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HOLOMORPHIC SOLUTIONS ABOUT AN IRREGULAR SINGULAR POINT OF AN ORDINARY LINEAR DIFFERENTIAL EQUATION

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Abstract

It is shown that that an ordinary linear differential equation may possess a holomorphic solution in a neighbourhood of an irregular singular point even though the usual linearly independent solutions corresponding to the two roots of the indicial equation both have zero radius of convergence.

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1. Introduction

The literature concerning solutions of an ordinary linear differential equation about an irregular singular point has been concerned primarily with the possible number of regular solutions and the nature of the remaining solutions. Ince ([8], page 422) states that if $z = 0$ is an irregular singular point of the equation

$$L(w) \equiv \sum_{k=0}^{n} d_{n-k}(z) w^{(k)}(z) = 0,$$

where the d_j are all holomorphic in a neighbourhood of the origin, then there can exist a regular solution

$$w = z^{\rho} \sum_{n=0}^{\infty} f_n z^n \tag{1}$$

about the origin only if all but a finite number of the constants f_n vanish. Otherwise the series $\Sigma f_n z^n$ has zero radius of convergence. Ince's argument is not entirely rigorous and can only be taken as an indication of what happens in the main.

Forsyth [3] establishes the same result. However, referring to the possibility of a non-terminating series $\Sigma f_n z^n$, he notes (page 236) that "it is not inconceivable that for special values $\cdots$ the series would converge: and an exception to the general theorem would occur. But it is clear that such an exception is of a very special character."

In fact such instances do exist. Examples are provided by the equations

$$z^2 w'' - (1 - z^2) w' - w = 0$$

and

$$z^2 w' + a w' - b(a + b z^2) w = 0$$

which have respective solutions $w_1 = \exp(-z)$ and $w_1 = \exp(bz)$ about the irregular singular point $z = 0$.

The problem of characterising the non-regular solutions proved remarkably obstinate. Two important lines of research have arisen. The first of these starts with the full set of formal solutions (in general divergent)

$$w_i = z^{r_i} \sigma_i(z) \exp Q_i(z), \qquad 1 \leqslant i \leqslant n,$$

which is known from the work of Fabry [2] to exist in all cases. Here

$$Q_i(z) = \sum_{\nu=0}^{l_i - 1} q_\nu^i z^{(l_i - \nu)/k_i}, \qquad 1 \leqslant i \leqslant n,$$

$$\sigma_1(z) = \sum_{h=0}^{m_i} (\log z)^h {}_h\eta^{m_i}(z),$$

$${}_h\eta^{m_i}(z) = \sum_{s=0}^{\infty} {}_h\eta^{m_i} z^{-s/k_i}, \qquad 0 \leqslant h \leqslant m_i,$$

and l_i, m_i, k_i are integers ($m_i \geqslant 0$, $k_i = r_i' p_i$ for integer $r_i' \geqslant 1$). A general theory of such asymptotic series has been established by Trjitzinsky [11] following ideas inspired by the work of Birkhoff.

The other development has used Poincaré's formal Laplace integral representation $\int_0^\infty \exp(-zt) u_i(t)\, dt$ or $\oint \exp(zt) u_i(t)\, dt$ for σ_i to lead to expansions for σ_i in terms of convergent factorial series

$$a_0 + \sum_{m=1}^{\infty} \frac{a_m m!}{z(z+\gamma) \cdots (z + (m-1)\gamma)}.$$

We note the work of Horn ([6], [7]), Trjitzinsky [12], Evans [1] and in particular Turrittin [13] who gives an admirable rigorous treatment of this difficult problem.

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Hille [5] gives a clear summary and biblography (pages 198–209) and discusses also the problem in a more general Banach algebra setting (pages 250–272).

Either of the above developments leads to an expression for an arbitrary solution about an irregular singular point as a sum of solutions of known asymptotic behaviour. This general approach is not very convenient for the investigation of regular solutions. In practice, the determination of regular solutions (1) centres about the indicial equation. If ρ is a root of the indicial equation, a corresponding regular solution will have a branch point at the origin if ρ is non-integral and a pole at the origin if ρ is a negative integer. In the event that two roots of the indicial equation differ by an integer, logarithmic solutions also may arise. It may also happen (the "usual" case) that the power series $\Sigma f_n z^n$ has zero radius of convergence.

Let us restrict our attention to holomorphic solutions. In the event that the indicial equation has roots $\rho = 0, 1$ general theory provides as candidates two corresponding power series developments $w_1 = \Sigma_{n=0}^{\infty} f_{1n} z^n$, $w_2 = \Sigma_{n=1}^{\infty} f_{2n} z^n$. Usually we choose w_1 with $f_{11} = 0$.

In this paper we shall furnish a constructive example in which both these power series are divergent (have zero radius of convergence) but which nevertheless possesses a holomorphic solution in the vicinity of the origin in the form of a non-terminating power series in non-negative integral powers of z with positive radius of convergence. This is a linear combination of the two formal divergent power series solutions. This phenomenon does not appear to have been remarked previously in the literature. As the number of roots of the indicial equation must be less than the order of the differential equation for the origin to be an irregular singular point, such behaviour can arise only with differential equations of order at least three.

2. Basic construction

THEOREM 1. *Suppose that* $(\psi(n);\, n > 0)$, $(\zeta(n);\, n \geqslant 0)$ *are sequences of positive numbers satisfying*

(2) $$\psi(n) \to \infty \quad \text{as } n \to \infty \text{ with } \psi \text{ strictly monotone for } n \geqslant n_0$$

and

(3) $$\zeta(n)/\psi(n) < c \quad \text{for all } n \geqslant n_0$$

for some constant $c > 0$ *and integer* $n_0 > 0$. *Then to a scale factor there is a unique non-trivial sequene* $(f(n);\, n \geqslant 0)$ *satisfying the recurrence relations*

(4) $$f(n+2) = \psi(n+1) f(n+1) - \zeta(n) f(n), \qquad n \geqslant 0,$$

for which

$$(5) \qquad |f(n+1)/f(n)| \nrightarrow \infty \quad \text{as } n \to \infty.$$

This unique sequence has the property that the series $\sum_0^\infty f(n)z^n$ *is convergent for* $|z| < c^{-1}$ *and is manifestly the only non-trivial sequence for which the series has positive radius of convergence.*

PROOF. Suppose that $(f(n); n \geqslant 0)$ is any not identically vanishing sequence satisfying (4) and (5). For brevity we shall refer to such a sequence as *regular*. The assumption that such a sequence exists leads to a construction for $(f(n))$ which manifests the desired properties. We note *a priori* that regularity implies that it is never the case that two consecutive terms $f(j)$, $f(j+1)$ both vanish, for then a backward and forward recursion based on (4) and the non-vanishing of $(\psi(n))$ and $(\zeta(n))$ would yield that $f(n) \equiv 0$. In particular, a regular sequence cannot possess only a finite number of non-vanishing terms.

Let M be the set of all real-valued maps defined on the non-negative integers with

$$(6) \qquad g(n+1) > g(n) \geqslant 1 \quad \text{for } n \geqslant 0$$

and satisfying $g(n) \to \infty$ as $n \to \infty$ and

$$(7) \qquad g(n)/\psi(n) \to 0 \quad \text{as } n \to \infty.$$

For $g_0 \in M$, define

$$(8) \qquad M(g_0) = \{ g \in M | g(n) \leqslant g_0(n) \text{ for all } n \geqslant 0 \}.$$

Select $0 < d < c^{-1}$ and $g_0 \in M$. Then for $n \geqslant n_0$,

$$\psi(n+1) > cd\psi(n) + (1-cd)\psi(n+1),$$

so that there exists an integer $N(d, g_0) \geqslant n_0$ such that $n \geqslant N(d, g_0)$ implies by (3) and (7) that

$$\psi(n+1) > d\zeta(n) + g_0(n+1)/d$$

and hence by (6) and (8) that

$$(9) \qquad \psi(n+1) > d\zeta(n)/g(n) + g(n+1)/d$$

for all $g \in M(g_0)$.

If our regular sequence satisfies

$$(10) \qquad f(n_1+1) > f(n_1)g_1(n_1)/d > 0$$

for some integer $n_1 \geqslant N(d, g_0)$ and $g_1 \in M(g_0)$, then (4), (9) and (10) imply

$$\begin{aligned} f(n_1+2) &\geqslant [\psi(n_1+1) - d\zeta(n_1)/g_1(n_1)] f(n_1+1) \\ &> f(n_1+1)g_1(n_1+1)/d > 0. \end{aligned}$$

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Then by mathematical induction

$$f(n+1) \geqslant f(n)g_1(n)/d > 0 \quad \text{for all } n \geqslant n_1$$

and so $f(n+1)/f(n) \to \infty$ as $n \to \infty$, in contradiction to $(f(n))$ being regular. Thus for every $n_1 \geqslant N(d, g_0)$, relation (10) cannot occur for any $g_1 \in M(g_0)$. As g_1 is an arbitrary element of $M(g_0)$, it follows that for any $\varepsilon > c^{-1}$, there exists an integer $N(\varepsilon)$ such that $n \geqslant N(\varepsilon)$ implies $f(n+1) \geqslant \varepsilon f(n) > 0$ never occurs. A similar argument holds for the inequality $f(n+1) \leqslant \varepsilon f(n) < 0$. Since $\psi(n) \to \infty$ as $n \to \infty$, equation (4) implies that there exists an N such that $n \geqslant N$ entails that the possibilities $f(n+1)$, $f(n)$ are of opposite sign or $f(n) = 0$, $f(n+1) \neq 0$ never occur. Since no two consecutive terms $f(n)$, $f(n+1)$ can vanish, the terms $f(n)$ must ultimately be of fixed sign and without loss of generality we may take this sign as plus.

Fix $\varepsilon > c^{-1}$. Then for all n sufficiently large, $n \geqslant N_1 \geqslant N$, n_0 say, we may assume

$$0 < f(n+1) < \varepsilon f(n) \tag{11}$$

and

$$\varepsilon < \psi(N_1). \tag{12}$$

Then for $m \geqslant N_1$,

$$\varepsilon f(m+1) > f(m+2) = \psi(m+1)f(m+1) - \zeta(m)f(m) > 0,$$

whence from (11)

$$\zeta(m)f(m)/\psi(m+1) < f(m+1) < \zeta(m)f(m)/[\psi(m+1) - \varepsilon]. \tag{13}$$

This relation can be employed as a basis for a backward recursion using (4) to show that for $N_1 < j \leqslant m+1$, the value of $f(j)$ lies between

$$f(j-1)\left[\frac{\zeta(j-1)}{\psi(j)-}\,\frac{\zeta(j)}{\psi(j+1)-}\cdots\frac{\zeta(m)}{\psi(m+1)}\right] \tag{14}$$

and an expression (15) which is the same except that $\psi(m+1) - \varepsilon$ replaces $\psi(m)$. Here

$$\frac{x(1)}{y(1)-}\,\frac{x(2)}{y(2)-}\cdots\frac{x(r)}{y(r)}$$

is used to denote the continued fraction

$$-\left[\frac{-x(l)}{y(l)}\right]_1^r \equiv x(1)\left[y(1) - x(2)\left[y(2) - \cdots - x(r)/y(r)\right]^{-1}\right]^{-1}\cdots\Big]^{-1} \tag{16}$$

(see Khovanskii [10]).

It is readily verified that (2) and (3) guarantee that as $m \to \infty$ the continued fractions in (14) and (15) converge to a common limit

$$-\left[\frac{-\zeta(l-1)}{\psi(l)}\right]_j^\infty = \lim_{m\to\infty} -\left[\frac{-\zeta(l-1)}{\psi(l)}\right]_j^{m+1}$$

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and if $c' > c$ then

$$(17) \qquad 0 < -\left[\frac{-\zeta(l-1)}{\psi(l)}\right]_j^\infty < c' \quad \text{for all } j \text{ sufficiently large.}$$

We thus have

$$(18) \qquad f(j) = -f(j-1)\left[\frac{-\zeta(l-1)}{\psi(l)}\right]_j^\infty \quad \text{for } j > N_1,$$

and a further backward recursion extends this result to hold for $j > n_0$. If

$$(19) \qquad \psi(s) \neq -\left[\frac{-\zeta(r-1)}{\psi(r)}\right]_{s+1}^\infty, \qquad 1 \leqslant s \leqslant n_0,$$

the recurrence relations (4) may be solved to yield that (18) holds also for $1 \leqslant j \leqslant n_0$, that is,

$$(20) \qquad f(j) = -f(j-1)\left[\frac{-\zeta(r-1)}{\psi(r)}\right]_j^\infty, \qquad j > 0.$$

More generally, suppose that

$$(21) \qquad \psi(s_i) = -\left[\frac{-\zeta(r-1)}{\psi(r)}\right]_{s_i+1}^\infty$$

for one or more values $1 \leqslant s_1 < s_2 < \cdots < s_t \leqslant n_0$. In this event the recursion argument leads to (20) for $j > s_t$ with $f(s_t - 1) = 0$. Further,

$$f(s_t) = -\zeta(s_t - 2)f(s_t - 2)$$

and

$$f(s_t - 2) = \zeta(s_t - 3)f(s_t - 3)/\psi(s_t - 2) = -\left[\frac{-\zeta(r-1)}{\psi(r)}\right]_{s_t-3}^\infty f(s_t - 3),$$

by virtue of (21). An elementary induction leads to the recurrence relations

$$\begin{aligned} &f(s_i - 1) = 0, && 1 \leqslant i \leqslant t,\\ &f(s_i) = -\zeta(s_i - 2)f(s_i - 2), && 1 \leqslant i \leqslant t,\\ &f(j) = -f(j-1)\left[\frac{-\zeta(r-1)}{\psi(r)}\right]_j^\infty, && j \neq s_i, s_i - 1;\ 1 \leqslant i \leqslant t. \end{aligned}$$

It follows that $(f(n))$ is uniquely determined to a scale factor.

Thus $(f(n))$ is uniquely determined to a scale factor whether or not (19) holds and it is trivial that the sequence so constructed does in fact satisfy (4). Inequality (17) ensures that $\Sigma f(n)z^n$ is convergent for $|z| < c^{-1}$ and the theorem is proved.

COROLLARY. *Under the conditions of Theorem* 1, *the unique sequence concerned is prescribed by*

$$(22)\qquad \begin{aligned} &f(n) = 0 && \textit{if}\ \psi(n+1) = -\left[\frac{-\zeta(r-1)}{\psi(r)}\right]_{n+2}^{\infty}, \\ &f(n) = B\prod_{j=1}^{n} A_j && \textit{otherwise}, \end{aligned}$$

where

$$(23)\qquad A_j = \begin{cases} 1 & \textit{if}\ \psi(j) = -\left[\dfrac{\zeta(r-1)}{\psi(r)}\right]_{j+1}^{\infty}, \\ -\zeta(j-1) & \textit{if}\ \psi(j+1) = -\left[\dfrac{-\zeta(r-1)}{\psi(r)}\right]_{j+2}^{\infty}, \\ -\left[\dfrac{-\zeta(r-1)}{\psi(r)}\right]_{j}^{\infty} & \textit{otherwise}. \end{cases}$$

3. Minimal solutions

A three-term recurrence solution

$$(24)\qquad y_{n+1} = b_n y_n + a_n y_{n-1}, \qquad a_n \neq 0, n \geqslant 1,$$

is said to have a *minimal solution* (h_n) if (h_n) is non-trivial and if for some solution (k_n) we have the relation

$$h_n/k_n \to 0 \quad \text{as } n \to \infty$$

(cf. Jones and Thron [9], page 163).

It is readily seen that if (y_n) is a solution not proportional to (h_n) then

$$(25)\qquad h_n/y_n \to 0 \quad \text{as } n \to \infty.$$

A minimal solution, if it exists, is unique up to a scale factor. As (25) indicates, a minimal solution may be regarded as a difference equation analogue of a principal solution for a linear second order differential equation (cf. Hartman [4], page 355, Theorem 6.4).

By a theorem of Pincherle (see [9], page 403, Theorem B.4) the recurrence relations (24) have a minimal solution if and only if $[a_r/b_r]_1^{\infty}$ is convergent, and in the event that this occurs $h_m/h_{m-1} = -[a_r/b_r]_m^{\infty}$. It is evident that the regular solution constructed in the previous section is a minimal solution for the recurrence relation (4).

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4. Main Theorem

THEOREM 2. *There exists a linear ordinary differential equation $L(w) = 0$ with an irregular singular point at the origin and possessing the following properties:*

(i) *corresponding to roots $\rho = 0, 1$ of the indicial equation there are formal power series solutions*

$$w_1 = \sum_{n=0}^{\infty} f_{1,n} z^n \text{ with } f_{1,1} = 0, \quad w_2 = \sum_{n=1}^{\infty} f_{2,n} z^n,$$

both of which have zero radius of convergence;

(ii) *there is a holomorphic solution about the origin in the form of a non-terminating power series with a positive radius of convergence.*

PROOF. Consider the equation

$$\begin{aligned} z^2(1 - \phi z)w'''(z) - (a_0 - a_1 z + a_2 z^2)w''(z) \\ + (b_0 - b_1 z)w'(z) - c_0 w(z) = 0, \end{aligned} \tag{26}$$

where the a_i, b_j and c_0 are all positive and $\phi \geqslant 0$. It is immediate that the origin is an irregular singular point with class (characteristic index) unity and that the indicial equation has only the roots $\rho = 0, 1$.

Formal series substitution of $\Sigma f_n z^n$ for w in (26) furnishes the recurrence relations

$$\begin{aligned} -f_{n+2} a_0 (n+2)(n+1) + f_{n+1}(n+1)[n(n-1) + a_1 n + b_0] \\ -f_n[\phi n(n-1)(n-2) + a_2 n(n-1) + b_1 n + c_0] = 0, \qquad n \geqslant 0, \end{aligned}$$

which is of the form (4) with

$$\psi(n) = [(n-1)(n-2) + a_1(n-1) + b_0]/[a_0(n+1)],$$
$$\zeta(n) = [\phi n(n-1)(n-2) + a_2 n(n-1) + b_1 n + c_0]/[a_0(n+2)(n+1)].$$

Because of the stated positivity of the coefficients in equation (26), the sequences $(\psi(n))$, $(\zeta(n))$ can be seen trivially to be eligible candidates to act as the forcing sequences in Theorem 1 for any value of c satisfying $c > \phi$ and the theorem follows at once.

We remark that the results of Theorem 1 give convergence of $\Sigma f_n z^n$ for $|z| < \phi^{-1}$. Inspection of (26) shows that this result is best possible in general.

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