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ON LINEAR FUNCTIONALS AND SUMMABILITY FACTORS FOR STRONG SUMMABILITY II

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Introduction. The first part of this paper, which will be referred to by I, appeared in Volume 30 of this journal. The present paper will use the same bibliography as I.

Theorem 1 in I shows that the knowledge of all continuous linear functionals in $o[A]_p$ is essential in determining convergence and summability factors for strong summability. So far, Theorem 7 in I was for a general A the only tool in deciding whether a given sequence ϵ generates such a functional. We mentioned in a remark following Theorem 7 the difficulties in verifying the conditions of this theorem (two parameters are involved). In the present paper, we study continuous linear functionals in $o[A]_1$ in more detail, and we obtain in a corollary to Theorem 22 a condition which appears to be a more satisfactory answer to the question, whether a given sequence ϵ generates a continuous linear functional in $o[A]_1$.

In Section 7, the first section of this paper, we estimate a form

$$\left(\sum_{k=0}^{N} \eta_k |s_k|^p\right)^{1/p}$$

(with $\eta_k \geqq 0$) in terms of the norm of $s = (s_k)$ in $o[A]_1$ and the value of the form at certain extremal sequences (Theorem 14). If we only consider the $(N+1)$-dimensional subspace of $o[A]_1$, consisting of all $s = (s_k)$ with $s_k = 0$ for $k > N$, then the unit sphere with respect to the norm in $o[A]_1$ can be seen to be bounded by parts of hyperplanes, and the extremal sequences mentioned above turn out to be the extreme points of the part of the sphere containing the sequences with all non-negative coordinates. One consequence of Theorem 14 is a comparison theorem for strong summability (Theorem 15), another consequence (and the more important one for the present paper) is a characterization of all continuous linear functionals in $o[A]_1$ in terms of extremal sequences (Theorems 16, 17).

In Section 8 we give another characterization of these functionals by stating that ϵ generates a functional in $o[A]_1{}^*$ if and only if it is monotone in a certain sense (Theorem 18). This result refines I, Theorem 7, (d) by

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characterizing sequences s for which the functional attains its (functional-) norm. Theorems 19 and 20 discuss some properties of this type of monotonicity.

In Section 9 we show that to every $\epsilon \in o[A]_1^*$ there exists a uniquely determined majorant which is monotone in a certain sense, and the construction of this majorant leads to the characterization of $\epsilon \in o[A]_1^*$ (Theorem 22 resp. its corollary) which we mentioned above. In the concluding Section 10 we indicate how some of our results can be extended to the case $p > 1$.

7. Linear functionals on $o[A]_1$**.** In what follows the following assumptions on a matrix A will frequently be used:

(7.1) $\quad A$ is normal, $0 \leqq a_{nk} \downarrow$ as $n \uparrow$, $n \geqq k$,

(7.2) $\quad A$ is normal, $0 \leqq a_{nk} \downarrow 0$ as $n \uparrow$, $n \geqq k$,

(7.3) $\quad A$ is normal, $0 < a_{nk} \downarrow 0$ strictly as $n \uparrow$, $n \geqq k$.

Let $A = (a_{nk})$, $n, k \in \mathbf{N}_0$ be normal. For any index set $\mathbf{J}$, i.e., any subset of $\mathbf{N}_0$, there exists a unique sequence $s(\mathbf{J}) = (s_0(\mathbf{J}), s_1(\mathbf{J}), \ldots)$ such that

$$(7.4) \qquad \sum_k a_{nk}s_k(\mathbf{J}) = 1 \quad \text{if } n \in \mathbf{J}, \quad s_k(\mathbf{J}) = 0 \quad \text{if } k \notin \mathbf{J}.$$

Clearly, $s_k(\mathbf{J})$ is independent of those elements of $\mathbf{J}$ which are greater than k.

LEMMA 5. *Let A satisfy* (7.1).

a) $s_n(\mathbf{J}) \geqq 0, \sum_k a_{nk}s_k(\mathbf{J}) \leqq 1$ *for $n \in \mathbf{N}_0$ and every* $\mathbf{J}$.

b) *If $s = (s_0, s_1, \ldots, s_N)$ satisfies*

$$(7.5) \qquad s_n \geqq 0, \sum_k a_{nk}s_k \leqq 1 \quad \textit{for } n = 0, 1, \ldots, N,$$

then

$$(7.6) \qquad s_k = \sum_{\mathbf{J}} \alpha_{\mathbf{J}} s_k(\mathbf{J}) \quad \textit{for } k = 0, 1, \ldots, N,$$

where $\mathbf{J}$ runs through all subsets of $\{0, 1, \ldots, N\}$, and where the $\alpha_{\mathbf{J}}$ are independent of k and satisfy $\alpha_{\mathbf{J}} \geqq 0, \sum_{\mathbf{J}} \alpha_{\mathbf{J}} = 1$.

Proof. Statement a) is an immediate consequence of (7.1) and (7.4). In order to prove b) we use induction with respect to N. If $N = 0$, then the only $\mathbf{J}$'s in (7.6) are $\mathbf{J}_0 = \emptyset$, $\mathbf{J}_1 = \{0\}$, and (7.6) is satisfied when $\alpha_{\mathbf{J}_1} = s_0/s_0(\mathbf{J}_1)$, $\alpha_{\mathbf{J}_0} + \alpha_{\mathbf{J}_1} = 1$. In order to carry out the induction, let

$$t_N = a_{NN}^{-1}\left(1 - \sum_{k=0}^{N-1} a_{Nk}s_k\right).$$

It follows from (7.5) that $s_N \leqq t_N$. Let $\beta_1 + \beta_2 = 1$, $\beta_2 = s_N/t_N$ ($\beta_2 = 0$ if $t_N = 0$).

We may then assume that

$$s_k = \sum_{\mathbf{K}} \tilde{\alpha}_{\mathbf{K}} s_k(\mathbf{K}), \quad k = 0, 1, \ldots, N-1,$$

$\tilde{\alpha}_{\mathbf{K}} \geqq 0$ and independent of k, $\sum_{\mathbf{K}} \tilde{\alpha}_{\mathbf{K}} = 1$, and $\mathbf{K}$ runs through all subsets of $\{0, \ldots, N-1\}$. If $\mathbf{J}$ is a subset of $\{0, \ldots, N\}$, we define

$$\alpha_{\mathbf{J}} = \begin{cases} \beta_1 \tilde{\alpha}_{\mathbf{J}} & \text{if } N \notin \mathbf{J}, \\ \beta_2 \tilde{\alpha}_{\mathbf{J}-\{N\}} & \text{if } N \in \mathbf{J}. \end{cases}$$

We have $\alpha_{\mathbf{J}} \geqq 0$, independent of k and $\sum_{\mathbf{J}} \alpha_{\mathbf{J}} = 1$. Furthermore, if $k = 0, 1, \ldots, N-1$,

$$\sum_{\mathbf{J}} \alpha_{\mathbf{J}} s_k(\mathbf{J}) = (\beta_1 + \beta_2) \sum_{\mathbf{K}} \tilde{\alpha}_{\mathbf{K}} s_k(\mathbf{K}) = s_k,$$

and (by (7.4))

$$\sum_{\mathbf{J}} \alpha_{\mathbf{J}} s_N(\mathbf{J}) = \beta_2 \sum_{\mathbf{K}} \tilde{\alpha}_{\mathbf{K}} s_N(\mathbf{K} \cup \{N\})$$

$$= \beta_2 \sum_{\mathbf{K}} \tilde{\alpha}_{\mathbf{K}} a_{NN}^{-1} \left(1 - \sum_{k=0}^{N-1} a_{Nk} s_k(\mathbf{K})\right) = \beta_2 a_{NN}^{-1} \left(1 - \sum_{k=0}^{N-1} a_{Nk} \sum_{\mathbf{K}} \tilde{\alpha}_{\mathbf{K}} s_k(\mathbf{K})\right)$$

$$= \beta_2 t_N = s_N.$$

THEOREM 14. *Let A satisfy* (7.1), *let $N \in \mathbf{N}_0$, $p \in [1, \infty)$, and let $s_k \geqq 0$, $\eta_k \geqq 0$ for $k = 0, 1, \ldots, N$. Then*

$$\left(\sum_{k=0}^{N} \eta_k s_k^{\,p}\right)^{1/p} \leqq \max_{\mathbf{J}} \left(\sum_{k=0}^{N} \eta_k s_k^{\,p}(\mathbf{J})\right)^{1/p} \max_{n \leqq N} \sum_{k=0}^{n} a_{nk} s_k, \tag{7.7}$$

where $\mathbf{J}$ runs through all subsets of $\{0, \ldots, N\}$.

Proof. We may assume that

$$\max_{n \leqq N} \sum_{k=0}^{n} a_{nk} s_k = 1.$$

It follows from (7.6) and Minkowski's inequality that

$$\left(\sum_{k=0}^{N} \eta_k s_k^{\,p}\right)^{1/p} \leqq \sum_{\mathbf{J}} \alpha_{\mathbf{J}} \left(\sum_{k=0}^{N} \eta_k s_k^{\,p}(\mathbf{J})\right)^{1/p} \leqq \max_{\mathbf{J}} \left(\sum_{k=0}^{N} \eta_k s_k^{\,p}(\mathbf{J})\right)^{1/p}.$$

Remark. It can be shown that (7.7) is not true in general if one $\mathbf{J}$ (except $\mathbf{J} = \emptyset$) is left out in the maximum.

The following Theorems 15 and 16 discuss applications of Theorem 14. Theorem 15 and its corollary give an answer to the problems mentioned at the end of I, Section 4.

THEOREM 15. *Let A satisfy* (7.2), *let $p \in [1, \infty)$, and let $B \geqq 0$. Then $o[A]_1 \subseteq o[B]_p$ if and only if $b_{nk} \to 0$ for $n \to \infty$, k fixed, and*

$$\sup_{\mathbf{J}} \left(\sum_{k} b_{nk} s_k^{\,p}(\mathbf{J})\right)^{1/p} = \mathbf{O}(1) \quad \text{as } n \to \infty, \tag{7.8}$$

where $\mathbf{J}$ runs through all finite subsets of $\mathbf{N}_0$.

Proof. It follows from (7.2) that $s(m) = (\delta_{mk}) \in o[A]_1$, hence if $o[A]_1 \subseteq o[B]_p$, then $s(m) \in o[B]_p$, i.e., $b_{nm} \to 0$ as $n \to \infty$, m fixed, and (note that then $(\sum_k b_{mk}|s_k|^p)^{1/p}$ is a continuous seminorm on $o[A]_1$) there exists $C > 0$ such that

$$(7.8') \qquad \left(\sum_k b_{mk}|s_k|^p\right)^{1/p} \leqq C \sup_n \sum_k a_{nk}|s_k|$$

for all $m \in \mathbf{N}_0$ and $s \in o[A]_1$. Conversely, if this inequality holds and $b_{nm} \to 0$ as $n \to \infty$, m fixed, then the identity mapping, considered as a mapping of the set of all sequences with finitely many non-zero terms (as a subset of $o[A]_1$) into $o[B]_p$, is bounded. Using the fact that these sequences form a dense subset of $o[A]_1$ (c.f. I), we conclude that the identity maps even $o[A]_1$ boundedly into $o[B]_p$, which means $o[A]_1 \subseteq o[B]_p$. Since (7.8) is equivalent to (7.8′), this proves the theorem.

COROLLARY. *A sequence λ is a sequence-to-sequence factor from $o[A]_1$ to $o[B]_p$ if and only if $o[A]_1 \subseteq o[\tilde{B}]_p$, where $\tilde{B} = (b_{nk}|\lambda_k|^p)$, hence Theorem* 15 *also gives a general characterization of sequence-to-sequence factors from $o[A]_1$ to $o[B]_p$.*

Our next theorem characterizes absolute convergence factors in $o[A]_1$.

THEOREM 16. *Let A satisfy* (7.2). *Then $\epsilon = (\epsilon_k)$ is an absolute convergence factor for $o[A]_1$ if and only if*

$$(7.9) \qquad \sup_{\mathbf{J}} \sum_k |\epsilon_k| s_k(\mathbf{J}) < \infty,$$

where $\mathbf{J}$ runs through all finite subsets of $\mathbf{N}_0$.

Proof. This is an immediate consequence of Theorem 14.

Remark. The quantity in (7.9) is also the norm of $f_\epsilon \in o[A]_1{}^*, f_\epsilon(s) = \sum \epsilon_k s_k$. It follows from the isomorphy between $o[A]_1{}^*$ and the absolute convergence factors (see I, Section 1) that

$$(7.10) \qquad \|\epsilon\| = \|f_\epsilon\| = \sup_{\mathbf{W}} \sum_k |\epsilon_k| s_k(\mathbf{J}).$$

Our next theorem shows that the supremum in (7.9) is attained for some $\mathbf{J}$ which is not necessarily finite.

THEOREM 17. *Let A satisfy* (7.2), *and let ϵ be an absolute convergence factor for $o[A]_1$ (i.e., $\epsilon \in o[A]_1{}^*$). Then there exists an index set $\mathbf{J} = \mathbf{J}(\epsilon)$ such that*

$$(7.11) \qquad \|\epsilon\| = \sum_k |\epsilon_k| s_k(\mathbf{J}).$$

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Proof. It follows from (7.10) that

$$\|\epsilon\| = \lim_{n\to\infty} \sum_k |\epsilon_k| s_k(\mathbf{J}_n) \tag{7.12}$$

for some sequence $\mathbf{J}_n$ of finite index sets. For fixed k, the sequence $(s_k(\mathbf{J}_n))$ is bounded (since $a_{kk}s_k(\mathbf{J}_n) \leqq \sum_m a_{km}s_m(\mathbf{J}_n) \leqq 1$), hence we may assume that $\lim_n s_k(\mathbf{J}_n) = s_k$ exists for every $k \in \mathbf{N}_0$. Let

$$\mathbf{J} = \liminf \mathbf{J}_n = \bigcup_m \bigcap_{n\geqq m} \mathbf{J}_n.$$

If $k \notin \mathbf{J}$ then $k \notin \mathbf{J}_n$ for infinitely many n, and $s_k = \lim_n s_k(\mathbf{J}_n) = 0$. If $k \in \mathbf{J}$, hence $k \in \mathbf{J}_n$ for $n \geqq n_0$, then

$$\sum_{m=0}^{k} a_{km}s_m = \lim_{n\to\infty} \sum_{m=0}^{k} a_{km}s_m(\mathbf{J}_n) = 1.$$

It follows that $s = (s_k) = s(\mathbf{J})$. By Theorem 7, (d) (note that the theorem can be seen to hold even if A is not regular but satisfies (7.2)), we have

$$|\epsilon_k| \leqq \sum_{m=k}^{\infty} \alpha_m a_{mk} \quad \text{for some } \alpha = (\alpha_n) \in l_1,\ \alpha_n \geqq 0.$$

Hence, given $\delta > 0$ we can find $N \in \mathbf{N}$ such that for all n

$$\sum_{k=N}^{\infty} |\epsilon_k| s_k(\mathbf{J}_n) \leqq \sum_{m=N}^{\infty} \alpha_m \sum_{k=N}^{m} a_{mk}s_k(\mathbf{J}_n) \leqq \sum_{m=n}^{\infty} \alpha_m \leqq \delta.$$

This and (7.12) show that (7.11) holds.

Remark. We could also have taken $\mathbf{J} = \limsup \mathbf{J}_n$ in the proof above. This already indicates that $\mathbf{J}$ with (7.11) is not unique. For details see Theorem 20.

8. $A_{\mathbf{J}}$-monotone sequences. Let A satisfy (7.2). Given an index set $\mathbf{J}$ we call a sequence $\epsilon = (\epsilon_k)$, $\epsilon_k \geqq 0$, an *$A_{\mathbf{J}}$-monotone sequence* if a sequence $\alpha = (\alpha_n)$ exists such that

$$\begin{cases} \alpha_n \geqq 0,\ \alpha_n = 0 & \text{if } \sum_k a_{nk}s_k(\mathbf{J}) < 1, \\ \epsilon_k \leqq \sum_m \alpha_m a_{mk} & \text{for } k \in \mathbf{N}_0, \\ \epsilon_k = \sum_m \alpha_m a_{mk} & \text{if } s_k(\mathbf{J}) > 0. \end{cases} \tag{8.1}$$

If ϵ is $A_{\mathbf{J}}$-monotone for $\mathbf{J} = \mathbf{N}_0$, then it is called *$A$-monotone*. As an example we mention that ϵ is C_1-monotone if and only if $0 \leqq \epsilon_k \downarrow 0$, and it is C_1-monotone with $\alpha \in l_1$ if and only if in addition $\sum_k \epsilon_k < \infty$.

The following theorem gives a characterization of absolute convergence factors for $o[A]_1$ in terms of $A_{\mathbf{J}}$-monotonicity.

THEOREM 18. *Let A satisfy* (7.2). *A sequence $\epsilon = (\epsilon_k)$, $\epsilon_k \geqq 0$, belongs to $o[A]_1^*$ if and only if ϵ is $A_{\mathbf{J}}$-monotone for some $\mathbf{J}$ and $\alpha \in l_1$. Moreover,*

$\sum \alpha_n = \|\epsilon\|$ *and the index sets* $\mathbf{J}$ *for which* ϵ *is* $A_{\mathbf{J}}$*-monotone are exactly those for which* (7.11) *holds.*

Proof. Let $\epsilon \in o[A]_1^*$. It follows from Theorem 7, (b) (note again that Theorem 7 also holds if A satisfies (7.2)) that $\mathbf{H} = (h_{nk})$ with

$$\|\mathbf{H}\| = \sum_n \sup_k |h_{nk}| < \infty$$

exists such that

$$(8.2) \qquad \epsilon_k = \sum_n h_{nk} a_{nk}, \quad k \in \mathbf{N}_0,$$

and the proof of Theorem 7 shows that h_{nk} may be chosen such that $\|\mathbf{H}\| = \|\epsilon\|$ (this follows from the Hahn-Banach-Theorem). Also, we may assume that $h_{nk} = \sup_m |h_{nm}|$ whenever $a_{nk} = 0$. Using (7.11) we find

$$\begin{aligned}\|\epsilon\| &= \sum_k \epsilon_k s_k(\mathbf{J}) = \sum_n \sum_k s_k(\mathbf{J}) h_{nk} a_{nk} \\ &\leqq \sum_n \sup_m |h_{nm}| \sum_k a_{nk} s_k(\mathbf{J}) \leqq \|\mathbf{H}\| = \|\epsilon\|,\end{aligned}$$

so that equality must hold in each estimate. Let $\alpha_n = \sup_k |h_{nk}|$. If $\sum_k a_{nk} s_k(\mathbf{J}) < 1$, then $\alpha_n = 0$, i.e., the first condition in (8.1) is satisfied. If $s_k(\mathbf{J}) > 0$, then

$$h_{nk} = \sup_m |h_{nm}| = \alpha_n \quad \text{for all } n \in \mathbf{N}_0,$$

i.e., the third condition in (8.1) follows from (8.2), and the second condition is an immediate consequence of (8.2). This part of the proof shows that (7.11) for some $\mathbf{J}$ implies that ϵ is $A_{\mathbf{J}}$-monotone for this $\mathbf{J}$ and $\alpha \in l_1$ (even $\sum \alpha_n = \|\epsilon\|$). Next, assume that ϵ is $A_{\mathbf{J}_0}$-monotone, and let $\alpha \in l_1$. It follows from (8.1) that

$$\sum_k \epsilon_k s_k(\mathbf{J}_0) = \sum_k s_k(\mathbf{J}_0) \sum_n \alpha_n a_{nk} = \sum_n \alpha_n \sum_k a_{nk} s_k(\mathbf{J}_0) = \sum_n \alpha_n,$$

and similarly

$$\sum_k \epsilon_k s_k(\mathbf{J}) \leqq \sum \alpha_n \quad \text{for any } \mathbf{J}.$$

Hence $\epsilon \in o[A]_1^*$ (Theorem 10), $\sum \alpha_n = \|\epsilon\|$ (by (7.10)) and (7.11) holds for $\mathbf{J}_0$. This completes the proof.

There is another way of characterizing $A_{\mathbf{J}}$-monotone sequences, which uses the inverse of A and related matrices. This will be discussed next.

Let A be normal. Given an index set $\mathbf{J}$ we define $A_{\mathbf{J}} = (a_{nk}(\mathbf{J}))$ by

$$(8.3) \qquad a_{nk}(\mathbf{J}) = \begin{cases} a_{nk} & \text{if } n \in \mathbf{J}, \\ \delta_{nk} & \text{if } n \notin \mathbf{J}. \end{cases}$$

The inverse of the matrix $A_{\mathbf{J}}$ will be denoted by $A_{\mathbf{J}}' = (a_{nk}'(\mathbf{J}))$.

LEMMA 6. *Let* A *satisfy* (7.3), *and let* $\epsilon \in o[A]_1^*$. *Then*

$$\sum_k |\epsilon_k a_{km}'(\mathbf{J})| < \infty \quad \textit{for every } m \in \mathbf{N}_0 \textit{ and every } \mathbf{J}.$$

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Proof. We consider the cases $m \in \mathbf{J}$ and $m \notin \mathbf{J}$ separately. Let $m \in \mathbf{J}$, hence $s_m(\mathbf{J}) > 0$ (observe (7.3)), and let $\tilde{\mathbf{J}} = \mathbf{J} - \{m\}$, $s = (s_k)$, $s_k = s_k(\mathbf{J}) - s_k(\mathbf{J})$. It follows from (7.10) that

$$\sum |\epsilon_k s_k| < \infty.$$

A short calculation shows that $b = (b_k) = A_{\mathbf{J}}s$ is given by

$$b_k = \delta_{km}a_{mm}s_m(\mathbf{J}),$$

hence $s = A_{\mathbf{J}}'b$, i.e.,

$$s_k = a_{km}'(\mathbf{J})a_{mm}s_m(\mathbf{J})$$

which implies

$$\sum_k |\epsilon_k a_{km}'(\mathbf{J})| < \infty.$$

Let $m \notin \mathbf{J}$. If $\tilde{\mathbf{J}} = \mathbf{J} \cup \{m\}$, then $m \in \tilde{\mathbf{J}}$, hence

$$\sum_k |\epsilon_k a_{km}'(\mathbf{J})| < \infty.$$

A short verification shows that

$$a_{km}'(\tilde{\mathbf{J}}) = a_{mm}^{-1}a_{km}'(\mathbf{J}),$$

and Lemma 6 follows.

THEOREM 19. *Let A satisfy* (7.3) *and let $\epsilon = (\epsilon_k)$, $\epsilon_k \geqq 0$, $k \in \mathbf{N}_0$ be given. Then ϵ is $A_{\mathbf{J}}$-monotone and $\alpha \in l_1$ if and only if $\sum \epsilon_k s_k(\mathbf{J}) < \infty$ and the numbers*

$$\beta_m = \sum_k \epsilon_k a_{km}'(\mathbf{J}), \quad m \in \mathbf{N}_0$$

exist and satisfy

(8.4) $\quad \beta_m \geqq 0 \quad$ *if* $m \in \mathbf{J}$, $\quad \beta_m \leqq 0 \quad$ *if* $m \notin J$.

If this is the case, the sequence α of (8.1) *is unique and satisfies*

$$\alpha_m = \beta_m \quad \text{if } m \in \mathbf{J}, \quad \alpha_m = 0 \quad \text{if } m \notin \mathbf{J}.$$

Note that (7.3) implies $\sum_k a_{nk}s_k(\mathbf{J}) < 1$ if and only if $n \notin \mathbf{J}$ and $s_k(\mathbf{J}) > 0$ if and only if $k \in \mathbf{J}$. It follows that (8.1) can be written in the form

$$(8.5) \quad \begin{cases} \epsilon_k \leqq \sum_m \alpha_m a_{mk}, \quad \alpha_m \geqq 0, \\ \alpha_n \neq 0 \text{ implies } n \in \mathbf{J}, \text{ and this implies } \epsilon_n = \sum_m \alpha_m a_{mn}. \end{cases}$$

Proof of Theorem 19. Let ϵ be $A_{\mathbf{J}}$-monotone and $\alpha \in l_1$, and note that $a_{kn}'(\mathbf{J}) = \delta_{kn}$ if $k \notin J$. If $m \in \mathbf{J}$, then

$$\begin{aligned} \beta_m &= \sum_{k \in \mathbf{J}} \epsilon_k a_{km}'(\mathbf{J}) = \sum_k a_{km}'(\mathbf{J}) \sum_n \alpha_n a_{nk}(\mathbf{J}) \\ &= \sum_n \alpha_n \sum_k a_{nk}(\mathbf{J}) a_{km}'(\mathbf{J}) = \alpha_m, \end{aligned}$$

and if $m \notin J$, then

$$\begin{aligned}\beta_m &= \textstyle\sum_{k\in\mathbf{J}} \epsilon_k a_{km}'(\mathbf{J}) + \epsilon_m a_{mm}'(\mathbf{J}) \\ &\leqq \textstyle\sum_{k\in\mathbf{J}} a_{km}'(\mathbf{J}) \sum_n \alpha_n a_{nk}(\mathbf{J}) + a_{mm}'(\mathbf{J}) \sum_n \alpha_n a_{nm} = \alpha_m = 0.\end{aligned}$$

It follows that (8.4) holds, that α is unique and $\alpha_m = \beta_m$ if $m \in \mathbf{J}$. From Theorem 18 we conclude $\sum \epsilon_k s_k(\mathbf{J}) < \infty$.

To prove the converse, assume that $\sum \epsilon_k s_k(\mathbf{J}) < \infty$ and (8.4) hold for some $\mathbf{J}$. From Theorem 18 we see that ϵ is $A_{\mathbf{J}}$-monotone with $\alpha \in l_1$ if and only if

$$\textstyle\sum \epsilon_k s_k(J) \geqq \sum \epsilon_k s_k(\tilde{\mathbf{J}})$$

for all index sets $\tilde{\mathbf{J}}$. Therefore we assume $\sum \epsilon_k s_k(\mathbf{J}) < \sum \epsilon_k s_k(\tilde{\mathbf{J}})$ for some $\tilde{\mathbf{J}}$, and we furthermore may assume $\sum \epsilon_k s_k(\tilde{\mathbf{J}}) < \infty$ (we actually may take $\tilde{\mathbf{J}}$ finite). Choose N such that

$$\sum_0^\infty \epsilon_k s_k(\mathbf{J}) < \sum_0^N \epsilon_k s_k(\tilde{\mathbf{J}}),$$

and concatenate $\mathbf{J}$ and $\tilde{\mathbf{J}}$ by introducing

$$\hat{\mathbf{J}} = (\tilde{\mathbf{J}} \cap \{0, 1, \ldots, N\}) \cup (\mathbf{J} \cap \{N+1, N+2, \ldots\}).$$

It follows that $s_k(\hat{\mathbf{J}}) = s_k(\tilde{\mathbf{J}})$ for $k \leqq N$, hence

$$(8.6) \qquad \textstyle\sum_k \epsilon_k(s_k(\hat{\mathbf{J}}) - s_k(\mathbf{J})) > 0.$$

Let $\lambda_n = \sum_k a_{nk}(\mathbf{J})(s_k(\hat{\mathbf{J}}) - s_k(\mathbf{J}))$. A short calculation shows that

$$(8.7) \qquad \lambda_n = \begin{cases} 0 \text{ if } n \in \hat{\mathbf{J}} \cap \mathbf{J} \text{ or } n \notin \hat{\mathbf{J}} \cup \mathbf{J}, \\ s_n(\hat{\mathbf{J}}) \geqq 0 \text{ if } n \in \hat{\mathbf{J}} - \mathbf{J}, \\ \sum_k a_{nk}(\mathbf{J}) s_k(\hat{\mathbf{J}}) - 1 \leqq 0 \text{ if } n \in \mathbf{J} - \mathbf{J}, \end{cases}$$

and especially $\lambda_n = 0$ if $n > N$.

It follows from $s_k(\hat{\mathbf{J}}) - s_k(\mathbf{J}) = \sum_n a_{kn}'(\mathbf{J})\lambda_n$ and (8.7) that

$$\begin{aligned}\sum_0^\infty \epsilon_k(s_k(\hat{\mathbf{J}}) - s_k(\mathbf{J})) &= \sum_k \epsilon_k \sum_n a_{kn}'(\mathbf{J})\lambda_n = \sum_n \lambda_n \sum_k \epsilon_k a_{kn}'(\mathbf{J}) \\ &= \sum_n \lambda_n \beta_n = \sum_{n\in\hat{\mathbf{J}}-\mathbf{J}} \lambda_n \beta_n + \sum_{n\in\mathbf{J}-\hat{\mathbf{J}}} \lambda_n \beta_n \leqq 0,\end{aligned}$$

and this contradicts (8.6).

To illustrate Theorem 19, let A be the weighted mean M_p, $p_n > 0$, $P_n \to \infty$, and let $\mathbf{J} = \{n_0, n_1, \ldots\}$. A short verification shows that

$$(8.8) \qquad a_{n_i k}'(\mathbf{J}) = \begin{cases} 0 & \text{if } k < n_{i-1}, \\ -P_{n_{i-1}}/p_{n_i} & \text{if } k = n_{i-1}, \\ -p_k/p_{n_i} & \text{if } n_{i-1} < k < n_i, \\ P_{n_i}/p_{n_i} & \text{if } k = n_i \end{cases}$$

(when $i = 0$, we define $n_{-1} = -1$), and $a_{nk}'(\mathbf{J}) = \delta_{nk}$ if $n \notin \mathbf{J}$.

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Let $\epsilon = (\epsilon_k)$, $\epsilon_k \geqq 0$ be given. Then

$$\beta_n = p_n(\epsilon_n/p_n - \epsilon_{n_k}/p_{n_k}) \quad \text{if } n_{k-1} < n < n_k,$$
$$\beta_{n_k} = P_{n_k}(\epsilon_{n_k}/p_{n_k} - \epsilon_{n_{k+1}}/p_{n_{k+1}}).$$

It is therefore easy to determine a $\mathbf{J}$ such that the β_n have the right signs, just by considering the sequence (ϵ_n/p_n). Then Theorem 19 can be easily applied by calculating $s(\mathbf{J})$.

Another case where Theorem 19 can be effectively used to calculate $\|\epsilon\|$ is when $\epsilon_k = 0$ for all but finitely many k. In this case the Simplex Method applies and gives $\mathbf{J}$ and $\|\epsilon\|$ effectively (c.f. Stoer, Einführung in die Numerische Mathematik I, Springer-Verlag, Berlin-Heidelberg-New York).

THEOREM 20. *Let A satisfy* (7.3), *and let $\epsilon \in o[A]_1^*$, $\epsilon_k \geqq 0$, $k \in \mathbf{N}_0$, and choose any $\alpha^* \in l_1$, $\alpha_n^* \geqq 0$, $n \in \mathbf{N}_0$ and $\mathbf{J}_0 \subseteq \mathbf{N}_0$ such that ϵ is $A_{\mathbf{J}_0}$-monotone with respect to α^*. Let*

$$(8.9) \qquad \mathbf{J}_1 = \{n | \alpha_n^* \neq 0\}, \quad \mathbf{J}_2 = \{n | \epsilon_n = \textstyle\sum_m \alpha_m^* a_{mn}\}.$$

Then the following statements hold:

(a) *If ϵ is $A_{\mathbf{J}}$-monotone with respect to α, then $\alpha = \alpha^*$ and $\mathbf{J}_1 \subseteq \mathbf{J} \subseteq \mathbf{J}_2$.*

(b) *If $\mathbf{J}_1 \subseteq \mathbf{J} \subseteq \mathbf{J}_2$, then ϵ is $A_{\mathbf{J}}$-monotone with respect to α^*.*

Proof. Let ϵ be $A_{\mathbf{J}}$-monotone with α and $A_{\mathbf{J}}$-monotone with $\tilde{\alpha}$ (in (8.1)). It follows from Theorem 18 that

$$\|\epsilon\| = \textstyle\sum_k \epsilon_k s_k(\tilde{\mathbf{J}}) \leqq \sum_k s_k(\tilde{\mathbf{J}}) \sum_n \alpha_n a_{nk}$$
$$= \textstyle\sum_n \alpha_n \sum_k a_{nk} s_k(\tilde{\mathbf{J}}) \leqq \sum_n \alpha_n = \|\epsilon\|,$$

hence (since equality must hold in each estimate) $\epsilon_k = \sum_n \alpha_n a_{nk}$ if $s_k(\tilde{\mathbf{J}}) > 0$, and $\alpha_n = 0$ if $\sum_k a_{nk} s_k(\tilde{\mathbf{J}}) < 1$. It follows that (8.1) with $\tilde{\mathbf{J}}$ holds for α and $\tilde{\alpha}$, and Theorem 19 shows that $\alpha = \tilde{\alpha} = \alpha^*$. The inclusion $\mathbf{J}_1 \subseteq \mathbf{J} \subseteq \mathbf{J}_2$ follows from (8.1). Conversely, if $\mathbf{J}$ satisfies $\mathbf{J}_1 \subseteq \mathbf{J} \subseteq \mathbf{J}_2$, then (8.1) holds.

9. The smallest $A_{\mathbf{J}}$-monotone majorant. As an application of Theorem 20 we prove

THEOREM 21. *Let A satisfy* (7.3), *and let $\epsilon \in o[A]_1^*$, $\epsilon_k \geqq 0$, $k \in \mathbf{N}_0$. Then there exists a uniquely defined sequence $\tilde{\epsilon} = (\tilde{\epsilon}_k)$ such that*

$$(9.1) \qquad \begin{cases} \tilde{\epsilon} \text{ is } A\text{-monotone and } \alpha \in l_1, \\ \epsilon_k \leqq \tilde{\epsilon}_k,\ k \in \mathbf{N}_0, \\ \|\epsilon\| = \|\tilde{\epsilon}\|. \end{cases}$$

Proof. Let α^* be the sequence which corresponds to ϵ by Theorem 20. If

$$\tilde{\epsilon}_k = \textstyle\sum_n \alpha_n^* a_{nk}, \quad k \in \mathbf{N}_0,$$

then (9.1) is satisfied (note that $\|\tilde{\epsilon}\| = \sum_k \tilde{\epsilon}_k s_k(\mathbf{N}_0) = \sum \alpha_n^* = \|\epsilon\|$). In order to prove that $\tilde{\epsilon}$ is unique, let $\bar{\epsilon}$ be any sequence satisfying (9.1). It follows that

$$\bar{\epsilon}_k = \sum_n \bar{\alpha}_n a_{nk}, \quad k \in \mathbf{N}_0 \text{ for some } \bar{\alpha} \in l_1.$$

Let $\|\epsilon\| = \sum \epsilon_k s_k(\mathbf{J})$. The relation

$$(9.2) \qquad \|\bar{\epsilon}\| \geqq \sum_k \bar{\epsilon}_k s_k(\mathbf{J}) \geqq \sum_k \epsilon_k s_k(\mathbf{J}) = \|\epsilon\| = \|\bar{\epsilon}\|$$

implies

(i) $\bar{\epsilon}_k = \epsilon_k$ for $k \in \mathbf{J}$,

(ii) $\bar{\epsilon}$ is $A_{\mathbf{J}}$-monotone (Theorem 18).

It follows from (i) and (ii) that ϵ is also $A_{\mathbf{J}}$-monotone with $\alpha = \bar{\alpha}$, therefore we see from Theorem 20 that $\bar{\alpha} = \alpha^*$, i.e., $\bar{\epsilon}$ is unique.

If $\epsilon \in o[A]_1^*$, $\epsilon_k \geqq 0$ for $k \in \mathbf{N}_0$, and if $\bar{\epsilon}$ is A-monotone and $0 \leqq \epsilon_k \leqq \bar{\epsilon}_k \leqq \tilde{\epsilon}_k$ for $k \in \mathbf{N}_0$, then $\bar{\epsilon} = \tilde{\epsilon}$. In fact, it follows as in (9.2) (with $\mathbf{J} = \mathbf{N}_0$) that $\|\epsilon\| \leqq \|\bar{\epsilon}\| \leqq \|\tilde{\epsilon}\| = \|\epsilon\|$, hence the uniqueness of $\tilde{\epsilon}$ implies $\bar{\epsilon} = \tilde{\epsilon}$. For this reason the sequence $\tilde{\epsilon}$ of Theorem 21 will be called the *smallest A-monotone majorant* of ϵ. The foregoing does not generally rule out the possibility that an A-monotone sequence $\tilde{\tilde{\epsilon}}$ exists with $0 \leqq \epsilon_k \leqq \tilde{\tilde{\epsilon}}$, $k \in \mathbf{N}_0$ and $\tilde{\tilde{\epsilon}}_k < \tilde{\epsilon}_k$ for some k (and then $\|\tilde{\tilde{\epsilon}}\| > \|\epsilon\|$ by Theorem 21). However, this is not the case if A satisfies in addition to (7.3) the condition $a_{nk}' \leqq 0$ for $k < n$, which is denoted by $A' \stackrel{\cdot}{\leqq} 0$. In this case a sequence ϵ which satisfies the assumptions of Theorem 21 is A-monotone by Theorem 19 if and only if

$$(9.3) \qquad \epsilon_k \geqq a_{kk} \sum_{n=k+1}^{\infty} \epsilon_n |a_{nk}'|, \quad k \in \mathbf{N}_0.$$

LEMMA 7. *Let A satisfy* (7.3) *and $A' \stackrel{\cdot}{\leqq} 0$. Consider a family $\epsilon(\gamma) = (\epsilon_n(\gamma)) \in o[A]_1^*$, $\gamma \in \Gamma$ of A-monotone sequences. Then $\epsilon = (\epsilon_k)$, $\epsilon_k = \inf_{\gamma \in \Gamma} \epsilon_k(\gamma)$ is in $o[A]_1^*$ and A-monotone.*

Proof. Obviously $\epsilon \in o[A]_1^*$. Moreover,

$$\epsilon_k(\gamma) \geqq a_{kk} \sum_{n=k+1}^{\infty} \epsilon_n |a_{nk}'|$$

for $k \in \mathbf{N}_0$ and every $\gamma \in \Gamma$, hence

$$\epsilon_k \geqq a_{kk} \sum_{n=k+1}^{\infty} \epsilon_n |a_{nk}'|.$$

In order to show that an $\tilde{\tilde{\epsilon}}$ as above does not exist, we introduce ϵ' by

$$\epsilon_k' = \min (\tilde{\epsilon}_k, \tilde{\tilde{\epsilon}}_k), \quad k \in \mathbf{N}_0.$$

This sequence is A-monotone by Lemma 7 and satisfies $\epsilon_k \leqq \epsilon_k'$, $k \in \mathbf{N}_0$,

$\|\epsilon\| \leqq \|\epsilon'\| \leqq \|\tilde{\epsilon}\| = \|\epsilon\|$. It follows from Theorem 21 that $\epsilon' = \tilde{\epsilon}$, hence $\tilde{\epsilon}_k \leqq \tilde{\tilde{\epsilon}}_k$ for $k \in \mathbf{N}_0$.

We next give the following construction of the smallest A-monotone majorant.

THEOREM 22. *Let A satisfy* (7.3) *and $A' \dot{\leqq} 0$. Let $\epsilon \in o[A]_1^*$, $\epsilon_k \geqq 0$ for $k \in \mathbf{N}_0$. Then its smallest A-monotone majorant $\tilde{\epsilon}$ is given by*

$$\tilde{\epsilon}_k = \max(\epsilon_k, \bar{\epsilon}_k),$$

$$\bar{\epsilon}_k = \sup_{\mathbf{J}} a_{kk}(\mathbf{J}) \sum_{n=k+1}^{\infty} \epsilon_n |a_{nk}'(\mathbf{J})|.$$

Proof. Let $C_{\mathbf{J}} = (c_{nk}(\mathbf{J})) = AA_{\mathbf{J}}'$, hence $C_{\mathbf{J}}' = (c_{nk}'(\mathbf{J})) = A_{\mathbf{J}}A'$ with

$$c_{nk}'(\mathbf{J}) = \begin{cases} \delta_{nk} & \text{if } n \in \mathbf{J}, \\ a_{nk}' & \text{if } n \notin \mathbf{J}. \end{cases}$$

It follows that $C_{\mathbf{J}}' \dot{\leqq} 0$, hence $c_{nk}(\mathbf{J}) \geqq 0$ (c.f. [15]). It also follows that $c_{nk}(\mathbf{J}) = \delta_{nk}$ if $n \in \mathbf{J}$. If $n > k$, then for $n \in \mathbf{J}$

$$a_{nk}'(\mathbf{J}) = \sum_m a_{nm}'c_{mk}(\mathbf{J}) = \sum_{m=k}^{n-1} a_{nm}'c_{mk}(\mathbf{J}) \dot{\leqq} 0$$

and $a_{nk}'(\mathbf{J}) = \delta_{nk}$ for $n \notin \mathbf{J}$. This shows that $A_{\mathbf{J}}' \dot{\leqq} 0$. Let $\mathbf{J}$ be any index set. Then

$$a_{kk}(\mathbf{J}) \sum_{n=k+1}^{\infty} \epsilon_n |a_{nk}'(\mathbf{J})| \leqq a_{kk}(\mathbf{J}) \sum_{n=k+1}^{\infty} \tilde{\epsilon}_n |a_{nk}'(\mathbf{J})|$$

$$= \tilde{\epsilon}_k - a_{kk}(\mathbf{J}) \sum_{n=k}^{\infty} \tilde{\epsilon}_n a_{nk}'(\mathbf{J})$$

and

$$\sum_n \tilde{\epsilon}_n a_{nk}'(\mathbf{J}) = \sum_n a_{nk}'(\mathbf{J}) \sum_m \alpha_m^* a_{mn} = \sum_m \alpha_m^* c_{mk}(\mathbf{J}) \geqq 0. \tag{9.4}$$

Therefore, $\bar{\epsilon}_k \leqq \tilde{\epsilon}_k$ for $k \in \mathbf{N}_0$.

Let ϵ be $A_{\mathbf{J}}$-monotone. If $k \in \mathbf{J}$, then $\epsilon_k = \tilde{\epsilon}_k$, hence

$$\max(\epsilon_k, \bar{\epsilon}_k) = \max(\tilde{\epsilon}_k, \bar{\epsilon}_k) = \tilde{\epsilon}_k.$$

Next, let $k \notin J$. Then (note that $\alpha_n^* = 0$ if $n \notin \mathbf{J}$ by Theorem 20)

$$\tilde{\epsilon}_k = \sum_n \alpha_n^* a_{nk} = \sum_{n \in \mathbf{J}} \alpha_n^* a_{nk}(\mathbf{J}) = \sum_n \alpha_n^* a_{nk}(\mathbf{J})$$

and (note that $a_{kn}'(\mathbf{J}) = \delta_{kn}$ if $k \notin \mathbf{J}$)

$$\tilde{\epsilon}_k a_{kk}'(\mathbf{J}) + \sum_{n=k+1}^{\infty} \epsilon_n a_{nk}'(\mathbf{J}) = \sum_{n=k}^{\infty} \tilde{\epsilon}_n a_{nk}'(\mathbf{J})$$

$$= \sum_{n=k}^{\infty} a_{nk}'(\mathbf{J}) \sum_m \alpha_m^* a_{mn}(\mathbf{J}) = \alpha_k^* = 0,$$

hence $\tilde{\epsilon}_k \leqq \bar{\epsilon}_k$, consequently $\tilde{\epsilon}_k = \bar{\epsilon}_k$ in this case. It follows that

$$\max(\epsilon_k, \bar{\epsilon}_k) = \max(\epsilon_k, \tilde{\epsilon}_k) = \tilde{\epsilon}_k$$

in this case, which completes the proof of Theorem 22.

The foregoing theorem has the following

COROLLARY. *Let A satisfy* (7.3) *and $A' \dot{\leqq} 0$. Then $\epsilon = (\epsilon_k)$ is in $o[A]_1^*$ if and only if*

$$(9.5)\qquad \sum_k s_k(\mathbf{N}_0) \sup_{\mathbf{J}} a_{kk}(\mathbf{J}) \sum_{n=k}^{\infty} |\epsilon_n|\,|a_{nk}'(\mathbf{J})| < \infty.$$

Proof. For the proof we may assume $\epsilon_k \geqq 0$ for $k \in \mathbf{N}_0$. Let $\epsilon \in o[A]_1^*$, and let $\tilde{\epsilon}$ be its smallest A-monotone majorant. Then $\sum_k \tilde{\epsilon}_k s_k(\mathbf{N}_0) < \infty$, hence (by Theorem 22)

$$\sum_k (\epsilon_k + \bar{\epsilon}_k) s_k(\mathbf{N}_0) \leqq 2 \sum_k \tilde{\epsilon}_k s_k(\mathbf{N}_0) < \infty.$$

But

$$\epsilon_k + \bar{\epsilon}_k = \sup_{\mathbf{J}} a_{kk}(\mathbf{J}) \sum_{n=k}^{\infty} \epsilon_n |a_{nk}'(\mathbf{J})|$$

for $k \in \mathbf{N}_0$, which proves (9.5).

Conversely, assume that (9.5) holds. If we define

$$\tilde{\epsilon}_k = \max(\epsilon_k, \bar{\epsilon}_k), \quad \bar{\epsilon}_k = \sup_{\mathbf{J}} a_{kk}(\mathbf{J}) \sum_{n=k+1}^{\infty} \epsilon_n |a_{nk}'(\mathbf{J})| \quad \text{for } k \in \mathbf{N}_0,$$

then $\sum \tilde{\epsilon}_k s_k(\mathbf{N}_0) < \infty$. If we show that $\sum_k \tilde{\epsilon}_k a_{km}' \geqq 0$ for $m \in \mathbf{N}_0$, then an application of Theorem 19 shows $\tilde{\epsilon} \in o[A]_1^*$, hence $\epsilon \in o[A]_1^*$. In order to prove the missing inequalities, we define $\epsilon^{(n)} = (\epsilon_k^{(n)})$ by

$$\epsilon_k^{(n)} = \begin{cases} \epsilon_k \text{ if } k \leqq n, \\ 0 \text{ if } k > n. \end{cases}$$

Then certainly $\epsilon^{(n)} \in o[A]_1^*$, and we denote by $\tilde{\epsilon}^{(n)}$ the smallest A-monotone majorant of $\epsilon^{(n)}$. Then by Theorem 22 we have

$$\tilde{\epsilon}_k^{(n)} = \max(\epsilon_k^{(n)}, \bar{\epsilon}_k^{(n)}), \quad \bar{\epsilon}_k^{(n)} = \sup_{\mathbf{J}} a_{kk}(\mathbf{J}) \sum_{m=k+1}^{n} \epsilon_m |a_{mk}'(\mathbf{J})|,$$

hence $\tilde{\epsilon}_k^{(n)} \leqq \tilde{\epsilon}_k$ for $k \in \mathbf{N}_0$. On the other hand, $\bar{\epsilon}_k^{(n)} \uparrow \bar{\epsilon}_k$ as $n \to \infty$, k fixed, and therefore $\tilde{\epsilon}_k^{(n)} \uparrow \tilde{\epsilon}_k$ as $n \to \infty$, k fixed.

Since $\tilde{\epsilon}_k^{(n)}$ is A-monotone, we have

$$\tilde{\epsilon}_k^{(n)} \geqq a_{kk} \sum_{m=k+1}^{\infty} \tilde{\epsilon}_m^{(n)} |a_{mk}'| \quad \text{for } k \in \mathbf{N}_0,$$

and by taking the limit as $n \to \infty$ on both sides, we see

$$\tilde{\epsilon}_k \geqq a_{kk} \sum_{m=k+1}^{\infty} \tilde{\epsilon}_k |a_{mk}'| \quad \text{for } k \in \mathbf{N}_0,$$

which completes the proof.

We apply the corollary to $A = M_p$, $p_n > 0$, $P_n \to \infty$. It follows from (8.8) that

$$\bar{\epsilon}_k = p_k \sup_{n>k} \epsilon_n/p_n,$$

consequently

$$\tilde{\epsilon}_k = p_k \sup_{n\geqq k} \epsilon_n/p_n,$$

and since $s_k(\mathbf{N}_0) \equiv 1$, we see that the corollary coincides with the case $p = 1$ of Theorem 8.

10. A remark on $p > 1$. In Sections 7–9 we have discussed the structure of $o[A]_1^*$. In particular, if A satisfies (7.3) the central result of Theorem 18 can be written as follows: $\epsilon \in o[A]_1^*$ if and only if $t = (t_k)$ and $\alpha = (\alpha_n)$ exist such that

$$(10.1)\quad \begin{cases} |t_k| \leqq 1,\ k \in \mathbf{N}_0 \\ \alpha_n \geqq 0,\ \alpha_n = 0 \text{ if } |t_n| < 1 \\ \epsilon_k = t_k \sum_n \alpha_n a_{nk} \text{ for } k \in \mathbf{N}_0 \\ \|\epsilon\| = \sum \alpha_n < \infty. \end{cases}$$

Given ϵ, the corresponding α is uniquely defined (by Theorem 20) and so is t if we set $t_k = 0$ when $\alpha_n = 0$ for all $n \geqq k$.

We wish to state without proof that the sequences $\epsilon \in o[A]_p^*$, $1 < p < \infty$ can be characterized analogously. We obtain a characterization of these sequences from (10.1) when the conditions $|t_k| \leqq 1$ and $|t_n| < 1$ are replaced by

$$\sum_m a_{km}|t_m|^q \leqq 1 \ (p^{-1} + q^{-1} = 1) \text{ and } \sum_k a_{nk}|t_k|^q < 1.$$

Again, the sequences α and t are uniquely defined by ϵ, if we again set $t_k = 0$ when $\alpha_n = 0$ for all $n \geqq k$.

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