ON THE UNIFORM CONTINUITY OF WIENER PROCESS WITH A MULTIDIMENSIONAL PARAMETER

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§1. Introduction

Let $X(A, \omega)$, $\omega \in \Omega$, be Wiener process on the probability space $(\Omega, \mathfrak{B}, P)$ depending on a point A of an N dimensional Euclidean space $\mathbf{E}_N$. It is defined by the following three conditions:

(1) For any positive integer n and any points $A_1, A_2, \ldots, A_n$ in $\mathbf{E}_N$, the joint distribution of $\{X(A_1), X(A_2), \ldots, X(A_n)\}$ is a non-degenerate n dimensional normal distribution with mean vector $\mathbf{O}$.

(2) $$E\{X(A)X(B)\} = \{r(O, A) + r(O, B) - r(A, B)\}/2,$$

where $r(A, B)$ denotes the distance between A and B, and O denotes the origin of $\mathbf{E}_N$.

(3) $$X(O, \omega) = 0, \qquad \text{for almost every } \omega \in \Omega.$$

Previously P. Lévy studied this process in his book [1] and obtained many properties of it. Concerning the uniform continuty, he proved the following:

If $a > 1$, for almost every $\omega \in \Omega$, there exists a positive number $\rho = \rho(\omega)$ such that

(4) $$r = r(A, B) < \rho \quad \textit{implies} \quad |X(A, \omega) - X(B, \omega)| \leqq a\sqrt{2Nr|\log r|}.$$

On the other hand, if $a < 1$, there exists no such $\rho(\omega)$. Here both A and B are arbitrarily chosen in the cube $C = \{A = (a_1, a_2, \ldots, a_N)\ ;\ |a_i| \leqq 1, i = 1, 2, \ldots, N\}$.

While T. Sirao [2] has proved a very precise continuity property of Wiener process with a parameter space $\mathbf{E}_1$. It is stated as follows:

If $c > 5$, there exists a positive number $\rho = \rho(\omega)$ such that

(5) $$r = |t' - t| < \rho \ \textit{implies}\ |X(t, \omega) - X(t', \omega)| \leqq \sqrt{2r(|\log r| + c\log|\log r|)},$$

with probability one. Here t and t' are arbitrary numbers in the interval $[0, 1]$. But, *for $c < -1$, there exists no such $\rho(\omega)$.*

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The aim of the present note is to improve the Lévy's result employing the method used by Sirao [2]. Our result may be regarded as the extension of the Sirao's result for the multidimensional parameter space. But, in our case where the paramter space is $\mathbf{E}_N$, we shall consider the *n-th bundle of lines* and every variation of $X(A)$ relating to it as will be seen at the proof of Theorem 1. Moreover, for the proof of Theorem 2 that gives the lower estimation of our process, we shall use the lemma and the method of the proof of Theorem 1 which appear in the paper written by K. L. Chung and P. Erdös [3].

§ 2. Upper estimation

Before we state the theorem, some preparations are necessary. Generally the direction of any straight line that runs through the origin is determined by the ordered set of angles $(\theta_1, \theta_2, \ldots, \theta_{N-1})$, $0 \leqq \theta_i < \pi$, $i = 1, 2, \ldots, N-1$. We consider n^{N-1} straight lines the direction of which are given such as $(j_1\pi/n^2, j_2\pi/n^2, \ldots, j_{N-1}\pi/n^2)$, $j_k = 0, 1, \ldots, n^2-1$, $k = 1, 2, \ldots, N-1$.

Taking one of those straight lines $\mathfrak{s}$, we consider the hyper plane $\mathfrak{H}$ that is orthogonal to $\mathfrak{s}$ and contains the origin. On this hyper plane we take $(2^n+1)^{N-1}$ lattice points, the coordinates of which are such as $(j_1/2^{n-1}, j_2/2^{n-1}, \ldots, j_{N-1}/2^{n-1})$, $j_k = 0, \pm 1, \pm 2, \ldots, \pm 2^{n-1}$, $k = 1, 2, \ldots, N-1$, for suitably chosen coordinate system. And then, for every one of those lattice points, we consider a straight line that is parallel with $\mathfrak{s}$ and runs through that point. The figure that consists of the hyper plane $\mathfrak{H}$ and $(2^n+1)^{N-1}$ straight lines considered above will be called the *n-th bundle of lines* and will be denoted by $\mathfrak{L}_n$.

In a similar way, we have n^{2N-2} n-th bundles of lines. The lattice point of $\mathfrak{L}_n$ is defined as the point that lies on some straight line of $\mathfrak{L}_n$ at the distance of $k/2^{n-1}$ (k is an integer and $|k| \leqq 2^{n-1}$) from the corresponding hyper plane. Thus $\mathfrak{L}_n$ has $(2^n+1)^N$ lattice points.

For any pair of lattice points A_1 and A_2 of $\mathfrak{L}_n$ such that they lie on the same straight line and $r(A_1, A_2) = 1/2^{n-1}$, there corresponds the difference $\Delta X(A) = X(A_1) - X(A_2)$, which we shall call the *variation of* $X(A)$ *relating to* $\mathfrak{L}_n$.

Let us put

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$$\varphi_c(r) = \{r(2N|\log r| + c \log|\log r|)\}^{1/2}.$$

It is easily seen that $\varphi_c(r)$ is monotone for small $r > 0$ and tends to 0 as $r \to +0$.

Thus we may state the following theorem for Wiener process $X(A)$, when A runs in the unit sphere of $\mathbf{E}_N$ with center O.

THEOREM 1. *If* $c > 8N+1$, *for almost every* ω, *there exists a positive number* $\rho = \rho(\omega)$ *such that*

(6) $$r = r(A, B) < \rho \quad \textit{implies} \quad |X(A) - X(B)| \leqq \varphi_c(r),$$

Proof. Let us put $c = 8N+1+\varepsilon$ with a positive number ε. First we take one of the straight lines belonging to $\mathfrak{L}_n$. The lattice points lying on it are arranged as $A_0, A_1, \ldots, A_{2^n}$ in a order of their position. It is noted that on that straight line $X(A)$ may be considered as a usual Wiener process with a linear parameter.

We shall define $\alpha_{n,m,l}$ as the probability that the inequality

(7) $$|X(A_{m+l}) - X(A_l)| > \varphi_c(m/2^{n-1}), \qquad m = 1, 2, \ldots, [n^2 \log n], \quad l = 0, 1, \ldots, 2^n - 1,$$

holds. Since $\alpha_{n,m,l}$ is independent of l, we may write simply $\alpha_{n,m}$. Then we have

$$\alpha_{n,1} \leqq \alpha_{n,2} \leqq \ldots \leqq \alpha_{n,[n^2 \log n]}$$

for sufficiently large n (See Sirao [2]).

Noting that $X(A) - X(B)$ is a normal random variable with mean 0 and variance $r = r(A, B)$, we have the following asymptotic relation.

(8) $$P(|X(A) - X(B)| > \varphi_c(r)) = O(1) r^N (|\log r|)^{-(c+1)/2},$$

for sufficiently small $r > 0$. Therefore we have

(9) $$\alpha_{n,[n^2 \log n]} = O(1) 2^{-nN} n^{-(2N+1+\varepsilon/2)} (\log n)^N.$$

If we consider such a probability for every straight line belonging to $\mathfrak{L}_n$, in a similar way, we have

(10) $$\sum_n n^{2N-2}(2^n+1)^{N-1} n^2 (\log n) 2^n \{O(1) 2^{-nN} n^{-(2N+1+\varepsilon/2)} (\log n)^N\} = O(1) n^{-(1+\varepsilon/2)} (\log n)^{N+1} < \infty.$$

The n-th term of the left hand side of (10) is not less than the probability

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that there exists at least one straight line belonging to $\mathfrak{L}_n$ and exists at least one pair (m, l) satisfying the inequality (7). The convergence property of (10) shows, by Borel-Cantelli's Lemma, that there exists a positive integer $n_0 = n_0(\omega)$ such that $n > n_0(\omega)$ implies the validity of (6) for every variation relating to $\mathfrak{L}_n$ with probability one. Hence the theorem is proved for the particular case where both A and B are the lattice points lying on the same straight line of $\mathfrak{L}_n$ $(n > n_0(\omega))$ and $r(A, B) = m/2^{n-1}$, $m = 1, 2, \ldots, [n^2 \log n]$.

Next we shall prove the theorem in a general case. For any points A and B in the unit sphere with center O, if $r = r(A, B)$ is sufficiently small, there exists a positive integer n such that

$$(11) \qquad [n^2 \log n]/2^n < r \leqq [(n-1)^2 \log (n-1)]/2^{n-1}.$$

Here r must be very small so that $n > n_0(\omega)$. Then there is some straight line $\mathscr{L}$ belonging to one of the n-th bundles of lines that is the nearest one to the segment AB. Let A_1 and B_1 be the projections of A and B on $\mathscr{L}$ respectively. Then there are some lattice points A_2 and B_2 lying on $\mathscr{L}$, which are the nearest ones to A_1 and B_1 respectively and satisfy the inequality $r(A_2, B_2) \leqq r(A, B)$. These facts imply

$$(12) \qquad r(A_2, B_2) = m/2^{n-1},$$
$$r_A = r(A, A_2) < c_1 \log n/2^{n-1} \quad \text{and} \quad r_B = r(B, B_2) < c_2 \log n/2^{n-1},$$

where c_1 and c_2 are absolute constants and m is an integer that is not less than $[2^{-1} n^2 \log (n-1)]$.

Let us put $c' = c - \varepsilon/2$. Then $c' > 8N + 1$. Since $n > n_0(\omega)$, by using the above conclusion for the particular case, we have

$$(13) \qquad |X(A_2) - X(B_2)| \leqq \varphi_{c'}(m/2^{n-1}) \leqq \varphi_{c'}(r).$$

The second inequality is derived from the monotony of $\varphi_{c'}(r)$. Therefore we can prove, from (4) and (13),

$$(14) \qquad |X(A) - X(B)| \leqq |X(A_2) - X(B_2)| + |X(A) - X(A_2)| + |X(B) - X(B_2)|$$
$$\leqq \varphi_{c'}(r) + a\sqrt{2N}(\sqrt{r_A}\,|\log r_A| + \sqrt{r_B}\,|\log r_B|)$$
$$< \sqrt{r}(2N|\log r| + c' \log|\log r|) + b\sqrt{n \log n/2^n},$$

where b is a constant determined by c_1, c_2 and a. Noting that r is of order $(n^2 \cdot \log n)/2^n$, we can easily prove by the simple computations that the last side of (14) is less than $\varphi_c(r)$. Thus we have completely proved the theorem.

§ 3. Lower estimation

Fist let us state the theorem.

THEOREM 2. *If $c<1$, for almost every ω, there exist infinitely many pairs (A, B) in C such that the inequality*

$$|X(A)-X(B)|>\varphi_c(r), \qquad (r=r(A, B)), \tag{15}$$

holds.

Before proceeding to the proof of this theorem, we shall state some simple lemmas.

LEMMA 1. *Let $\{F_k\}$, $k=1, 2, \ldots, n$, be an arbitrary sequence of events in $(\Omega, \mathfrak{B}, P)$. Then we have, if $P(\bigcup_{k=1}^{n} F_k)>0$,*

$$2\sum_{1\leq j<k\leq n} P(F_j\cap F_k) \geqq [P(\bigcup_{k=1}^{n} F_k)]^{-1}(\sum_{k=1}^{n} P(F_k))^2-\sum_{k=1}^{n} P(F_k). \tag{16}$$

The proof is found in K. L. Chung and P. Erdös [3].

LEMMA 2. *Let $\Delta_1 X(A)=X(A_1)-X(A_1')$ and $\Delta_2 X(A)=X(A_2)-X(A_2')$ be arbitrary variations relating to the n-th and $(n+m)$-th bundles of lines respectively and assume that A_1A_1' and A_2A_2' be parallel with each other. Then, if the distance between two segments is longer than $n/2^{n-1}$, the correlation coefficient $r_{1,2}$ of $\Delta_1 X(A)$ and $\Delta_2 X(A)$ is less than $(\lambda_n\lambda_{n+m})^{-1}$. Here λ_k is defined by*

$$\lambda_k=2^{(k-1)/2}\varphi_c(1/2^{k-1}). \tag{17}$$

Considering the results obtained by P. Lévy (See [1] § 61), we can prove this lemma by elementary considerations.

LEMMA 3. *Let $\Delta_1 X(A)$ and $\Delta_2 X(A)$ be what were defined in Lemma 2 and n be large. Let us put*

$$E_1=\{\omega\ ;\ \Delta_1 X(A)>\varphi_c(1/2^{n-1})\} \quad and \quad E_2=\{\omega\ ;\ \Delta_2 X(A)>\varphi_c(1/2^{n+m-1})\}. \tag{18}$$

Then, if $r_{1,2}<(\lambda_n\lambda_{n+m})^{-1}$, we can find an absolute constant d satisfying

$$P(E_1\cap E_2)<dP(E_1)P(E_2). \tag{19}$$

And, if $(\lambda_n\lambda_{n+m})^{-1}\leqq r_{1,2}\leqq 1/2$, we have

$$P(E_1\cap E_2)\leqq O(1)e^{-\alpha(n+m)}P(E_1) \tag{20}$$

for some $\alpha>0$.

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Proof. By the asymptotic relation (8), it is proved that

$$P(E_1)\,P(E_2) = O(1)\,(\lambda_n \lambda_{n+m})^{-1} e^{-(\lambda_n^2+\lambda_{n+m}^2)/2}. \tag{21}$$

Under the assumption $0 < r_{1,2} < (\lambda_n \lambda_{n+m})^{-1}$, the left side of (19) becomes

$$\begin{aligned} P(E_1 \cap E_2) &= (2\pi)^{-1}(1-r^2)^{-1/2}\int_{\lambda_n}^{\infty}\int_{\lambda_{n+m}}^{\infty} e^{-(x^2+y^2-2rxy)/2(1-r^2)}\,dx\,dy \qquad (r = r_{1,2}) \\ &= (2\pi)^{-1}(1-r^2)^{-1/2}\int_{\lambda_n}^{\sqrt{3}\lambda_{n+m}}\int_{\lambda_{n+m}}^{\sqrt{3}\lambda_{n+m}} e^{-(x-ry)^2/2} e^{-y^2/2}\,dx\,dy \\ &\quad + O\left(\frac{1}{\lambda_{n+m}} e^{-\lambda_{n+m}^2}\right) \\ &\leqq O(1)\,\lambda_{n+m}^{-1} e^{-\lambda_{n+m}^2/2}\int_{\lambda_n}^{\sqrt{3}\lambda_{n+m}} e^{-(x-\sqrt{2}/\lambda_n)^2/2}\,dx \\ &= O(1)\,(\lambda_n\lambda_{n+m})^{-1} e^{-(\lambda^2_n+\lambda^2_{n+m})/2} \end{aligned} \tag{22}$$

If $r_{1,2} \leqq 0$, (19) is a trivial inequality. Thus, from (21) and (22), we can find d satisfying the inequality (19).

If $r_{1,2} \leqq 1/2$, from the second equality of (22), we have

$$\begin{aligned} P(E_1 \cap E_2) &\leqq O(1)\,\lambda_n^{-1} e^{-\lambda^2_n/2}\lambda_{n+m}^{-1} e^{-(\lambda^2_{n+m}(1-\sqrt{3/2})^2)/2(1-r^2)} \\ &\leqq O(1)\,e^{-\alpha(n+m)}\,P(E_1), \qquad (\alpha > 0), \end{aligned}$$

which proves the inequality (20).

Proof of Theorem 2.[1] For every variation $\Delta_{n,j}X(A)$, $j = 1, 2, \ldots, 2^{nN}$, relating to the n-th bundle of lines,[2] let us put

$$E_{n,j} = \{\omega\ ;\ \Delta_{n,j}X(A) > \varphi_c(1/2^{n-1})\}, \qquad j = 1, 2, \ldots, 2^n(2^n+1)^{N-1}. \tag{23}$$

Then we have

$$\sum_n\sum_j P(E_{n,j}) = O(1)\sum_n 2^{nN}(2^{-nN} n^{-(1+c)/2}) = \infty, \tag{24}$$

from the assumption $c < 1$. Let us put

$$E_n = \bigcup_j E_{n,j} \quad \text{and} \quad B_m = \bigcup_{n=m}^{\infty} E_n.$$

In order to show that $\lim_{\infty\to m} P(B_m) = 1$, we shall prove that $P(B_m) = 1$ for everv m.

Suppose $P(B_m) < 1$ for some m. And let $P(B_m) = 1 - \delta$, $(\delta > 0)$. Then we have

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1) We owe the method of this proof to K. L. Chung and P. Erdös. But we need some modifications.

2) Hereafter, for every n, we take one of the n-th bundles of lines that contains the same hyper plane.

(25) $$P(B_m^c) = P(\bigcap_{n=m}^{\infty} E_n^c) = \delta.$$

For any ε such that $1-\delta > \varepsilon > 0$, there exists an integer l satisfying the following inequality:

(26) $$P(\bigcup_{n=m}^{l} E) > 1-\delta-\varepsilon.$$

If we write $G_{m,l} = B_m - \bigcup_{n=m}^{l} E_n$, we have

(27) $$P(G_{m,l}) < \varepsilon.$$

As is easily seen, for the given event $\bigcup_{n=m}^{l} E_n$, there exists a large $K(>m)$ such that

(28) $$P(E_{k,p} \mid (\bigcup_{n=m}^{l} E_n)^c) > \frac{1}{2} P(E_{k,p})$$

holds for any $k > K$ and any $p > 0$. Hence, from (24) and (28), we have

$$\sum_k \sum_p P(E_{k,p} \cap (\bigcup_{n=m}^{l} E_n)^c) = \infty.$$

Therefore we can choose an integer $K' > K$ such that

(29) $$1 < \sum_{k=K}^{K'} \sum_p P(E_{k,p} \cap (\bigcup_{n=m}^{l} E_n)^c) \leqq 2,$$

where the sum $\sum_p$ should be properly added when $k = K$ or $= K'$. Thus we obtain, from (25), (28) and (29),

(30) $$\sum_{k=K}^{K'} \sum_p P(E_{k,p}) < 4/\delta.$$

Furthermore we consider the following relation:

(31) $$\sum_{K \leqq k < k' \leqq K'} \sum_{p,p'} P(E_{k,p} \cap E_{k',p'}) \leqq \sum{}^1 P(E_{k,p} \cap E_{k',p'}) + d \sum{}^2 P(E_{k,p}) P(E_{k',p'}) + \sum_{k=K}^{K'}{}^3 P(E_{k,p} \cap E_{k+1,p'}),$$

where $\sum^1$ or $\sum^2$ indicates the summation extending over those k, k', p and p' for which the corresponding correlation coefficient r satisfies the inequality $r < (\lambda_k \lambda_{k'})^{-1}$ or $1/2 \geqq r \geqq (\lambda_k \lambda_{k'})^{-1}$ respectively, and $\sum^3$ indicates the summation of the rest, that is, the case $r > 1/2$.

The sum $\sum^3$ is not larger than $2 \sum_{k=K}^{K'} \sum_p P(E_{k,p}) < 2 \frac{4}{\delta}$, since such a circumstance may happen only in the case where one of the corresponding segments is a sub-segment with half length of the other.

Concerning $\sum^2$, the number of k appearing in the summand is at most Min $([\lambda \log k], K')$ with some $\lambda > 0$,[3] and for each pair (k, p) the number of p' appearing there is at most of order $(2^{k'-k} k)^N$, which is deduced from Lemma 2. Hence we have

$$
\begin{aligned}
(32) \qquad \sum^2 P(E_{k,p} \cap E_{k',p'}) &\leqq \sum_{k=K}^{K'} \sum_{k'=k}^{k+[\lambda \log k]} (\sum_{p,p'}^2 P(E_{k,p} \cap E_{k',p'})) \\
&\leqq O(1) \sum_{k=K}^{K'} \sum_{k'=k}^{k+[\lambda \log k]} (\sum_{p,p'}^2 P(E_{k,p}) e^{-\alpha k'}) \\
&= O(1) \sum_{k=K}^{K'} \sum_{k'=k}^{k+[\lambda \log k]} (2^{k'-k} k)^N e^{-\alpha k'} 2^{Nk} P(E_{k,p}) \\
&= O(1) \sum_{k=K}^{K'} e^{-\alpha k} k^N 2^{Nk} P(E_{k,p}) \sum_{j=1}^{[\lambda \log k]} (2^N e^{-\alpha})^j,
\end{aligned}
$$

which is less than a certain absolute constant M. Combining the obvious inequality

$$\sum_{K \leqq k < k' \leqq K'} \sum_{p,p'} P(E_{k,p}) P(E_{k',p'}) \leqq \{ \sum_{K \leqq k \leqq K'} \sum_{p} P(F_{k,p}) \}^2 \leqq (4/\delta)^2,$$

we have

$$(33) \qquad \sum_{K \leqq k < k' \leqq K'} \sum_{p,p'} P(E_{k,p} \cap E_{k',p'}) \leqq d(4/\delta)^2 + M + 8/\delta.$$

Now let us put $F_{k,p} = E_{k,p} \cap (\bigcup_{n=m} E_n)^c$. $\bigcup_{k=K}^{K'} \bigcup_{p} F_{k,p}$ being the subset of $G_{m,l}$, we have

$$(34) \qquad P(\bigcup_{k=K}^{K'} \bigcup_{p} F_{k,p}) < \varepsilon.$$

From (29) and (30), we have

$$(35) \qquad 1 < \sum_{k=K}^{K'} \sum_{p} P(F_{k,p}) \leqq \sum_{k=K}^{K'} \sum_{p} P(E_{k,p}) < 4/\delta.$$

Applying the Lemma 1 to $F_{k,p}$, we obtain, from (34) and (35),

$$(36) \qquad 2 \sum_{K \leqq k < k' \leqq K'} \sum_{p,p'} P(E_{k,p} \cap E_{k',p'}) \geqq 2 \sum_{K \leqq k < k' \leqq K'} \sum_{p,p'} P(F_{k,p} \cap F_{k',p'}) \geqq \frac{1}{\varepsilon} - \frac{4}{\delta}.$$

Since ε may be chosen arbitrarily small, (33) and (36) are incompatible. This

[3] λ may be taken as 3. In fact, we can easily prove that $k' < [\lambda \log k]$ implies $r < (\lambda_k \lambda_{k'})^{-1}$.

contradiction proves that $\delta = 0$. Hence $P(B_m) = 1$ for every m, which proves the theorem for the particular case where both A and B are lattice points. This completes the proof.

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