A CLOSURE CRITERION FOR ORTHOGONAL FUNCTIONS

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1. Introduction. In this paper we give a simple, necessary, and sufficient condition for a sequence of orthogonal functions to be closed in L_2. In theory the question of closure is reduced to the evaluation of certain integrals and the summation of an infinite series whose terms depend only upon the index n. Our principal result is

THEOREM I. *Let $p(t)$ be a function whose zeros and discontinuities have Jordan content zero, such that for each $x \in (a, b)$, $p(t) \in L_2$ on* $\min(c, x) < t < \max(c, x)$, *where $a \leqslant c \leqslant b$. ($a$, b, and c may be infinite.) Let $w(x)$ be a measurable function almost everywhere finite and positive, and such that*

$$w(x)\int_c^x |p(t)|^2 dt \in L_1$$

on (a, b). Then for any family of functions $\{\phi_n\}$ orthogonal and normal on (a, b).

$$\sum_{n=1}^{\infty} \int_a^b \left| \int_c^x p(t)\phi_n(t)dt \right|^2 w(x)dx \leqslant \int_a^b \left| \int_c^x |p(t)|^2 dt \right| w(x)dx, \tag{1.1}$$

where equality holds if and only if $\{\phi_n\}$ is closed in L_2 on (a, b).

The insertion of the functions $p(t)$ and $w(x)$ serves two purposes. First, it enables us to deal with the case where the interval (a, b) is infinite, and second, proper choice of these functions greatly facilitates the calculation of the integrals involved and the summation of the series on the left side of (1.1).

Several special cases of Theorem I were published by Dalzell [**2**, **3**]. Inasmuch as Dalzell's results were stated only for special cases and under rather stringent hypotheses, it seemed desirable to publish the criterion in the above general form in which it was rediscovered by the author.

Actually Theorem I is a special case of a more general result. Before stating this result, it is necessary to attend to certain matters of notation. Note that we distinguish carefully between open intervals (a, b) and closed intervals $[a, b]$. The distinction is necessary in order that the statements and proofs should be valid for both finite and infinite intervals. We use $|a, b|$ to denote the open interval $\min(a, b) < x < \max(a, b)$, while $\chi_E(x)$ is the characteristic function of the set E and $||f||$ is the L_2 norm of f.

Definition 1.1. $\mathfrak{W}_{a,b}$ is the class of measurable functions $w(x)$ which are positive and finite almost everywhere on (a, b).

Received November 20, 1950.

https://doi.org/10.4153/CJM-1952-018-6 Published online by Cambridge University Press

Definition 1.2. $\mathfrak{P}_{a,b}{}^c$, where $c \in [a, b]$, is the class of measurable functions $p(t)$ on (a, b) such that

A. For each $x \in (a, b)$, $p(t) \in L_2$ on $|c, x|$.

B. The class of functions of the form

$$f(t) = \sum_{k=1}^{m} c_k p(t) \chi_{(a_k, b_k)}(t); \qquad a_k, b_k \in (a, b), \tag{1.2}$$

is dense in L_2 on the interval (a, b).

We are now in a position to state the generalization of Theorem I:

THEOREM II. *Let* $p(t) \in \mathfrak{P}_{a,b}{}^c$ *and* $w(x) \in \mathfrak{W}_{a,b}$ *be such that*

$$w(x)\int_c^x |p(t)|^2 dt \in L_1$$

on (a, b), *and let* $\{\phi_n\}$ *be a family of orthonormal functions on* (a, b). *Then*

$$\sum_{n=1}^{\infty} \int_a^b \left| \int_c^x p(t)\phi_n(t)dt \right|^2 w(x)dx \leqslant \int_a^b \left| \int_c^x |p(t)|^2 dt \right| w(x)dx, \tag{1.3}$$

where equality holds if and only if $\{\phi_n\}$ *is closed*[1] *in* L_2 *on* (a, b).

The fact that condition B of Definition 1.2 is not nearly as restrictive as it appears is a consequence of the next theorem:

THEOREM III. *Let* $p(t)$ *be a function whose discontinuities and zeros are of Jordan content zero. Then the class of functions of the form*

$$f(t) = \sum_{k=1}^{m} c_k p(t) \chi_{(a_k, b_k)}(t); \qquad a_k, b_k \in (a, b),$$

is dense in L_2 *on the interval* (a, b).

Theorem I is, of course, an immediate corollary of Theorems II and III. We devote §§2 and 3 to the proofs of these theorems, while in §4 we apply Theorem I to establish the closure of the Hermite functions. For further applications see Dalzell [**2**; **3**], where the method is used to prove closure of the trigonometric, Legendre, Jacobi, and Laguerre functions, and also for Dini's series in the theory of Bessel functions.

2. Proof of Theorem II. We first establish the inequality (1.3). Except at most for a matter of sign,

$$\int_c^x \overline{p(t)}\ \overline{\phi_n(t)}dt$$

is the nth orthogonal coefficient of

$$\overline{p(t)}\chi_{|c,x|}(t).$$

[1]It will be seen from the proof that, if $\{\phi_n\}$ is closed in L_2 on (a, b), then the equality holds in (1.3) even without the restriction that the functions of the form (1.2) be dense in L_2 on (a, b).

https://doi.org/10.4153/CJM-1952-018-6 Published online by Cambridge University Press

By Bessel's inequality we have

$$(2.1)\qquad \sum_{n=1}^{\infty}\left|\int_c^x \overline{p(t)}\ \overline{\phi_n(t)}dt\right|^2 \leqslant \int_a^b \left|\overline{p(t)}\chi_{|c,x|}(t)\right|^2 dt = \left|\int_c^x |\overline{p(t)}|^2 dt\right|,$$

whence

$$(2.2)\qquad \sum_{n=1}^{\infty}\left|\int_c^x \overline{p(t)}\ \overline{\phi_n(t)}dt\right|^2 w(x) \leqslant \left|\int_c^x |\overline{p(t)}|^2 dt\right| w(x).$$

Integration of (2.2) yields

$$(2.3)\qquad \sum_{n=1}^{\infty}\int_a^b\left|\int_c^x \overline{p(t)}\ \overline{\phi_n(t)}dt\right|^2 w(x)dx \leqslant \int_a^b\left|\int_c^x |\overline{p(t)}|^2 dt\right| w(x)dx,$$

which is equivalent to (1.3).

In case $\{\phi_n\}$ is closed in L_2 on (a, b), the equalities hold in (2.1), (2.2), and (2.3), and hence equality holds in (1.3).

Suppose now that equality holds in (1.3), and thus in (2.3). Then equality must hold almost everywhere in (2.2), and hence almost everywhere in (2.1); that is, for almost all x's Parseval's equality holds for the functions

$$\overline{p(t)}\chi_{|c,x|}(t),$$

and in particular for a set of x's dense in (a, b). We conclude at once that $\{\phi_n\}$ is closed with respect to all functions of the form

$$\overline{p(t)}\chi_{|c,x|}(t);$$

thus $\{\phi_n\}$ is closed with respect to all functions of the form

$$(2.4)\qquad f(t) = \sum_{k=1}^{m} c_k\overline{p(t)}\chi_{(a_k,b_k)}(t); \qquad a_k, b_k \in (a,b).$$

But

$$p(t) \in \mathfrak{P}_{a,b}{}^c$$

implies that

$$\overline{p(t)} \in \mathfrak{P}_{a,b}{}^c.$$

Since by this remark the functions of the form (2.4) are dense in L_2, the L_2 closure of $\{\phi_n\}$ follows immediately.

3. Proof of Theorem III. As the class of step functions vanishing outside a finite interval is dense in L_2 and as every such function is a linear combination of characteristic functions of finite intervals, it will suffice to show that the characteristic function of a finite interval can be approximated arbitrarily closely in norm by functions of the form (1.2). Since the set of zeros and discontinuities of $p(t)$ can be covered by a finite number of intervals of arbitrarily small total measure, it will certainly suffice to show that we can approximate $\chi_{(\alpha,\beta)}(t)$ arbitrarily closely in L_2 norm by functions of the form (1.2), where $[\alpha,\beta]$ is a finite interval which contains no zeros or discontinuities of $p(t)$.

Let $\epsilon > 0$ be given, let m be the minimum of $|p(t)|$ on $[\alpha,\beta]$, and let $\delta > 0$ be chosen so that the oscillation of $p(t)$ on every subinterval of $[\alpha,\beta]$ whose length does not exceed δ is less than $\epsilon m(\beta - \alpha)^{-\frac{1}{2}}$. Let

$$\alpha = t_0 < t_1 < \ldots < t_n = \beta$$

be a partition of $[\alpha,\beta]$ whose norm does not exceed δ. For $t \in [t_{k-1},t_k]$ we have $|p(t_k) - p(t)| < \epsilon m(\beta - \alpha)^{-\frac{1}{2}}$, whence

$$\left|1 - \frac{p(t)}{p(t_k)}\right| < \epsilon (\beta - \alpha)^{-\frac{1}{2}}, \qquad t \in [t_{k-1}, t_k]. \tag{3.1}$$

Define

$$f(t) = \sum_{k=1}^{n} \frac{p(t)}{p(t_k)} \chi_{(t_{k-1}, t_k)}(t). \tag{3.2}$$

We see from (3.1) and (3.2) that, for $t \in (t_{k-1},t_k)$,

$$\left|\chi_{(\alpha,\beta)}(t) - f(t)\right| = \left|1 - \frac{p(t)}{p(t_k)}\right| < \epsilon (\beta - \alpha)^{-\frac{1}{2}}, \tag{3.3}$$

while both $f(t)$ and $\chi_{(\alpha,\beta)}(t)$ vanish outside of (α,β). From this last remark and (3.3) we have $||\chi_{(\alpha,\beta)} - f|| \leqslant \epsilon$, as was to be proved.

4. Closure of the Hermite functions. The normalized Hermite functions $\{\phi_n\}$ are defined as

$$\phi_n(x) = (\pi^{\frac{1}{2}} n! 2^n)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} H_n(x) \qquad (n = 0,1,2, \ldots), \tag{4.1}$$

where the Hermite polynominals $H_n(x)$ are given by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \qquad (n = 0,1,2, \ldots). \tag{4.2}$$

We will use the facts that [**1**, pp. 77-79; **4**, pp. 143-144]

$$H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}, \quad H_{2n+1}(0) = 0 \qquad (n = 0,1,2, \ldots) \tag{4.3}$$

and

$$H_n'(x) = 2nH_{n-1}(x) \qquad (n = 1,2, \ldots). \tag{4.4}$$

To establish the closure of the Hermite functions, we choose $p(t) = \exp\{\frac{1}{2}t^2\}$, $w(x) = \exp\{-2x^2\}$. According to Theorem I, the closure is equivalent to the equality

$$\sum_{n=0}^{\infty} \frac{1}{\pi^{\frac{1}{2}} n! 2^n} \int_{-\infty}^{\infty} \left[\int_0^x H_n(t) dt\right]^2 e^{-2x^2} dx = 2\int_0^{\infty} \left[\int_0^x e^{t^2} dt\right] e^{-2x^2} dx.$$

A transformation to polar coordinates and an elementary integration yield

$$2\int_0^{\infty} \left[\int_0^x e^{y^2} dy\right] e^{-2x^2} dx = 2^{-\frac{1}{2}} \log (1 + 2^{\frac{1}{2}}),$$

so that if we set

$$J_n = (2n)^2 \int_{-\infty}^{\infty} \left[\int_0^x H_{n-1}(t)dt \right]^2 e^{-2x^2} dx \qquad (n = 1,2, \ldots),$$

it remains only to show that

$$\sum_{n=0}^{\infty} \frac{J_{n+1}}{\pi^{\frac{1}{2}} n! 2^n (2n+2)^2} = 2^{-\frac{1}{2}} \log (1 + 2^{\frac{1}{2}}). \tag{4.5}$$

Use of (4.4) yields

$$J_n = \int_{-\infty}^{\infty} H_n^2(x) e^{-2x^2} dx - 2H_n(0) \int_{-\infty}^{\infty} H_n(x) e^{-2x^2} dx + H_n^2(0) \int_{-\infty}^{\infty} e^{-2x^2} dx.$$

Upon integrating by parts n times and using (4.2) we obtain

$$\int_{-\infty}^{\infty} H_n^2(x) e^{-2x^2} dx = \int_{-\infty}^{\infty} \left[\frac{d^n}{dx^n} e^{-x^2} \right]^2 dx = (-1)^n \int_{-\infty}^{\infty} H_{2n}(x) e^{-2x^2} dx,$$

from which

$$J_n = (-1)^n I_{2n} - 2H_n(0) I_n + \left(\tfrac{1}{2}\pi \right)^{\frac{1}{2}} H_n^2(0), \tag{4.6}$$

where we have written

$$I_n = \int_{-\infty}^{\infty} H_n(x) e^{-2x^2} dx.$$

We have

$$I_{2n} = \int_{-\infty}^{\infty} e^{-x^2} \frac{d^{2n}}{dx^{2n}} e^{-x^2} dx = \left[\frac{d^{2n}}{dt^{2n}} \int_{-\infty}^{\infty} e^{-x^2} e^{-(x+t)^2} dx \right]_{t=0} \tag{4.7}$$

$$= \left[\left(\tfrac{1}{2}\pi \right)^{\frac{1}{2}} \frac{d^{2n}}{dt^{2n}} e^{-t^2/2} \right]_{t=0} = \left(\tfrac{1}{2}\pi \right)^{\frac{1}{2}} \frac{(-1)^n (2n)!}{n! 2^n},$$

while, as $H_{2n+1}(x)$ is an odd function,

$$I_{2n+1} = 0. \tag{4.8}$$

Use of (4.3), (4.6), (4.7), and (4.8) yields after some reductions

$$\sum_{n=0}^{\infty} \frac{J_{n+1}}{\pi^{\frac{1}{2}} n! 2^n (2n+2)^2} \tag{4.9}$$

$$= 2^{-3/2} \left\{ \frac{3}{2} \sum_{n=1}^{\infty} \frac{(1/2)(3/2) \ldots (n - \frac{1}{2})}{(n!)n} - \sum_{n=1}^{\infty} \frac{(1/2)(3/2) \ldots (n - \frac{1}{2})}{(n!)n} (\tfrac{1}{2})^n \right\}$$

$$= 2^{-3/2} \left\{ \frac{3}{2} K(1) - K(\tfrac{1}{2}) \right\},$$

where we have set

$$K(a) = \sum_{n=1}^{\infty} \frac{(1/2)(3/2) \ldots (n - \frac{1}{2})}{(n!)n} a^n.$$

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The series defining $K(\alpha)$ converges absolutely for $|\alpha| \leqslant 1$, and for $|\alpha| \leqslant 1$,

$$K(\alpha) = \int_0^\alpha \frac{1}{x} \sum_{n=1}^{\infty} \frac{(1/2)(3/2)\dots(n-\frac{1}{2})}{n!} x^n dx \tag{4.10}$$

$$= \int_0^\alpha \frac{1}{x}\left[(1-x)^{-\frac{1}{2}} - 1\right] dx = 2 \log\left[\frac{2}{1+(1-\alpha)^{\frac{1}{2}}}\right].$$

It follows from (4.10) that

$$2^{-3/2}\left\{\frac{3}{2}K(1) - (\tfrac{1}{2})\right\} = 2^{-\frac{1}{2}} \log(1+2^{\frac{1}{2}}). \tag{4.11}$$

From (4.9) and (4.11) we obtain (4.5), and the closure of the Hermite functions is established.

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https://doi.org/10.4153/CJM-1952-018-6 Published online by Cambridge University Press