_{2n-1}$. Similarly, it follows that $d_{2n+1} \leq d_{2n}$ for each $n \in I^+$. Consequently, $\{d_n\}$ is a nonincreasing sequence in R^+ and hence there is a $r \in R^+$ such that $d_n \to r$. Clearly $r = 0$, for otherwise, by (3),

$$r \leq \phi(r, r, 0, 2r, r) < r ,$$

a contradiction. Thus

(4) $$d_n = d(x_n, x_{n+1}) \to 0 .$$

We show that $\{x_n\}$ is a Cauchy sequence in X . In view of (4) it suffices to show that the sequence $\{x_{2n}\}$ is Cauchy. Suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then there is an $\varepsilon > 0$ such that for each even integer $2k$, $k \in I^+$, there exist integers $2n(k)$ and $2m(k)$ with $2k \leq 2n(k) < 2m(k)$ such that

(5) $$d(x_{2n(k)}, x_{2m(k)}) > \varepsilon .$$

Let, for each integer $2k$, $k \in I^+$, $2m(k)$ be the least integer

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exceeding $2n(k)$ satisfying (5); that is

(6) $$d\left(x_{2n(k)}, x_{2m(k)-2}\right) \le \varepsilon \text{ and } d\left(x_{2n(k)}, x_{2m(k)}\right) > \varepsilon .$$

Then, for each integer $2k$, $k \in I^+$,

$$\varepsilon < d\left(x_{2n(k)}, x_{2m(k)}\right) \le d\left(x_{2n(k)}, x_{2m(k)-2}\right) + d_{2m(k)-2} + d_{2m(k)-1} .$$

Therefore, by (4) and (6), we obtain

(7) $$d\left(x_{2n(k)}, x_{2m(k)}\right) \to \varepsilon \text{ as } k \to \infty .$$

It now follows immediately from the triangular inequality that

$$\left|d\left(x_{2n(k)}, x_{2m(k)-1}\right) - d\left(x_{2n(k)}, x_{2m(k)}\right)\right| \le d_{2m(k)-1} ,$$

and

$$\left|d\left(x_{2n(k)+1}, x_{2m(k)-1}\right) - d\left(x_{2n(k)}, x_{2m(k)}\right)\right| \le d_{2m(k)-1} + d_{2n(k)} ,$$

and hence, by (6) as $k \to \infty$,

(8) $$d\left(x_{2n(k)}, x_{2m(k)-1}\right) \to \varepsilon , \quad d\left(x_{2n(k)+1}, x_{2m(k)-1}\right) \to \varepsilon .$$

For simplicity of the notation, let, for each $k \in I^+$,

$$r(2k) = d\left(x_{2n(k)}, x_{2m(k)}\right) ,$$

$$s(2k) = d\left(x_{2n(k)}, x_{2m(k)-1}\right) ,$$

and

$$t(2k) = d\left(x_{2n(k)+1}, x_{2m(k)-1}\right) .$$

Then, since $r(2k) \le d_{2n(k)} + d\left(fx_{2n(k)}, gx_{2m(k)-1}\right)$, it follows by (1) that

$$r(2k) \le d_{2n(k)} + \phi\left(d_{2n(k)}, d_{2m(k)-1}, r(2k), t(2k), s(2k)\right) ,$$

and hence it follows by (2), (7), and (8) that

$$\varepsilon \le \phi(0, 0, \varepsilon, \varepsilon, \varepsilon) < \varepsilon ,$$

contradicting the existence of an $\varepsilon > 0$. Consequently, $\{x_n\}$ is a Cauchy sequence and hence, by completeness, there is a $u \in X$ such that $x_n \to x$. We show that $f(u) = g(u) = u$. Now, since $x_{2n} = gx_{2n-1}$,

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$$d(fu, x_{2n}) \le \phi\big(d(u, fu), d_{2n-1}, d(u, x_{2n}), d(x_{2n-1}, fu), d(x_{2n-1}, u)\big) .$$

Therefore, as $n \to \infty$ in the above inequality, we obtain

$$d(fu, u) \le \phi\big(d(u, fu), 0, 0, d(u, fu), 0\big) ,$$

and hence, by the nondecreasing property of ϕ, it follows that $fu = u$. A similar argument applied to $d(x_{2n+1}, gu)$ yields $gu = u$. This proves *(a)*. To prove *(b)* suppose there is a $v \ne u$ for which $gv = v$. Let $r = d(u, v) > 0$. Then

$$r = d(fu, gv) \le \phi(0, 0, r, r, r) < r ,$$

contradicting $r > 0$. Thus $v = u$. A similar argument shows that u is the unique fixed point of f also. This proves *(b)*.

2.

In the following, let F denote a family of self mappings of X and, for each $f, g \in F$, let $a = a(f, g)$ indicate that a depends on f and g.

The following is an immediate consequence of Theorem 1.

THEOREM 2. *Let F satisfy the condition: for each pair $f, g \in F$, there exists a $\phi = \phi(f, g) \in \psi$ satisfying* (1) *and* (2). *Then there is a $u \in X$ such that*

(a) $fu = u$ for each $f \in F$ and

(b) u is the unique fixed point for each $f \in F$.

The following special case of Theorem 2 provides an extension of Theorem 1 in [*1*].

COROLLARY 1. *Let F satisfy the condition: for each pair $f, g \in F$ there exist nonnegative reals $\alpha = \alpha(f, g)$, $\beta = \beta(f, g)$, and a $\gamma = \gamma(f, g)$ with $2\alpha + 2\beta + \gamma < 1$ such that for all $x, y \in X$,*

$$d(fx, gy) \le \alpha\big(d(x, fx)+d(y, gy)\big) + \beta\big(d(x, gy)+d(y, fx)\big) + \gamma d(x, y) .$$

Then F has a unique common fixed point.

Proof. Define $\phi = \phi(f, g) : (R^+)^5 \to R^+$ by

$$\phi(t_1, t_2, t_3, t_4, t_5) = \alpha(t_1+t_2) + \beta(t_3+t_4) + \gamma t_5 .$$

Then $\phi \in \psi$ and satisfies (2). Clearly, each pair $f, g \in F$ satisfies (1) with respect to $\phi = \phi(f, g)$. The conclusion now follows by Theorem 2.

The following result contains some of the results of Srivastava and Gupta [5], Reich [2], Sehgal [3, 4], and others.

COROLLARY 2. *Let* F *satisfy the condition: for each pair* $f, g \in F$, *there exist positive integers* $m = m(f, g)$ *and* $n = n(f, g)$ *and a* $\phi = \phi(f, g) \in \psi$ *satisfying* (2) *such that for all* $x, y \in X$,

$$(9) \quad d(f^m x, g^n y) \le \phi\big(d(x, f^m x), d(y, g^n y), d(x, g^n y), d(y, f^m x), d(x, y)\big) .$$

Then F *has a common fixed point which is the unique fixed point of each* $f \in F$.

Proof. Let $f_1 = f^m$ and $g_1 = g^n$. Then the pair f_1, g_1 satisfies the conditions of Theorem 1 and hence there is a $u \in X$ with $f^m u = g^n u = u$ and u is the unique fixed point of f^m and g^n. Since $f^m(fu) = f(f^m u) = fu$, it follows that $fu = u$ and, similarly, $gu = u$ and u is the unique fixed point of f and g. If $h \in F$, then by the above argument, the pair f, h has a common fixed point $v \in X$ and v being a fixed point of f, it follows that $v = u$.

3.

In this section we obtain some generalizations of Theorem 2 and Theorem 3 in [1].

THEOREM 3. *Let* g *and a sequence* $\{f_n\}$ *be self mappings of* X *such that* $f_n \to g$ *uniformly. Suppose for each* $n \ge 1$, f_n *has a fixed point* x_n *and* g *satisfies the condition: for all* $x, y \in X$,

$$(10) \quad d(gx, gy) \le \phi\big(d(x, gx), d(y, gy), d(x, gy), d(y, gx), d(x, y)\big) ,$$

for some $\phi \in \psi$ *satisfying* (2). *If* x_0 *is the fixed point of* g *and* $\sup d(x_n, x_0) < \infty$, *then* $x_n \to x_0$.

Proof. Note that g has a unique fixed point x_0 by Theorem 1.

Since $f_n x_n = x_n$ and $f_n \to g$ uniformly, it follows that $d(f_n x_n, gx_n) = d(x_n, gx_n) \to 0$ as $n \to \infty$. Let $\varepsilon = \limsup d(x_n, x_0)$. Then, since $d(gx_n, x_0) \le d(gx_n, x_n) + d(x_n, x_0)$, it follows by (10) that

$$d(x_n, x_0) \le d(x_n, gx_n) + d(gx_n, gx_0)$$
$$\le d(x_n, gx_n) + \phi\big(d(x_n, gx_n), 0, d(x_n, x_0), d(gx_n, x_n) + d(x_n, x_0), d(x_n, x_0)\big) .$$

This implies that

$$\varepsilon \le \phi(0, 0, \varepsilon, \varepsilon, \varepsilon)$$

and hence $\varepsilon = 0$ and, consequently, $x_n \to x_0$.

REMARK. If in Theorem 3, condition (10) is replaced by

$$d(gx, gy) \le \alpha\big(d(x, gx)+d(y, gy)\big) + \beta\big(d(x, gy)+d(y, gx)\big) + \gamma d(x, y) ,$$

where α, β, γ are some nonnegative reals with $2\alpha + 2\beta + \gamma < 1$, then it is easy to show [1] that $\sup d(x_n, x_0) < \infty$. Thus Theorem 3 improves Theorem 2 in [1].

THEOREM 4. *Let $\{f_n\}$ be a sequence of self mappings of X satisfying the condition: there is a $\phi \in \psi$ satisfying* (2) *such that for all x $y \in X$ and $n \ge 1$,*

$$d(f_n x, f_n y) \le \phi\big(d(x, f_n x), d(y, f_n y), d(x, f_n y), d(y, f_n x), d(x, y)\big) .$$

Let x_n be the fixed point of f_n (given by Theorem 1) *and let $g : X \to X$ such that $f_n \to g$. If x_0 is any cluster point of the sequence $\{x_n\}$ then $gx_0 = x_0$.*

Proof. Let $x_{n_i} \to x_0$. Since $f_n \to g$, therefore $d\left(f_{n_i} x_0, gx_0\right) \to 0$. Furthermore, for each $i \ge 1$,

$$d\left(x_{n_i}, f_{n_i} x_0\right) \le a_i = d\left(x_{n_i}, x_0\right) + d(x_0, gx_0) + d\left(gx_0, f_{n_i} x_0\right) \to d(x_0, gx_0)$$

and

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$$d\left(x_0, f_{n_i}x_0\right) \le b_i = d\left(x_0, gx_0\right) + d\left(gx_0, f_{n_i}x_0\right) \to d\left(x_0, gx_0\right) .$$

Thus for each $i \ge 1$,

$$\begin{aligned} d\left(x_0, gx_0\right) &\le d\left(x_0 x_{n_i}\right) + d\left(f_{n_i}x_{n_i}, f_{n_i}x_0\right) + d\left(f_{n_i}x_0, gx_0\right) \\ &\le d\left(x_0, x_{n_i}\right) + \phi\left(0, b_i, a_i, d\left(x_{n_i}, x_0\right), d\left(x_{n_i}, x_0\right)\right) \\ &\qquad + d\left(f_{n_i}x_0, gx_0\right) . \end{aligned}$$

Therefore, as $i \to \infty$,

$$d\left(x_0, gx_0\right) \le \phi\left(0, d\left(x_0, gx_0\right), d\left(x_0, gx_0\right), 0, 0\right) ,$$

which implies $gx_0 = x_0$.

References

[1] Kiyoshi Iséki, "On common fixed points of mappings", *Bull. Austral. Math. Soc.* 10 (1974), 365-370.

[2] Simeon Reich, "Some remarks concerning contraction mappings", *Canad. Math. Bull.* 14 (1971), 121-124.

[3] V.M. Sehgal, "On fixed and periodic points for a class of mappings", *J. London Math. Soc.* (2) 5 (1972), 571-576.

[4] V.M. Sehgal, "Some fixed and common fixed point theorems in metric spaces", *Canad. Math. Bull.* 17 (1974), 257-259.

[5] Pramila Srivastava and Vijay Kumar Gupta, "A note on common fixed points", *Yokohama Math. J.* 19 (1971), 91-95.

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