PROPERTIES OF THE COEFFICIENTS OF ORTHONORMAL SEQUENCES

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1. Introduction. In this paper we consider complete orthonormal sequences defined on the interval [0, 1] and satisfying an inequality of the type

$$
\begin{aligned}
(1)\qquad J_\nu(\phi_n) &= \left(\int_0^1 |\phi_n|^\nu dx\right)^{1/\nu} \leqslant F_n, \qquad 2 \leqslant \nu < \infty,\\
&= \sup_{0 \leqslant x \leqslant 1} |\phi_n| \qquad\quad \leqslant F_n, \qquad \nu = \infty,
\end{aligned}
$$

for all n and some sequence $\{F_n\}$. Such sequences were first considered by Zygmund and Marcinkiewicz **(8)**. They extended the well-known results of Hausdorff–Young and Paley, originally proved for the case $\nu = \infty$, $F_n = M$ for all n **(12)**. We will consider cases of equality in the Hausdorff–Young theorems and certain limiting cases of the Paley theorems. Application of these results and the results in **(8)** will be made to functions harmonic in the unit α-sphere.

2. If $p \geqslant 1$ then p' will denote the conjugate index, $1/p + 1/p' = 1$.

If $c_1, c_2, \ldots,$ are the Fourier coefficients of a function in L_q, with respect to $\{\phi_n\}$ satisfying (1) define

$$(2)\qquad d_n(q) = c_n F_n^{(\nu')/(2-\nu')(1-2/q)}, \qquad n = 1, 2, \ldots,$$

$$
\begin{aligned}
(3)\qquad U_r(c) = U_r(d) &= \left\{\sum_{n=1}^{\infty} |d_n(s)|^r\right\}^{1/r} = \left\{\sum_{n=1}^{\infty} |c_n|^r F_n^{2-r}\right\}^{1/r},\\
&= \max_n |d_n(\nu')| = \max_n (|c_n| F_n^{-1}), \; r = \infty,
\end{aligned}
$$

where r, s are related by

$$(4)\qquad \frac{\nu'}{s} + \frac{2-\nu'}{r} = 1.$$

The Fourier coefficients are replaced in this general situation by the sequence $\{d_n\}$. For instance, the following extension of Mercer's theorem can be proved along the lines of the original theorem **(6**, p. 155**)**.

THEOREM 1. *If* $f \in L_{\nu'}$ *then* $d_n(\nu') = o(1)$.

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3. Cases of equality.

3.1. The cases of equality in the Hausdorff–Young theorems were first discussed by Hardy and Littlewood **(4)** for the trigonometric case. Their results were extended to the case $\nu = \infty$, $F_n = M$ for all n, by Verblunsky **(10)** and Calderón and Zygmund **(1)**. We will use the methods of the last authors to prove the general Hausdorff–Young result and then to discuss equality.

3.2. THEOREM 2. (a) *If* $f \in L_p$, $\nu' \leqslant p \leqslant 2$, *with Fourier coefficients* $c_1, c_2, \ldots,$ *with respect to* $\{\phi_r\}$ *then*

$$U_q(d) \leqslant J_p(f) \tag{5}$$

where

$$\frac{\nu'}{p} + \frac{2-\nu'}{q} = 1.$$

(b) *If for a sequence* $\{c_n\}$, $U_p(d) < \infty$, $1 \leqslant p \leqslant 2$, *then there exists a function* $f \in L_q$ *such that* c_n *is the Fourier coefficient of* f *with respect to* ϕ_n, $n = 1, 2, \ldots,$ *and*

$$J_q(f) \leqslant U_p(d), \tag{6}$$

where

$$\frac{\nu'}{q} + \frac{2-\nu'}{p} = 1.$$

It is known that (a) implies (b) by a conjugacy argument and that it is sufficient to prove (a) under the assumption that $\{\phi_n\}$ has N terms, f is a simple function with $J_p(f) = 1$, **(1)**.

Let $\{\alpha_n\}$ be a sequence such that

$$U_q(d) = \sum_{n=1}^{N} c_n \alpha_n F_n^{2-q}.$$

Define $\{A_n\}$ and $F(t)$ by

$$\alpha_n = A_n^{1/q} F_n^{(q-2)/q} \epsilon_n, \qquad A_n \geqslant 0, \qquad |\epsilon_n| = 1,$$
$$f(t) = F^{1/p}(t)\eta(t), \qquad F(t) \geqslant 0, \qquad |\eta(t)| = 1.$$

Putting $1/p = z$ in $U_q(d)$ it becomes

$$\Phi(z) = \sum_{n=1}^{N} A_n^{(1-\nu'(1-z))/(2-\nu')} F_n^{\nu'/(2-\nu')(1-2z)} \epsilon_n \left\{ \int_0^1 F^z \eta \phi_n dt \right\}, \tag{7}$$

a function continuous and bounded in every strip, $x_1 \leqslant x \leqslant x_2$, of finite width. It is not difficult to show that

$$F_n \geqslant 1, \sum_{n=1}^{N} A_n = 1, \int_0^1 F(t)dt = 1. \tag{8}$$

Hence, by simple applications of Hölder's inequality and Bessel's inequality, it can be shown that neither $|\Phi(1/\nu' + iy)|$, nor $|\Phi(1/2 + iy)|$ exceeds 1. This, by the Phragmén–Lindelöf theorem, implies $|\Phi(z)| \leqslant 1$ in the whole strip $1/2 \leqslant x \leqslant 1/\nu'$, which proves the theorem.

3.3. We are now in a position to discuss cases of equality in Theorem 2, excluding the trivial cases $f = 0$, p.p., and $c_n = 0$ for all n.

We can deduce (7) with no restrictions except $J_p(f) = 1$ and again $\Phi(z)$ is continuous and bounded, $\frac{1}{2} \leqslant x \leqslant 1/\nu'$, and regular, $\frac{1}{2} < x < 1/\nu'$, and (8) holds (with $N = \infty$ of course.)

THEOREM 3. (a) *A necessary condition for equality in* (5) *is that*

$$f(x) = \sum_{k=1}^{N} c_{nk}\phi_{nk}(x), \qquad n_1 < n_2 < \ldots < n_N. \tag{9}$$

For such functions we have equality if and only if

(i) $$|c_{n_k}|F_{n_k}^{-1} = \lambda,$$

independenf of k,

(ii) *f is constant in a set of measure*

$$\left(\sum_{k=1}^{N} F_{nk}^2\right)^{-1} \qquad \text{and} \qquad f = 0 \text{ in } \mathscr{C}E.$$

(b) *A necessary condition for equality in* (6) *is that only a finite number of c_n differ from zero, and satisfy* (i). *The function is then of form* (9) *and we have equality if and only if it satisfies* (ii).

A conjugacy argument shows that (a) implies (b). Let us assume then that $\Phi(1/p) = 1$, that is, that we have equality in (5). Then the Phragmén–Lindelöf theorem implies that $\Phi(z) = 1$ for $\frac{1}{2} \leqslant x \leqslant 1/\nu'$. In particular $\Phi(1/\nu') = 1$. Further, (8) implies that

$$\left| F_n^{-1}\epsilon_n\left\{\int_0^1 F^{1/\nu'}\eta\bar{\phi}_n dt\right\}\right| \leqslant 1.$$

Hence for all n for which $A_n \neq 0$,

$$F_n^{-1}\epsilon_n\left\{\int_0^1 F^{1/\nu'}\eta\bar{\phi}_n dt\right\} = 1. \tag{10}$$

But $F^{1/\nu'} \in L_{\nu'}$ and so by Theorem 1 the left-hand side of (10) is $o(1)$. Therefore there is at most a finite number of non-zero A_n, which proves (9).

From (10) we also get that

$$\int_0^1 F^{1/\nu'}|\phi_{nk}| \ (\operatorname{sign} f)(\operatorname{sign} \bar{c}_{nk}\bar{\phi}_{nk})dt = F_{nk} > 0,$$

which implies two important facts about the set E where f is non-zero.

(a) $\operatorname{sign} f = \operatorname{sign} (\bar{c}_{nk}\bar{\phi}_{nk})$, p.p. in E, $k = 1, \ldots, N$.

(b) $F(x) = \{F_{nk}^{-1}|\phi_{nk}(x)|\}^{\nu}$ p.p. in E, $k = 1, 2, \ldots, N$.

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Hence,

$$|c_{nk}| = \int_E |f|\, |\phi_{nk}|\, dt = F_{nk} \int_E |f|\, F^{1/\nu}\, dt,$$

which proves (i). Also

$$|f(x)| = F^{1/\nu}(x) \sum_{k=1}^{N} |c_{nk}|\, F_{nk}.$$

Let r be any number, $\nu' \leqslant r \leqslant 2$, then the initial remark of the proof implies

$$\left\{\sum_{n=1}^{\infty} |d_n(r)|^s\right\}^{1/s} = J_r(f),$$

which, using the above results, gives

$$\left(\sum_{k=1}^{N} F_{nk}^2\right)^{-1/s} = \left(\int_E F^{r/\nu} dt\right)^{1/r}.$$

Applying this equality for $r = \nu'$, 2, and p, where p is any value between 2 and ∞, to the Hölder inequality

$$\int_E F^{p/\nu} dt \leqslant \left(\int_E F^{2/\nu} dt\right)^{(p-\nu')/(2-\nu')} \left(\int_E F^{\nu'/\nu} dt\right)^{(2-p)/(2-\nu')}$$

we see that it reduces to equality. Hence F, and so f, is constant p.p. in E.

This proves the necessity, the sufficiency is immediate.

4. Star theorems. Given a sequence $\{c_n\}$ such that $c_n = o(1)$ the sequence $\{c^*_n\}$ denotes $\{|c_n|\}$ arranged in descending order.

The proof **(8)** of the extension of Paley's theorems requires F_n to satisfy

$$(11) \qquad F_1 \leqslant F_2 \leqslant F_3 \leqslant \ldots$$

or, at least, that for some $a > 1$ and all i, j, $i < j$,

$$(11)' \qquad \max_{a^i+1 \leqslant a^{i+1}} F_n \leqslant K \min_{a^j+1 \leqslant a^{j+1}} F_n.$$

Whether this is essential is not known. If $F_n = M$ for all n the order of the sequence ϕ_n is immaterial and the Paley theorems can be improved to the Paley star theorems **(8)**. However, because of (11) (or (11)′), no such simple argument is possible in general. We conjecture the following star theorem. It would follow immediately from the unstarred result if (11)′ could be dropped. Let, $d = d_n$ and define

$$(12) \quad V_r(d) = V_r = \left\{\sum_{n=1}^{\infty} |d_n(r)|^r\, n^{(r-2)/(2-\nu')}\right\}^{1/r}, \nu > 2, 1 < \nu' \leqslant r \leqslant \nu < \infty$$

$$= \max_n \{|d_n(\infty)|n\},\ r = \nu = \infty.$$

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THEOREM 4. (a) *Let* $d_n = o(1)$ *be such that*, $2 \leqslant q < \nu$, $V_q(d^*) < \infty$. *Then there exists an* $f \in L_q$ *such that*

$$c_n = d_n F_n^{\nu'/(2-\nu')((2'/q)-1)}$$

is the Fourier coefficient of f *with respect to* ϕ_n, $n = 1, 2, \ldots$, *and*

$$J_q(f) \leqslant A_{q,\nu} V_q(d^*). \tag{13}$$

(b) *If* $f \in L_p$, $\nu' < p \leqslant 2$, *has Fourier coefficients* $c_1, c_2 \ldots$, *with respect to* $\{\phi_n\}$ *then*

$$V_p(d^*) \leqslant A_{p,\nu} J_p(f). \tag{14}$$

These theorems were first mentioned in a paper by Littlewood **(7)**, and the following comments are of some interest.

(i) The hypothesis of (a) implies the existence of an $f \in L_2$ with the required Fourier coefficients.

(ii) The hypothesis of (b) implies, by Theorem 2, that $d_n = o(1)$, and hence that starring is possible.

(iii) By a conjugacy argument (a) implies (b).

(iv) In § 5 the cases $q = \nu = \infty$, $p = \nu' = 1$ are shown to hold in a modified form.

(v) In § 6 Theorem 4 is used to prove a known result.

(vi) A similar argument to that in Zygmund (12) shows that Theorem 4 implies Theorem 2 although in a slightly less precise form.

(vii) If d_n takes only the values 0, 1, -1, (a) is true. Because, let $N < \infty$ be the number of non-zero terms, then by Theorem 2

$$J_q^q(f) \leqslant \{U_p(d)\}^q = N \leqslant K_{p,\nu} \sum_{n=1}^{N} n^{(q-2)/(1-\nu')} = K_{p,\nu} V_q^q(d^*).$$

(viii) Similarly (b) is true if f is a function such that d_n takes only the values 0, 1, -1.

(ix) Finally we have the following weaker result.

THEOREM 5. (a) *Let* $d_n = o(1)$ *be such that for an* $\epsilon > 0$, $2 \leqslant q < \nu$, $V_q(n^\epsilon d^*_n) < \infty$. *Then Theorem* 4 (a) *holds with* (13) *replaced by*

$$J_q(f) \leqslant A_{q,\nu,\epsilon} V_q(n^\epsilon d^*_n).$$

(b) *With the hypothesis of Theorem* 4(b) *we have*

$$V_p(n^{-\epsilon} d^*_n) \leqslant A_{p,\nu,\epsilon} J_p(f), \textit{ for all } \epsilon > 0.$$

As the usual conjugacy argument shows that (a) implies (b) it is sufficient to prove (a). By Theorem 2:

$$J_q(f) \leqslant U_p(d) = U_p(d^*) \leqslant A_{q,\nu,\epsilon} V_q(n^\epsilon d^*).$$

5. Some limiting cases of Paley's theorem. It is known that the Paley results are not valid for the extreme values of p and q, that is, $p = \nu'$, $q = \nu$. Zygmund, **(11)**, has extended the results to these cases for uniformly bounded $\{\phi_n\}$ by slightly modifying the hypotheses and conclusions.

Let us, for convenience, number the orthonormal sequence $\phi_2, \phi_3, \ldots$, and also let $\nu = \infty$. By $f \in L_{p,q}$ we shall mean that $|f|^p (\log^+|f|)^q \in L$. We place no restriction on the sequence $\{F_n\}$ and so the star theorems follow immediately from the unstarred results. The proofs follow Zygmund's closely enough for them to be omitted here.

THEOREM 6. *Let $\{d_n\}$ be any sequence satisfying*

$$d_n' \leqslant n^{-1}(\log n)^{\alpha-1}, \quad \alpha < 0, \; n = 2, 3, \ldots,$$

where $\{d_n'\}$ is some ordering of $\{|d_n|\}$. Then $c_n = d_n F_n^{-1}$ is the coefficient with respect to ϕ_n of a function such that for $\lambda > 0$, small enough,

$$\int_0^1 \exp\{\lambda|f|^{1/\alpha}\}\, dx < A,$$

THEOREM 7. *If $f \in L_{1,\alpha}$, $\alpha > 0$, and if $\{c_n\}$ are the Fourier coefficients of f with respect to $\{\phi_n\}$ and if $d_n = d_n$ (1) then*

(a) $$\sum_{n=2}^{\infty} n^{-1}(\log n)^{\alpha-1} d^*_n \leqslant A \int_0^1 |f|(\log^+|f|)^\alpha\, dx + B = C,$$

(b) $$\sum_{n=2}^{\infty} \exp(-k\, d^*_n{}^{-1/\alpha}) < \infty, \qquad \text{for every } k > 0,$$

(c) if in addition $\alpha \leqslant 1$, $\sum_{n=2}^{\infty} n^{-1} d^*_n{}^{1/\alpha} \leqslant K_\alpha C^{1/\alpha}$.

THEOREM 8. If *$\{d_n\}$ be any sequence such that*

$$\sum_{n=2}^{\infty} |d_n|(\log 1/|d_n|)^\alpha < \infty, \alpha > 0,$$

then $c_n = d_n F_n^{-1}$ is the Fourier coefficient with respect to ϕ_n of a function f such that exp $(k|f|^{1/\alpha}) \in L$, *for every $k > 0$.*

THEOREM 9. *If $\{d_n\}$ is such that $d_n = o(1)$ and*

$$\sum_{n=2}^{\infty} n^{r-1} d^{*r}_n < \infty,$$

$r > 1$, then $c_n = d_n F_n^{-1}$ is the Fourier coefficient of a function f such that exp $(k|f|^{r'}) \in L$, for all $k > 0$.

5.2. The following theorem generalizes results due to Verblunsky, **(10)**.

THEOREM 10.

(a) If $\mu = \dfrac{1}{r} - \dfrac{1}{q}, \dfrac{\nu'}{q} + \dfrac{2-\nu'}{p} = 1$, and $p \leqslant r \leqslant q$ then

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(i) $\left(\int_0^1 f^{*r} x^{-\mu r} dx\right)^{1/r} \leqslant A_{p,\nu} U_p(c),$

(ii) $\left(\sum_{n=1}^{\infty} |c_n|^r \, n^{(-\nu'\mu r)/(2-\nu')} \, F_n^{2-r-(2\mu r\nu')/(2-\nu')}\right)^{1/r} \leqslant A_{p,\nu} J_p(f).$

(b) If $\mu = \dfrac{1}{r} - \dfrac{1}{p}, \dfrac{\nu'}{p} + \dfrac{2-\nu'}{q} = 1, p \leqslant r \leqslant q$ then

(i) $U_q(c) \leqslant A_{q,\nu} \left(\int_0^1 f^{*r} x^{-\mu r} dx\right)^{1/r},$

(ii) $J_q(f) \leqslant A_{q,\nu} \left(\sum_{n=1}^{\infty} |c_n|^r \, n^{(-\nu'\mu r')(2-\nu')} F_n^{2-r-(2\mu r\nu')/(2-\nu')}\right)^{1/r}.$

[$f^*(x)$ is a non-increasing rearrangement of $|f(x)|$, **(8)**.]

Extreme values of r give known theorems. For instance if, in (a), $r = p$ then (i) reduces to the integral analogue of Theorem 4 (b), and (ii) becomes the unstarred form of Theorem 4 (b). If $r = q$ then (i) and (ii) of (a) reduce to parts (a) and (b) of Theorem 2 respectively.

The proof of (a) is by an application of Hölder's inequality using these extreme forms.

(b) follows by a similar argument or by a conjugacy argument from (a).

5.3. Further extensions of Paley's theorems are obtainable by integrating with respect to q, or by multiplying through by a function $K(q)$ and integrating, **(9)**. For example, integration of the unstarred form of (13) gives

$$\left(\int_0^1 \frac{|f|^q - |f|^2}{\log |f|} dx\right)^{1/q}$$

$$\leqslant A_{q,\nu} \left\{ \sum_{n=1}^{\infty} \frac{(|c_n| F_n^{\nu'/(2-\nu')} n^{1/(2-\nu')})^q - (|c_n| F_n^{\nu'/(2-\nu')} n^{1/(2-\nu')})^2}{F_n^{\nu'/(2-\nu')} n^{2/(2-\nu')} \log (|c_n| F_n^{\nu'/(2-\nu')} n^{1/(2-\nu')})} \right\}^{1/q}.$$

5.4. The Paley theorems were originally proved for the trigonometric system by Hardy and Littlewood **(4)**, where they arose out of the following problem. If $f \in L_r$ and

$$f \sim \sum c_n \phi_n,$$

for what value of Y and X does

$$\sum n^{-X} |c_n|^Y$$

converge? Using the above results we can solve this problem in the case of an orthonormal sequence satisfying

$$J_\nu(\phi_n) \leqslant K \, n^\alpha, \alpha \geqslant 0.$$

THEOREM 11. *If* $f \in L_r$, $r > 1$, *and* $(\nu'/p) + (2 - \nu'/q)$, *then the series*

$$\sum_{n=1}^{\infty} n^{-X} |c_n|^Y$$

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is convergent if

(i) $r > 2,\ Y \geqslant 2, X \geqslant 0,$

(ii) $r > 2,\ Y < 2, X > 1 - \dfrac{Y}{2},$

(iii) $r = p \leqslant 2,\ Y > q, X > \alpha(q-2),$

(iv) $r = p \leqslant 2, p \leqslant Y \leqslant q, X \geqslant (1+\alpha Y) - \dfrac{Y}{q}(1+2\alpha).$

(v) $r = p \leqslant 2, 0 \leqslant Y < p, X > (1+\alpha Y) - \dfrac{Y}{q}(1+2\alpha),$

and, in general, it is not necessarily convergent in any other case.

The proof follows that of Hardy and Littlewood exactly.

6. Applications.

6.1. Let $f(P)$ be an integrable function defined on the surface, S, of the unit α-sphere, $\alpha > 1$. Any such function can be expanded in terms of the orthonormal sequence of ultraspherical polynomials $\{V_n^{(\alpha)}(P)\}$ having the property

$$|V_n^{(\alpha)}(P)| \leqslant K_\alpha\, n^{(\alpha/2)-1}. \tag{15}$$

If $f(P) \sim \sum c_n V_n^{(\alpha)}(P)$, then we define

$$f(r,P) = \sum_{n=1}^{\infty} c_n V_n^{(\alpha)}(P) r^n, \qquad 0 \leqslant r < 1; \tag{16}$$

$f(r, P)$ is the function harmonic in the unit α-sphere with $f(P)$ as boundary function. Series (16) can be summed to the Poisson integral taken over the surface, E, of the α-sphere of radius r. Using this representation du Plessis, **(3)**, has proved a radial extension of the Fejér–Riesz theorem. It is known, **(4)**, that when $\alpha = 2$ the Fejér–Riesz theorem can be deduced from the Paley theorems. We will show that this is so in general.

For reference we note that for orthonormal sequences satisfying (15)

$$\begin{aligned}
d_n(q) &= |c_n| n^{((\alpha/2)-1)(1-(2/q))} \\
U_r(c) &= \left(\sum_{n=1}^{\infty} |c_n|^r\, n^{((\alpha/2)-1)(2-r)}\right)^{1/r} && 1 \leqslant r < \infty, \\
&= \max\,(|c_n|\, n^{1-(\alpha/2)}), && r = \infty. \\
V_r(d) &= \left(\sum_{n=1}^{\infty} |c_n|^r\, n^{(\alpha/2)(r-2)}\right)^{1/r} && 1 \leqslant r < \infty, \\
&= \max\,(|c_n|\, n^{\alpha/2}), && r = \infty.
\end{aligned}$$

THEOREM 12. *If $f(r, P)$ is subharmonic in the unit α-sphere and*

$$\int_E f(r, P)^p dP \leqslant C, \qquad p > 1, r < 1, \tag{17}$$

then

$$\int_0^1 (1-r)^{\alpha-2} |f(r, P)|^p dr < K_{p,\alpha} C.$$

It is known that it is sufficient to prove this for f harmonic and p arbitrary but near to 1. Then it is an immediate consequence of the following lemma.

LEMMA. *Let $f(r, P)$ be given by* (16), *and define*

$$F(r) = \sum_{n=1}^{\infty} |c_n| \, n^{(\alpha/2)-1} r^n. \tag{18}$$

If $1 < p \leqslant 2$ and (17) *holds then*

$$\int_0^1 (1-r)^{\alpha-2} F^p(r) dr \leqslant K_{p,\alpha} C.$$

This lemma, an extension of one in **(4)**, is stronger than Theorem 12 when $1 < p \leqslant 2$, but is false if $p > 2$.

By Theorem 2 with $\nu = q = \infty$ we have

$$|c_n| \, n^{1-(\alpha/2)} r^n \leqslant K_\alpha \int_E |f(r, P)| \, dP,$$

and hence, from (17),

$$|c_n| \leqslant K_\alpha C \, n^{(\alpha/2)-1},$$

which gives

$$F(e^{-1}) \leqslant K_\alpha C.$$

Therefore, using Lemma 36 of **(4)** and the unstarred form of (14), we have

$$\begin{aligned} \int_0^1 (1-r)^{\alpha-2} F^p(r) dr &\leqslant K_{p,\alpha} C + \int_{e^{-1}}^1 (1-r)^{\alpha-2} F^p(r) dr \\ &\leqslant K_{p,\alpha} C + K \sum_{n=1}^{\infty} n^{-2} (1 - e^{-(1/n)})^{\alpha-2} F^p(e^{-(1/n+1)}) \\ &\leqslant K_{p,\alpha} C + K \sum_{n=1}^{\infty} n^{-\alpha} \left(\sum_{m=1}^{\infty} |c_m| m^{(\alpha/2)-1} e^{(m/n)} \right)^p \\ &\leqslant K_{p,\alpha} C + K \, V_p(d) \leqslant K_{p,\alpha} C. \end{aligned}$$

It is known that Theorem 12 is false if $p = 1$ but the following result can be proved.

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THEOREM 13. *Let $f(r, P)$ be subharmonic in the unit α-sphere. If, $p \geqslant 1$,*

$$\int_E |f(r, P)| (\log^+ |f(r, P)|)^{1/p}\, dP \leqslant C, r < 1, \tag{19}$$

then

$$\int_0^1 (1 - r)^{p(\alpha-1)-1} |f(r, P)|^p\, dr \leqslant AC + B.$$

This follows from the lemma,

LEMMA. *Let $f(r, P)$ be given by* (16) *and $F(r)$ by* (18) *then if* (19) *holds*

$$\left(\int_0^1 (1 - r)^{p(\alpha-1)-1} F^p(r) dr\right)^{1/p} \leqslant AC + B.$$

The proof of this lemma is similar to the above proof using Theorem 7 (c) in place of the unstarred form of (14). The case $p = 1$ of this theorem has been proved by du Plessis, **(3)**, who considers diametral as well as radial theorems.

6.2. If $f(P) \sim \sum c_n V_n^{(\alpha)}(P)$

then

$$f_\beta(P) \sim \sum n^{-\beta} c_n V_n^{(\alpha)}(P)$$

is called the βth integral of f. If $\alpha = 2$, then Hardy and Littlewood, **(5)**, proved that if $f \in L_p$ then $f_\beta \in L_q$ where $\beta = 1/p - 1/q$. This result has been extended by du Plessis, **(2)**, to general α. Zygmund **(12)** has shown that, in the case $\alpha = 2$, the result follows from the Paley star theorems provided $p \leqslant 2 \leqslant q$. We will show that this is the case in general, assuming the truth of Theorem 4.

THEOREM 14. *If $f \in L_p$, $p > 1$, $0 < \beta < (\alpha - 1)/p$, then $f_\beta \in L_q$ where q is given by $\beta = (\alpha - 1)(1/p - 1/q)$. Further*

$$\left(\int_S |f_\beta|^q\, dP\right)^{1/q} \leqslant K_{p,q,\alpha} \left(\int_S |f|^p\, dP\right)^{1/p}.$$

We may assume that the right-hand integral has value 1. From Theorem 2,

$$\left(\int_S |f_\beta|^q\, dP\right)^{1/q} \leqslant K_{p,\alpha} U_p(d) \leqslant K_{p,\alpha} U_p(d^*)$$

$$\leqslant K_{p,\alpha} \max(d_n^* n^{1/p'})^{1-(p/q')} V_p^{(p/q')}(d^*),$$

since $p' > q'$, and provided $q' \geqslant p$.

Since $d^{*p}_n n^{p-2}$ decreases monotonically we have

$$d_n^* n^{1/p'} = 0(1)$$

with bound not exceeding $V_p{}^p(d^*)$.

Hence if $q' \geqslant p$,

$$\left(\int_S |f_\beta|^q\, dP\right)^{1/q} \leqslant K_{p,q,\alpha}\, V_p^{(p(1-p+q')/q')}(d^*) \leqslant K_{p,q,\alpha}$$

by Theorem 4. The completion to all $p, q, p \leqslant 2 \leqslant q$, follows as in Zygmund, **(12)**.

It is known that Theorem 14 is false if $p = 1$ but a modified theorem can be proved, again subject to $q \geqslant 2$, although the result is probably true without this restriction.

THEOREM 15. (i) *If* $f \in L_{1,(1/q)}$, $q > 1$, *then* $f_\beta \in L_q$ *where* β *is given by* $\beta = (\alpha - 1)/q'$. *Moreover*

$$\left(\int_S |f_\beta|^q\, dP\right)^{1/q} \leqslant A \int_S |f|\, (\log^+|f|)^{1/q}\, dP + B.$$

(ii) *If* $f \in L_{1,1}$, *then* $f_\beta \in L_q$ *where* q *is given by* $\beta = (\alpha - 1)/q'$, *and moreover*

$$\left(\int_S |f_\beta|^q\, dP\right)^{1/q} \leqslant A \int_S |f|\, (\log^+|f|)\, dP + B,$$

This is a generalization of a result due to Zygmund **(11)** although his proof is different. We deduce it from Theorem 7 (c) and the unstarred Theorem 4. Let $d_n = d_n(1)$, then these two results imply that

$$\left(\int_S |f_\beta|^q\, dP\right)^{1/q} \leqslant K_{q,\alpha}\left(\sum_{n=1}^{\infty} |d_n|^q\, n^{-1}\right)^{1/q} \leqslant A \int_S |f|\, (\log^+|f|)^{1/q}\, dP + B.$$

which is (i). In a similar manner (ii) follows, but is also a consequence of (i) since $f \in L_{1,1}$ implies $f \in L_{1,(1/q)}$ for all $q > 1$.

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