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A CHARACTERISATION OF TRACIALLY NUCLEAR C*-ALGEBRAS

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Dedicated to the memory of Uffe Haagerup

Abstract

We give two characterisations of tracially nuclear C*-algebras. The first is that the finite summand of the second dual is hyperfinite. The second is in terms of a variant of the weak* uniqueness property. The necessary condition holds for all tracially nuclear C*-algebras. When the algebra is separable, we prove the sufficiency.

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1. Introduction

Suppose $\mathcal{A}$ is a unital C*-algebra, $\mathcal{M}$ is a von Neumann algebra and $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are unital $*$-homomorphisms. We say that π and ρ are *weak* approximately unitarily equivalent in* $\mathcal{M}$ if and only if there are nets $\{U_\lambda\}$ and $\{V_\lambda\}$ of unitary operators in $\mathcal{M}$ such that, for every $a \in \mathcal{A}$,

$$U_\lambda \pi(a) U_\lambda^* \to \rho(a) \quad \text{and} \quad V_\lambda \rho(a) V_\lambda^* \to \pi(a)$$

in the weak*-topology. In [3], Ding and Hadwin defined the $\mathcal{M}$-rank(T) of an operator T in $\mathcal{M}$ as the Murray–von Neumann equivalence class of the projection onto the closure of the range of T.

In [1], Ciuperca, Giordano, Ng and Niu proved that if $\mathcal{A}$ is a separable C*-algebra, then the following properties are equivalent:

(1) for every separably acting von Neumann algebra $\mathcal{M}$ and all representations $\pi, \rho : \mathcal{A} \to \mathcal{M}$, the homomorphism π is weak* approximately unitarily equivalent to ρ if and only if $(\mathcal{M}\text{-rank}) \circ \pi = (\mathcal{M}\text{-rank}) \circ \rho$;
(2) $\mathcal{A}$ is nuclear.

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In this paper we investigate how the Ciuperca–Giordano–Ng–Niu theorem changes if in statement (1) we restrict $\mathcal{M}$ to be a finite von Neumann algebra. The answer turns out to be the condition that $\mathcal{A}$ is *tracially nuclear*, a condition defined in [5].

It is known that a C*-algebra is *nuclear* if and only if, for every Hilbert space H and every unital $*$-homomorphism $\pi : \mathcal{A} \to B(H)$, the von Neumann algebra $\pi(\mathcal{A})''$ generated by $\pi(\mathcal{A})$ is hyperfinite [8]. In [5] a unital C*-algebra $\mathcal{A}$ was defined to be *tracially nuclear* if, for every tracial state τ on $\mathcal{A}$, if π_τ is the GNS representation for τ, then $\pi_\tau(\mathcal{A})''$ is hyperfinite. Tracially nuclear algebras also play a key role in the theory of tracially stable C*-algebras [7].

We give two new characterisations of tracially nuclear C*-algebras: the first (Theorem 2.1) in terms of the second dual of the algebra, and the second (Theorem 4.3) in terms of weak* approximate equivalence of representations into finite von Neumann algebras. In one direction, we show (Theorem 3.3) that if $\mathcal{A}$ is any tracially nuclear C*-algebra and $\mathcal{M}$ is any finite von Neumann algebra, then the rank condition in [3] on two representations $\pi, \rho : \mathcal{A} \to \mathcal{M}$ implies a strong version of weak* approximate equivalence of π and ρ. When $\mathcal{A}$ is separable we prove the converse (Theorem 4.3). Thus the second characterisation is an analogue of the characterisation of nuclearity given in [1].

When $\mathcal{A}$ is separable, we only need to check $\pi_\tau(\mathcal{A})''$ is hyperfinite when τ is an *infinite-dimensional factor state*, that is, $\pi_\tau(\mathcal{A})''$ is a II_1 factor von Neumann algebra.

Lemma 1.1. *Suppose $\mathcal{A}$ is a separable unital C*-algebra. Then $\mathcal{A}$ is tracially nuclear if and only if, for every infinite-dimensional factor tracial state τ on $\mathcal{A}$, $\pi_\tau(\mathcal{A})''$ is hyperfinite.*

Proof. We let $N = \pi_\tau(\mathcal{A})''$. Since $\mathcal{A}$ is separable, N acts on a separable Hilbert space. Using the central decomposition, we can write $N = \int_\Omega^\oplus N_\omega \, d\mu(\omega)$ where each N_ω is a factor von Neumann algebra, and we can write $\pi_\tau = \int_\Omega^\oplus \pi_\omega \, d\mu(\omega)$ and $\tau = \int_\Omega^\oplus \tau_\omega \, d\mu(\omega)$ with each τ_ω a factor state, each $\pi_\omega = \pi_{\tau_\omega}$ and each $\pi_{\tau_\omega}(\mathcal{A})'' = N_\omega$. Since N is hyperfinite if and only if almost every N_ω is hyperfinite, and since every finite-dimensional factor is hyperfinite, the lemma is proved. □

2. The second dual $\mathcal{A}^{\#\#}$

If $\mathcal{R} \subset B(H)$ is a finite von Neumann algebra, then we can write $H = \sum_{\gamma\in\Gamma}^\oplus H_\gamma$ and $\mathcal{R} = \sum_{\gamma\in\Gamma}^\oplus \mathcal{R}_\gamma$, where each $\mathcal{R}_\gamma \subset B(H_\gamma)$ has a faithful normal tracial state τ_γ. We can extend each τ_γ to a tracial state on $\mathcal{R}$ by $\tau_\gamma(T) = \tau_\gamma(T_\gamma)$, where $T = \sum_{\lambda\in\Gamma}^\oplus T_\lambda$. Each τ_γ gives a seminorm $\|T\|_{2,\gamma} = \tau_\gamma(T^*T)^{1/2}$. It is a simple fact that on bounded subsets of $\mathcal{R}$, the strong (SOT) and $*$-strong ($*$-SOT) operator topologies coincide and are generated by the family $\{\|\cdot\|_{2,\gamma} : \gamma \in \Gamma\}$. Thus a bounded net $\{T_n\}$ in $\mathcal{R}$ converges in SOT or $*$-SOT to $T \in \mathcal{R}$ if and only if, for every $\gamma \in \Gamma$,

$$\|T_n - T\|_{2,\gamma} \to 0.$$

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Also every von Neumann algebra $\mathcal{R}$ can be decomposed uniquely into a direct sum $\mathcal{R} = \mathcal{R}_f \oplus \mathcal{R}_i$, where $\mathcal{R}_f$ is a finite von Neumann algebra and $\mathcal{R}_i$ has no finite direct summands. Equivalently, $\mathcal{R}_i$ has no normal tracial states. Relative to this decomposition, we write $Q_{f,\mathcal{R}} = 1 \oplus 0$.

If $\mathcal{A}$ is a unital C*-algebra, then $\mathcal{A}^{\#\#}$ is a von Neumann algebra, and, using the universal representation, we can assume $\mathcal{A} \subset \mathcal{A}^{\#\#} \subset B(\mathcal{H})$ where the weak* topology on $\mathcal{A}^{\#\#}$ coincides with the weak operator topology, so that $\mathcal{A}'' = \mathcal{A}^{\#\#}$. Moreover, for every von Neumann algebra $\mathcal{R}$ and every unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{R}$, there is a weak*-weak* continuous unital $*$-homomorphism $\hat{\pi} : \mathcal{A}^{\#\#} \to \mathcal{R}$ such that $\hat{\pi}|_{\mathcal{A}} = \pi$. Moreover, $\ker \hat{\pi}$, being a weak* closed two-sided ideal in $\mathcal{A}^{\#\#}$, has the form

$$\ker \hat{\pi} = (1 - P_\pi)\mathcal{A}^{\#\#} \quad \text{with } P_\pi = P_\pi^2 = P_\pi^* \in \mathcal{Z}(\mathcal{A}^{\#\#}),$$

where $\mathcal{Z}(\mathcal{M})$ denotes the centre of a von Neumann algebra $\mathcal{M}$. Thus

$$\mathcal{A}^{\#\#} = P_\pi \mathcal{A}^{\#\#} \oplus \ker \hat{\pi}.$$

The following theorem contains our first characterisation of tracially nuclear C*-algebras.

Theorem 2.1. *If $\mathcal{A}$ is a unital C*-algebra, then:*

(1) *$P_\pi \le Q_{f,\mathcal{A}^{\#\#}}$ for every unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{M}$ with $\mathcal{M}$ a finite von Neumann algebra;*

(2) *$\mathcal{A}$ is tracially nuclear if and only if $(\mathcal{A}^{\#\#})_f$ is a hyperfinite von Neumann algebra.*

Proof. (1) Assume, by way of a contradiction, that $\hat{\pi}(1 - Q_{f,\mathcal{A}^{\#\#}}) \neq 0$. Since $\mathcal{M}$ is finite, there is a normal tracial state τ on $\mathcal{M}$ such that

$$s = \tau(\hat{\pi}(1 - Q_{f,\mathcal{A}^{\#\#}})) \neq 0.$$

Hence the map $\gamma : (\mathcal{A}^{\#\#})_i \to \mathbb{C}$ defined by

$$\gamma(T) = \frac{1}{s}\hat{\pi}(0 \oplus T)$$

is a faithful normal tracial state on $(\mathcal{A}^{\#\#})_i$, which is a contradiction. Thus

$$\hat{\pi}(1 - Q_{f,\mathcal{A}^{\#\#}}) = 0,$$

which means that $P_\pi \le Q_{f,\mathcal{A}^{\#\#}}$.

(2) Suppose $\mathcal{A}$ is tracially nuclear: $(\mathcal{A}^{\#\#})_f = \sum_{\lambda\in\Lambda}^{\oplus} (\mathcal{R}_\lambda, \tau_\lambda)$, where τ_λ is a faithful normal tracial state on $\mathcal{R}_\lambda$. Then $\mathcal{A}^{\#\#} = (\mathcal{A}^{\#\#})_f \oplus (\mathcal{A}^{\#\#})_i = \sum_{\lambda\in\Lambda}^{\oplus} \mathcal{R}_\lambda \oplus (\mathcal{A}^{\#\#})_i$ relative to $\mathcal{H} = \sum_{\lambda\in\Lambda}^{\oplus} \mathcal{H}_\lambda \oplus \mathcal{H}_i$. Viewing $\mathcal{A} \subset \mathcal{A}^{\#\#}$, we define $\pi_\lambda : \mathcal{A} \to \mathcal{R}_\lambda$ by $\pi_\lambda(A) = A|_{\mathcal{H}_\lambda}$. Then $\psi_\lambda = \tau_\lambda \circ \pi_\lambda$ is a tracial state on $\mathcal{A}$ and $\pi_{\psi_\lambda}(\mathcal{A})^{-\text{weak}*} = \mathcal{R}_\lambda$ since $\mathcal{A}$ is weak*-dense in $\mathcal{A}^{\#\#}$. Since $\mathcal{A}$ is tracially nuclear, $\mathcal{R}_\lambda$ must be hyperfinite. Hence, $(\mathcal{A}^{\#\#})_f = \sum_{\lambda\in\Lambda}^{\oplus} \mathcal{R}_\lambda$ is hyperfinite.

Conversely, suppose $(\mathcal{A}^{\#\#})_f$ is hyperfinite, and suppose τ is a tracial state on $\mathcal{A}$. Since $\pi_\tau(\mathcal{A})''$ has a faithful normal tracial state, it must be finite. Thus $P_{\pi_\tau} \le Q_{f,\mathcal{A}^{\#\#}}$. This means that $P_{\pi_\tau}\mathcal{A}^{\#\#}$ is a direct summand of $(\mathcal{A}^{\#\#})_f$, and is therefore hyperfinite. But this summand is isomorphic to $\pi_\tau(\mathcal{A})''$. Thus $\mathcal{A}$ is tracially nuclear. □

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3. Weak* approximate equivalence in finite von Neumann algebras

Suppose $\mathcal{A}$ is a unital C*-algebra, $\mathcal{R}$ is a von Neumann algebra and $\pi, \rho : \mathcal{A} \to \mathcal{R}$ are unital $*$-homomorphisms. Following [1], π and ρ are *weak* approximately equivalent* if there are nets $\{U_\lambda\}$ and $\{V_\lambda\}$ of unitary operators in $\mathcal{R}$ such that, for every $A \in \mathcal{A}$,

$$U_\lambda^* \pi(A) U_\lambda \overset{\text{weak*}}{\to} \rho(A) \quad \text{and} \quad V_\lambda^* \rho(A) V_\lambda \overset{\text{weak*}}{\to} \pi(A).$$

As observed in [1], it follows that the convergence above actually occurs in the $*$-strong operator topology.

Suppose $\mathcal{M}$ is a von Neumann algebra and $T \in \mathcal{M}$. Following [3], $\mathcal{M}$-rank(T) is defined to be the Murray–von Neumann equivalence class in $\mathcal{M}$ of the projection onto the closure of the range of T. In [1] it was shown that if $\mathcal{A}$ is a separable nuclear C*-algebra and $\mathcal{M}$ is a von Neumann algebra acting on a separable Hilbert space, then two unital $*$-homomorphisms $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are weak* approximately equivalent if and only if $(\mathcal{M}\text{-rank}) \circ \pi = (\mathcal{M}\text{-rank}) \circ \rho$. Moreover, this property for $\mathcal{A}$ is equivalent to nuclearity.

The following result is from [6]. For completeness we include a short proof.

Lemma 3.1 [6]. *Suppose $a = a^*$ in $\mathcal{B}(H)$, $0 \le a \le 1$ and $C_0^*(a)$ is the norm-closure of $\{p(a) : p \in \mathbb{C}[z], p(0) = 0\}$. Suppose $\mathcal{M}$ is a finite von Neumann algebra with a centre-valued trace $\Phi : \mathcal{M} \to \mathcal{Z}(M)$, and $\pi, \rho : C_0^*(a) \to \mathcal{M}$ are *-homomorphisms. Then the following properties are equivalent:*

(1) $\mathcal{M}$-rank $\pi(x) = \mathcal{M}$-rank $\rho(x)$ *for all* $x \in C_0^*(a)$*;*
(2) $\Phi \circ \pi = \Phi \circ \rho$.

Proof. (1) $\Rightarrow$ (2). Observe that we can extend π and ρ to weak*-weak* continuous *-homomorphisms $\hat{\pi}, \hat{\rho} : C_0^*(a)^{\#\#} \to \mathcal{M}$. Suppose $x \in C_0^*(a)$ and $0 \le x \le 1$. Suppose $0 < \alpha < 1$ and define $f_\alpha : [0, 1] \to [0, 1]$ by

$$f(t) = \text{dist}(t, [0, \alpha]).$$

Since $f(0) = 0$, we see that $f(x) \in \mathcal{A}$ and $\chi_{(\alpha,1]}(x) = \text{weak*-lim}_{n\to\infty} f(x)^{1/n} \in \mathcal{A}^{\#\#}$, so

$$\hat{\pi}(\chi_{(\alpha,1]}(x)) \quad \text{and} \quad \hat{\rho}(\chi_{(\alpha,1]}(x))$$

are the range projections for $\pi(f(x))$ and $\rho(f(x))$, respectively. Since

$$\mathcal{M}\text{-rank}\, \pi(f(x)) = \mathcal{M}\text{-rank}\, \rho(f(x)),$$

we see that $\hat{\rho}(\chi_{(\alpha,1]}(x))$ and $\hat{\pi}(\chi_{(\alpha,1]}(x))$ are Murray–von Neumann equivalent. Hence

$$\Phi(\hat{\pi}(\chi_{(\alpha,1]}(x))) = \Phi(\hat{\rho}(\chi_{(\alpha,1]}(x))).$$

Suppose $0 < \alpha < \beta < 1$. Since $\chi_{(\alpha,\beta]} = \chi_{(\alpha,1]} - \chi_{(\beta,1]}$,

$$\Phi(\hat{\pi}(\chi_{(\alpha,\beta]}(x))) = \Phi(\hat{\rho}(\chi_{(\alpha,\beta]}(x))).$$

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Thus, for all $n \in \mathbb{N}$,

$$\Phi\Big(\hat{\pi}\Big(\sum_{k=1}^{n-1} \frac{k}{n}\chi_{(k/n,(k+1)/n]}(x)\Big)\Big) = \Phi\Big(\hat{\rho}\Big(\sum_{k=1}^{n-1} \frac{k}{n}\chi_{(k/n,(k+1)/n]}(x)\Big)\Big).$$

For every $n \in \mathbb{N}$,

$$\Big\|x - \sum_{k=1}^{n-1} \frac{k}{n}\chi_{(k/n,(k+1)/n]}(x)\Big\| \leq 1/n,$$

and it follows that

$$\Phi(\pi(x)) = \Phi(\hat{\pi}(x)) = \Phi(\hat{\rho}(x)) = \Phi(\rho(x)).$$

Since $\mathcal{A}$ is the linear span of its positive contractions, $\Phi \circ \pi = \Phi \circ \rho$.

(2) $\Rightarrow$ (1). Since Φ, $\hat{\pi}$ and $\hat{\rho}$ are weak*-weak* continuous, $\Phi \circ \hat{\pi} = \Phi \circ \hat{\rho}$. So, for any $x \in C_0^*(a)$,

$$\Phi(\hat{\pi}(\chi_{(0,\infty)}(|x|))) = \Phi(\hat{\rho}(\chi_{(0,\infty)}(|x|))),$$

which implies $\chi_{(0,\infty)}(|\pi(x)|)$ and $\chi_{(0,\infty)}(|\rho(x)|)$ are Murray–von Neumann equivalent. Thus $\mathcal{M}$-rank $\pi(x) = \mathcal{M}$-rank $\rho(x)$. □

The following lemma is from [3].

Lemma 3.2 [3]. *Suppose $\mathcal{B} = \sum_{m=1}^{t} \mathcal{M}_{k_m}(\mathbb{C})$ with matrix units $e_{i,j,m}$, $\mathcal{D}$ is a unital C*-algebra and $\pi, \rho : \mathcal{B} \to \mathcal{D}$ are unital *-homomorphisms such that $\pi(e_{i,i,m}) \sim \rho(e_{i,i,m})$. Then there exists a unitary $w \in \mathcal{D}$ such that $\pi(\cdot) = w^*\rho(\cdot)w$.*

Theorem 3.3. *Suppose $\mathcal{A}$ is a unital tracially nuclear C^*-algebra, $\mathcal{M}$ is a finite von Neumann algebra with centre-valued trace Φ and $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are unital *-homomorphisms. Then the following properties are equivalent:*

(1) *$\mathcal{M}$-rank $\pi(a) = \mathcal{M}$-rank $\rho(a)$ for every $a \in \mathcal{A}$;*
(2) *$\Phi \circ \pi = \Phi \circ \rho$;*
(3) *the representations π and ρ are weak* approximately equivalent;*
(4) *there is a net $\{U_n\}$ of unitary operators in $\mathcal{M}$ such that, for every $a \in \mathcal{A}^{\#\#}$,*
    (a) *$U_n\pi(a)U_n^* \to \rho(a)$ in the $*$-SOT, and*
    (b) *$U_n^*\rho(a)U_n \to \pi(a)$ in the $*$-SOT.*

Proof. Clearly, (4) $\Rightarrow$ (3) $\Rightarrow$ (2).

(1) $\Leftrightarrow$ (2). This is proved in Lemma 3.1.

(2) $\Rightarrow$ (4). Let $\hat{\pi}, \hat{\rho} : \mathcal{A}^{\#\#} \to \mathcal{M}$ be the weak*-weak* continuous extensions of π and ρ, respectively. Since Φ is weak*-weak* continuous, we see that $\Phi \circ \hat{\pi} = \Phi \circ \hat{\rho}$. Since $\mathcal{M}$ is finite, $\mathcal{M}$ can be written as $\mathcal{M} = \sum_{\gamma\in\Gamma}^{\oplus}(\mathcal{M}_\gamma, \beta_\gamma)$, where β_γ is a faithful normal tracial state of $\mathcal{M}_\gamma$. Similarly, we can write $(\mathcal{A}^{\#\#})_f = \sum_{\lambda\in\Lambda}^{\oplus}(\mathcal{R}_\lambda, \tau_\lambda)$ where τ_λ is a faithful normal tracial state on $\mathcal{R}_\lambda$ for each $\lambda \in \Lambda$. Thus $\mathcal{A}^{\#\#} = \sum_{\lambda\in\Lambda}^{\oplus}(\mathcal{R}_\lambda, \tau_\lambda) \oplus (A^{\#\#})_i$. If $S \in \mathcal{M}$ and $T \in \mathcal{A}^{\#\#}$, we write

$$S = \sum_{\gamma\in\Gamma} S(\gamma) \quad \text{and} \quad T = \sum_{\lambda\in\Lambda} T(\lambda) \oplus T(i).$$

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Since $\mathcal{M}$ is finite, we know from Theorem 2.1 that $\hat{\pi}(Q_{f,\mathcal{A}^{\#\#}}) = \hat{\rho}(Q_{f,\mathcal{A}^{\#\#}}) = 1$. We also know that $\hat{\pi}$ and $\hat{\rho}$ are continuous in the SOT. Thus if $\{T_j\}$ is a norm-bounded net in $\mathcal{A}^{\#\#}$, $T \in \mathcal{A}^{\#\#}$ and $T_j Q_{f,\mathcal{A}^{\#\#}} \to T Q_{f,\mathcal{A}^{\#\#}}$ in the SOT, then $\hat{\pi}(T_j) = \hat{\pi}(T_j Q_{f,\mathcal{A}^{\#\#}}) \to \hat{\pi}(T)$ and $\hat{\rho}(T_j) \to \hat{\rho}(T)$ in the SOT. This means that if $\|T_j(\lambda) - T(\lambda)\|_{2,\tau_\lambda} \to 0$ for every $\lambda \in \Lambda$, then

$$\|\hat{\pi}(T_j)(\gamma) - \hat{\pi}(T)(\gamma)\|_{2,\beta_\gamma} \to 0 \quad \text{and} \quad \|\hat{\rho}(T_j)(\gamma) - \hat{\rho}(T)(\gamma)\|_{2,\beta_\gamma} \to 0$$

for every $\gamma \in \Gamma$.

Suppose $A \subset \operatorname{ball}(\mathcal{A}^{\#\#})$ is finite, $L \subset \Lambda$ is finite and $\varepsilon > 0$. Then there exist a $\delta > 0$ and a finite subset $G \subset \Gamma$ such that, if $T \in A$, $S \in 2\operatorname{ball}(\mathcal{A}^{\#\#})$ and, for every $\lambda \in L$, we have $\|T(\lambda) - S(\lambda)\|_{2,\tau_\lambda} < \delta$, then

$$\sum_{\gamma\in G} [\|\hat{\pi}(S)(\gamma) - \hat{\pi}(T)(\gamma)\|_{2,\beta_\gamma} + \|\hat{\rho}(S)(\gamma) - \hat{\rho}(T)(\gamma)\|_{2,\beta_\gamma}] < \varepsilon/37.$$

Since $\mathcal{A}$ is tracially nuclear, $\mathcal{R}_\lambda$ is hyperfinite for every $\lambda \in \Lambda$. Thus, for each $\lambda \in L$, there is a finite-dimensional unital C*-subalgebra $\mathcal{B}_\lambda \subset \mathcal{R}_\lambda$ such that, for each $S \in A$, there is a $B_{\lambda,S} \in \mathcal{B}_\lambda$ such that $\|B_{\lambda,S}\| \le \|S(\lambda)\|$ and $\|S(\lambda) - B_{\lambda,S}\|_{2,\tau_\lambda} < \delta$. Then $\mathcal{B} = \sum_{\lambda\in L}^{\oplus} \mathcal{B}_\lambda$ is a finite-dimensional C*-subalgebra of $\mathcal{A}^{\#\#}$. For each $S \in A$, we define $B_s = \sum_{\lambda\in L}^{\oplus} B_{\lambda,S} \in \mathcal{B}$. It follows that

$$\sum_{S\in A} \sum_{\gamma\in G} [\|\hat{\pi}(S)(\gamma) - \hat{\pi}(B_S)(\gamma)\|_{2,\beta_\gamma} + \|\hat{\rho}(S)(\gamma) - \hat{\rho}(B_S)(\gamma)\|_{2,\beta_\gamma}] < \varepsilon/37.$$

From $\Phi \circ \hat{\pi} = \Phi \circ \hat{\rho}$ and Lemma 3.2, there is a unitary operator $U = U_{(A,G,\varepsilon)} \in \mathcal{M}$ such that, for every $W \in \mathcal{B}$,

$$U\hat{\pi}(W)U^* = \hat{\rho}(W).$$

Therefore

$$\begin{aligned}
&\sum_{S\in A} \sum_{\gamma\in G} \|U\hat{\pi}(S)U^*(\gamma) - \hat{\rho}(S)(\gamma)\|_{2,\beta_\gamma} \\
&\quad \le \sum_{S\in A} \sum_{\gamma\in G} [\|U(\hat{\pi}(S)(\gamma) - \hat{\pi}(B_S)(\gamma))U^*\|_{2,\beta_\gamma} + \|\hat{\rho}(B_S)(\gamma) - \hat{\rho}(S)(\gamma)\|_{2,\beta_\gamma}] \\
&\quad < \varepsilon/37 < \varepsilon.
\end{aligned}$$

Also

$$\sum_{S\in A} \sum_{\gamma\in G} \|\hat{\pi}(S)(\gamma) - U^*\hat{\rho}(S)U(\gamma)\|_{2,\beta_\gamma} = \sum_{S\in A} \sum_{\gamma\in G} \|U\hat{\pi}(S)U^*(\gamma) - \hat{\rho}(S)(\gamma)\|_{2,\beta_\gamma} < \varepsilon.$$

If we order the triples (A, G, ε) by $(\subset, \subset, >)$, we have a net $\{U_{(A,G,\varepsilon)}\}$ of unitary operators in $\mathcal{M}$ such that, for every $T \in \mathcal{A}^{\#\#}$,

$$U_{(A,G,\varepsilon)}\hat{\pi}(T)U^*_{(A,G,\varepsilon)} \to \hat{\rho}(T) \quad \text{and} \quad U^*_{(A,G,\varepsilon)}\hat{\rho}(T)U_{(A,G,\varepsilon)} \to \hat{\pi}(T)$$

in the SOT. □

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4. FWU algebras: a converse

In this section we prove a converse of Theorem 3.3 when $\mathcal{A}$ is separable. We say that a unital C*-algebra $\mathcal{A}$ is an *FWU algebra*, or that $\mathcal{A}$ has the *finite weak*-uniqueness property*, if, for every finite von Neumann algebra $\mathcal{M}$ with a faithful normal tracial state τ and every pair $\pi, \rho : \mathcal{A} \to \mathcal{M}$ of unital $*$-homomorphisms such that, for all $a \in \mathcal{A}$,

$$\mathcal{M}\text{-rank}(\pi(a)) = \mathcal{M}\text{-rank}(\rho(a)),$$

there is a net $\{U_i\}$ of unitary operators in $\mathcal{M}$, such that, for every $a \in \mathcal{A}$,

$$\|U_i \pi(a) U_i^* - \rho(a)\|_{2,\tau} \to 0.$$

Since every finite von Neumann algebra is a direct sum of algebras having a faithful normal tracial state [8], being an FWU algebra is equivalent to saying that for every finite von Neumann algebra and every pair $\pi, \rho : \mathcal{A} \to \mathcal{M}$ of unital $*$-homomorphisms such that, for all $a \in \mathcal{A}$,

$$\mathcal{M}\text{-rank}(\pi(a)) = \mathcal{M}\text{-rank}(\rho(a)),$$

it follows that π and ρ are weak* approximately unitarily equivalent.

A key ingredient is a characterisation of hyperfiniteness proved by Connes [2]. If $\mathcal{N}$ is a von Neumann algebra, then the *flip automorphism* $\pi : \mathcal{N} \otimes \mathcal{N} \to \mathcal{N} \otimes \mathcal{N}$ is the automorphism defined by $\pi(a \otimes b) = b \otimes a$.

THEOREM 4.1 [2]. *Suppose $\mathcal{N} \subset B(H)$ is a II_1 factor von Neumann algebra acting on a separable Hilbert space. Then the following properties are equivalent:*

(1) *$\mathcal{N}$ is hyperfinite;*
(2) *for $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathcal{N}$ and $y_1, \ldots, y_n \in \mathcal{N}'$,*

$$\Big\| \sum_{k=1}^{n} x_k y_k \Big\|_H = \Big\| \sum_{k=1}^{n} x_k \otimes y_k \Big\|_{H \otimes H};$$

(3) *the flip automorphism π on $\mathcal{N} \otimes \mathcal{N}$ is weak* approximately unitarily equivalent in $\mathcal{N} \otimes \mathcal{N}$ to the identity representation.*

If $\mathcal{N}$ is a von Neumann algebra and the flip automorphism π is weak* approximately equivalent to the identity, it easily follows that the implementing net $\{U_\lambda\}$ of unitaries simultaneously makes the maps $\rho_1, \rho_2 : \mathcal{N} \to \mathcal{N} \otimes \mathcal{N}$ defined by

$$\rho_1(a) = a \otimes 1, \quad \rho_2(a) = 1 \otimes a, \quad \text{for every } a \in \mathcal{N},$$

weak* approximately equivalent. Using Connes' proof, we obtain a stronger statement which fills in the details of [4, Remark 7]. This says that if $\mathcal{M} = W^*(X_1, \ldots, X_m)$ is a separably acting factor von Neumann algebra with trace τ, then $\mathcal{M}$ is nuclear whenever the following condition holds. For every $\varepsilon > 0$ there exist a $\delta > 0$ and a

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positive integer N such that, for every factor von Neumann algebra $\mathcal{N}$ with trace ρ, and for all $A_1, B_1 \ldots, A_n, B_n \in \mathcal{N}$, if

$$|\tau(m(X_1, \ldots, X_m)) - \rho(m(A_1, \ldots, A_m))| < \delta$$

and

$$|\tau(m(X_1, \ldots, X_m)) - \rho(m(B_1, \ldots, B_m))| < \delta$$

for all $*$-monomials m with degree$(m) \leq N$, then there is a unitary operator $U \in \mathcal{N}$ such that

$$\sum_{k=1}^{m} \|UA_kU^* - B_k\|_{2,\rho}^2 < \varepsilon.$$

Theorem 4.2. *Suppose $\mathcal{N} \subset B(H)$ is a finite factor von Neumann algebra acting on a separable Hilbert space H. Define $\rho_1, \rho_2 : \mathcal{N} \to \mathcal{N} \otimes \mathcal{N}$ by*

$$\rho_1(a) = a \otimes 1, \quad \rho_2(a) = 1 \otimes a, \quad \textit{for every } a \in \mathcal{N}.$$

Suppose ρ_1 and ρ_2 are weak approximately equivalent in $\mathcal{N} \otimes \mathcal{N}$. Then $\mathcal{N}$ is hyperfinite.*

Proof. Let τ be the unique faithful normal tracial state on $\mathcal{N}$. Then $\tau \otimes \tau$ is a faithful normal tracial state on the factor $\mathcal{N} \otimes \mathcal{N} \subset B(H \otimes H)$. Suppose ρ_1 and ρ_2 are weak* approximately equivalent in $\mathcal{N} \otimes \mathcal{N}$. We can choose a net $\{U_\lambda\}$ of unitary operators in $\mathcal{N} \otimes \mathcal{N}$ such that, for every $a \in N$,

$$\|U_\lambda^*(a \otimes 1)U_\lambda - (1 \otimes a)\|_{2,\tau\otimes\tau} \to 0.$$

Suppose $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathcal{N}$ and $y_1, \ldots, y_n \in \mathcal{N}'$. Since $U_\lambda \in \mathcal{N} \otimes \mathcal{N}$ and each $1 \otimes y_k \in (\mathcal{N} \otimes \mathcal{N})'$,

$$\begin{aligned} U_\lambda^*\Big(\sum_{k=1}^{n} x_k \otimes y_k\Big)U_\lambda &= \sum_{k=1}^{n} U_\lambda^*(x_k \otimes 1)(1 \otimes y_k)U_\lambda = \sum_{k=1}^{n} [U_\lambda^*(x_k \otimes 1)U_\lambda](1 \otimes y_k) \\ &\overset{\text{weak*}}{\to} \sum_{k=1}^{n} (1 \otimes x_k)(1 \otimes y_k) = 1 \otimes \Big(\sum_{k=1}^{n} x_k y_k\Big). \end{aligned}$$

Since, for every λ,

$$\Big\|U_\lambda^*\Big(\sum_{k=1}^{n} x_k \otimes y_k\Big)U_\lambda\Big\| = \Big\|\sum_{k=1}^{n} x_k \otimes y_k\Big\|,$$

it follows that

$$\Big\|\sum_{k=1}^{n} x_k y_k\Big\| \leq \Big\|\sum_{k=1}^{n} x_k \otimes y_k\Big\|.$$

It also follows that, for every $a \in \mathcal{N}$,

$$\|a \otimes 1 - U_\lambda(1 \otimes a)U_\lambda^*\|_{2,\tau\otimes\tau} = \|U_\lambda^*(a \otimes 1)U_\lambda - (1 \otimes a)\|_{2,\tau\otimes\tau} \to 0.$$

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Thus

$$U_\lambda\Big[1\otimes\Big(\sum_{k=1}^n x_ky_k\Big)\Big]U_\lambda^* = U_\lambda\sum_{k=1}^n(1\otimes x_k)(1\otimes y_k)U_\lambda^* = \sum_{k=1}^n U_\lambda(1\otimes x_k)U_\lambda^*(1\otimes y_k)$$
$$\overset{\text{weak*}}{\to}\sum_{k=1}^n(x_k\otimes 1)(1\otimes y_k) = \sum_{k=1}^n x_k\otimes y_k$$

and

$$\Big\|\sum_{k=1}^n x_k\otimes y_k\Big\| \le \Big\|\sum_{k=1}^n x_ky_k\Big\|.$$

Thus by Connes' theorem (Theorem 4.1), $\mathcal{N}$ is hyperfinite. □

We now prove our converse result.

THEOREM 4.3. *A separable unital C*-algebra is an FWU algebra if and only if it is tracially nuclear.*

PROOF. Suppose $\mathcal{A}$ is an FWU algebra. Suppose τ is a factor tracial state on $\mathcal{A}$. Let $\mathcal{N} = \pi_\tau(\mathcal{A})''$. Since $\mathcal{A}$ is separable and π_τ has a cyclic vector, $\mathcal{N}$ acts on a separable Hilbert space. If $\mathcal{N}$ is finite-dimensional, then $\mathcal{N}$ is hyperfinite. Thus we can assume that $\mathcal{N}$ is a II_1 factor. Then $\mathcal{N}\subset L^2(\mathcal{A},\tau)$ and $\pi_\tau(\mathcal{A})$ is $\|\ \|_{2,\tau}$-dense in $\mathcal{N}$. Define $\rho_1,\rho_2:\mathcal{N}\to\mathcal{N}\otimes\mathcal{N}$ by

$$\rho_1(b) = b\otimes 1,\quad \rho_2(b) = 1\otimes b,\quad \text{for every } b\in\mathcal{N}.$$

For $k = 1, 2$, let $\sigma_k = \rho_k\circ\pi_\tau : \mathcal{A}\to\mathcal{N}\otimes\mathcal{N}$. Since $(\tau\otimes\tau)\circ\rho_1 = (\tau\otimes\tau)\circ\rho_2$, we see that $(\tau\otimes\tau)\circ\sigma_1 = (\tau\otimes\tau)\circ\sigma_2$. Since $\mathcal{A}$ is an FWU algebra, σ_1 and σ_2 are weak* approximately unitarily equivalent in $\mathcal{N}\otimes\mathcal{N}$. Thus there is a net $\{U_\lambda\}$ of unitary operators in $\mathcal{N}\otimes\mathcal{N}$ such that, for every $b\in\pi_\tau(\mathcal{A})$,

$$\|U_\lambda^*(b\otimes 1)U_\lambda - (1\otimes b)\|_{2,\tau\otimes\tau}\to 0.$$

For each λ, the map

$$b\mapsto U_\lambda^*(b\otimes 1)U_\lambda - (1\otimes b)$$

is $\|\ \|_{2,\tau\otimes\tau}$-continuous and linear on $\mathcal{N}$ and has norm at most 2, and $\pi_\tau(\mathcal{A})$ is $\|\ \|_{2,\tau\otimes\tau}$ dense in $\mathcal{N}$. It follows that, for every $b\in\mathcal{N}$,

$$\|U_\lambda^*(b\otimes 1)U_\lambda - (1\otimes b)\|_{2,\tau\otimes\tau}\to 0.$$

Thus, by Theorem 4.2, $\mathcal{N}$ is hyperfinite and, from Lemma 1.1, $\mathcal{A}$ is tracially nuclear. The other direction is contained in Theorem 3.3. □

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