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THE DUAL OF FROBENIUS' RECIPROCITY THEOREM

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1. Introduction. In two preceding papers [**2**; **3**] the author has studied the algebras of the irreducible representations λ and the classes C_i of a finite group G. Integral representations $\{\lambda\}$ and $\{C_i\}$ of these algebras are derivable from the appropriate multiplication tables [**4**]. It should be emphasized, however, that the symmetry properties of the two sets of structure constants are not the same, and this leads to somewhat greater complexity in the formulae relating to classes as compared to representations. Nevertheless, it does seem possible to dualize Frobenius' Reciprocity Theorem in § 5 below. In § 6 we consider the important special case in which the subgroup $\hat{G} \simeq G/H$, utilizing the fact that no class of G splits in $\hat{G}$ so that φ (or F or Ω) has a *left inverse*. This makes it possible to invert the restricting and inducing processes as will be further explained in a subsequent paper.

What ties the representations of the two algebras together is the use of the character table as a matrix, leading to the definitions:

$$\{\lambda\} = XD^{\lambda'}X^{-1} = (g_{\beta\gamma'}{}^{\lambda}),$$
$$\{C_i\} = X^{-1}D_{i'}X = (c_{is'}{}^{r}).$$

Here X is the character matrix over the complex field, D^{λ} a matrix with the values of χ^{λ} over the classes C_i of G in the diagonal with zeros elsewhere; similarly D_i is a matrix with values of the class multiplier γ_i over the representations λ of G in the diagonal with zeros elsewhere. The idempotents of the dual algebras are particularly important:

$$\{T^{\lambda}\} = \frac{f^{\lambda}}{g} \sum_i \{c_i\}\chi_i^{\lambda}, \qquad \{S_i\} = \frac{g_i}{g} \sum_{\lambda} \chi_i^{\lambda}\{\lambda\},$$

of which the first is well-known [**4**]. Since the representations of the two algebras are of the same dimension we may identify the idempotents [**2**]

$$X^{-1}\{S_i\}X = X\{T^{\lambda}\}X^{-1},$$

and this is the basis of what follows.

The reader should be warned that associated with the representation spaces described above are operators of two types: (a) intertwining operators which effect the restricting and inducing processes, and (b) operators on $\{\lambda\}$ and $\{C_i\}$ considered as vectors. To distinguish the latter they will be enclosed in brackets.

When writing paper [**3**] the author was unaware of the work of Gamba [**5**]

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who describes his approach to the duality problem in terms of the important special case mentioned above. In conclusion the author would express his thanks to R. C. King for much stimulating discussion and for several suggestions which simplified the presentation of these ideas.

2. Technical preliminaries. Since the character table of G will be used as a matrix X we begin by relating it to the table of class multipliers Γ. We assume the irreducible representations α, β, ... of G are associated with the rows of X and Γ, while the classes C_1, C_2, ... of G are associated with the columns. It is usual to let α be the identity representation and C_1 the identity class of G, but further explicit associations are not in general possible. We write $\chi_1^\lambda = f^\lambda$ so that

$$X = \begin{bmatrix} 1 & 1 & \dots \\ f^\beta & \chi_2^\beta & \dots \\ \dots & & \end{bmatrix}, \qquad X^{-1} = \frac{1}{g}\begin{bmatrix} 1 & f^{\beta'} & \dots \\ g_2 & g_2\chi_2^{\beta'} & \dots \\ \dots & & \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} 1 & g_2 & \dots \\ 1 & \frac{g_2\chi_2^\beta}{f^\beta} & \dots \\ \dots & & \end{bmatrix}, \qquad \Gamma^{-1} = \frac{1}{g}\begin{bmatrix} 1 & f^\beta f^{\beta'} & \dots \\ 1 & f^\beta\chi_2^{\beta'} & \dots \\ \dots & & \end{bmatrix}.$$

Thus

$$\Gamma' = D(g_i)X'D(1/f^\lambda) = X_c^{-1}D(g/f^\lambda), \tag{2.1}$$

where we denote the conjugate of X by X_c and the transpose of X by X'.

If now we write $\{\lambda\}$ and $\{S\}$ as *column* vectors with components $\{\alpha\}$, $\{\beta\}$, ... and $\{S_1\}$, $\{S_2\}$, ..., and $\{C\}$ and $\{T\}$ as *row* vectors with components $\{C_1\}$, $\{C_2\}$, ... and $\{T^\alpha\}$, $\{T^\beta\}$, ... we have the important

2.2 THEOREM. (i) $\{S\} = (X^{-1}\{\lambda\})$ *and* (ii) $\{T\} = (\{C\}\Gamma^{-1})$.

Proof. It is only necessary to recall the definitions of the idempotents given in the Introduction, replacing S, λ, T, C by their representative matrices.

As we have seen elsewhere [**2**; **3**],

$$X^{-1}\{S_i\}X = \frac{g_i}{g}\sum_\lambda \chi_i^\lambda \cdot X^{-1}\{\lambda\}X = \frac{g_i}{g}\sum_\lambda \chi_i^\lambda D^{\lambda'} \tag{2.3}$$

$$X\{T^\lambda\}X^{-1} = \frac{f^\lambda}{g}\sum_i X\{C_i\}X^{-1}\cdot\chi_i^\lambda = \frac{f^\lambda}{g}\sum_i D_{i'}\chi_i^\lambda \tag{2.4}$$

are idempotents of the dual representation algebras with 1 in the $i(\lambda)$ position of the principle diagonal and zeros elsewhere.

Since it is desirable to illustrate these ideas in some detail we introduce a simple

2.5 *Example.* We set $G = S_3$ so that:

$$X = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \quad X^{-1} = \tfrac{1}{6}\begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & -3 \\ 2 & -2 & 2 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 0 & -1 \\ 1 & -3 & 2 \end{bmatrix},$$

$$\Gamma^{-1} = \tfrac{1}{6}\begin{bmatrix} 1 & 4 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\{[3]\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \{[2,1]\} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \{[1^3]\} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

with

$$\{S_{1^3}\} = \tfrac{1}{6}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}, \{S_{2,1}\} = \tfrac{3}{6}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \{S_3\} = \tfrac{2}{6}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

where

$$\{S_{1^3}\} + \{S_{2,1}\} + \{S_3\} = I_3$$

and

$$X^{-1}\{S_{1^3}\}X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, X^{-1}\{S_{2,1}\}X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, X^{-1}\{S_3\}X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Again

$$\{C_{1^3}\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \{C_{2,1}\} = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix}, \quad \{C_3\} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

with

$$\{T^{[3]}\} = \tfrac{1}{6}\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \end{bmatrix}, \quad \{T^{[2,1]}\} = \tfrac{2}{6}\begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix},$$

$$\{T^{[1^3]}\} = \tfrac{1}{6}\begin{bmatrix} 1 & -1 & 1 \\ -3 & 3 & -3 \\ 2 & -2 & 2 \end{bmatrix}$$

where

$$\{T^{[3]}\} + \{T^{[2,1]}\} + \{T^{[1^3]}\} = I_3,$$

and

$$X\{T^{[3]}\}X^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X\{T^{[2,1]}\}X^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$X\{T^{[1^3]}\}X^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

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3. Frobenius' Reciprocity Theorem. If $\hat{G}$ is a subgroup of G the process of *restricting* ($\downarrow$) and *inducing* ($\uparrow$) can be described [3] relative to the irreducible representations by means of a matrix F, so that the Theorem takes the form:

$$\{\lambda\} \downarrow (F\{\hat{\lambda}\}), \quad \{\hat{\lambda}\} \uparrow (F'\{\lambda\}), \tag{3.1a}$$

or using 2.2 in the form

$$\{S\} \downarrow (X^{-1}F\hat{X}\{\hat{S}\}) = (\varphi\{\hat{S}\}), \quad \{\hat{S}\} \uparrow (\hat{X}^{-1}F'X\{S\}) = (\varphi_1\{S\}). \tag{3.1b}$$

The matrices φ, φ_1 are vital in what follows. Like F, F' they are operators on the vectors $\{\hat{S}\}$ and $\{S\}$. Before proving Theorem 3.3 below we illustrate these ideas by taking $G = S_3$, $\hat{G} = A_3$ in

3.2 *Example.* If we denote the irreducible representations of A_3 by α, β, γ, then

$$\hat{X} = \begin{array}{c} \\ \alpha \\ \beta \\ \gamma \end{array} \!\!\begin{array}{c} I \quad (123) \quad (132) \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & \rho & \rho^2 \\ 1 & \rho^2 & \rho \end{bmatrix} \end{array} = \hat{\Gamma}, \qquad \hat{X}^{-1} = \tfrac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \rho^2 & \rho \\ 1 & \rho & \rho^2 \end{bmatrix} = \hat{\Gamma}^{-1}$$

with

$$\{S_I\} = \tfrac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \{S_{(123)}\} = \tfrac{1}{3}\begin{bmatrix} 1 & \rho & \rho^2 \\ \rho^2 & 1 & \rho \\ \rho & \rho^2 & 1 \end{bmatrix}, \quad \{S_{(132)}\} = \tfrac{1}{3}\begin{bmatrix} 1 & \rho^2 & \rho \\ \rho & 1 & \rho^2 \\ \rho^2 & \rho & 1 \end{bmatrix}$$

where

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \varphi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \varphi_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The importance of the matrix φ lies in the fact that it describes the splitting of the classes of G relative to $\hat{G}$. The relation of φ to φ_1 is given in

3.3 THEOREM. (i) *The matrix* $\varphi_1 = \hat{X}^{-1}F'X$ *has the same form as* $\varphi' = (X^{-1}F\hat{X})'$ *except that the* 1'*s in* φ *are replaced by integers associated with the classes of* $\hat{G}$. (ii) $\varphi\varphi_1$ *is a matrix with the permutation character of G induced by the identity representation of* $\hat{G}$ *in the diagonal with zeros elsewhere:*

$$\varphi\varphi_1 = D(gg_s{}^{\hat{G}}/\hat{g}g_s)$$

Proof. The proof of (i) is left to the reader. To prove (ii) we write an element of φ_1 in the form

$$\sum_\mu \frac{\hat{g}_{\hat{R}}}{\hat{g}}\, \hat{\chi}^{\mu'}(\hat{R}) \cdot \hat{\chi}^{\mu}(S)\, \frac{gg_S{}^{\hat{G}}}{gg_S} = \begin{cases} \dfrac{gg_{\hat{R}}{}^{\hat{G}}}{\hat{g}g_{\hat{R}}}, & \text{if } S \sim \hat{R} \\ 0, & \text{if } S \nsim \hat{R}. \end{cases} \tag{3.4}$$

Note that $F'X$ yields the representation induced by μ of $\hat{G}$. $S \sim \hat{R}$ implies that S and $\hat{R}$ belong to the same class of $\hat{G}$. Multiplying by φ adds these integers over the classes $\hat{R}$ of $\hat{G}$ which are conjugate to S in G, so yielding the permutation character as claimed.

We can look at 3.3 (ii) from another point of view writing

$$\varphi\hat{I}\varphi_1 = D(gg_i{}^{\hat{G}}/\hat{g}g_i). \tag{3.5}$$

If we consider $\hat{I}$ to be a sum of idempotents $\hat{X}^{-1}\{\hat{S}_i\}\hat{X}$, we may write

$$\varphi \cdot \hat{X}^{-1}\{\hat{S}\}\hat{X} \cdot \varphi_1 = (\varphi_1 X^{-1}\{S\}X) \tag{3.6}$$

which separates the contributions of the classes of G in 3.5. Transforming by X,

$$F\{\hat{S}\}F' = (\varphi_1\{S\}). \tag{3.7}$$

More generally, we may multiply each idempotent in 3.5 by the appropriate value of $\chi_i{}^{\hat{\lambda}}$ to yield

$$\varphi D^{\hat{\lambda}}\varphi_1 = D(gg_i{}^{\hat{G}}\chi_i{}^{\hat{\lambda}}/\hat{g}g_i) = (F'D^{\lambda}). \tag{3.8}$$

We illustrate this relation with reference to Example 3.2 above in

3.9 *Example.*

$$\begin{aligned} \varphi D^{\alpha}\varphi_1 &= (F'D^{\lambda}) = D^{[3]} + D^{[1^3]} \\ \varphi D^{\beta}\varphi_1 &= D^{[2,1]} = \varphi D^{\gamma}\varphi_1 \end{aligned}$$

as may easily be verified.

4. Representation algebra. In the preceding section we have derived certain analogues of the inducing part of the Reciprocity Theorem from 3.3. In order to obtain the corresponding formulae for restricting we have to resort again to the idempotents $X^{-1}\{S_i\}X$, using φ as an intertwining operator as well as an operator on the vector space thus:

$$X^{-1}\{S\}X \cdot \varphi = \varphi \cdot (\varphi\hat{X}^{-1}\{\hat{S}\}\hat{X}). \tag{4.1}$$

The significance of this relation is brought out by the

4.2 *Example.* From 3.2,

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\varphi = \varphi\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \quad \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\varphi = \begin{bmatrix}0&0&0\\0&0&0\\0&1&1\end{bmatrix} = \varphi\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

If now we transform 4.1 by X we have

$$\{S\}F = F(\varphi\{\hat{S}\}). \tag{4.3}$$

Substituting from 2.2(i)

$$(X^{-1}\{\lambda\})F = F(\varphi\hat{X}^{-1}\{\hat{\lambda}\}) \tag{4.4}$$

or

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$$(4.5)\qquad \{\lambda\}F = F(F\{\hat{\lambda}\})$$

which corresponds to the restricting part of 3.1a. Note that the φ which survives in 4.4 is an operator on the vector space which is changed to F by the application of 2.2(i). We derive the analogue of the inducing part of 3.1a by transforming 3.8 by X, which, along with 4.5 yields [**3**]

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$$\{\lambda\}F = F(F\{\hat{\lambda}\}), \quad F\{\hat{\lambda}\}F' = (F'\{\lambda\}).$$

4.7 *Example.* One easily verifies that

$$\{\alpha\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \{\beta\} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \{\gamma\} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

from 3.2 so that from 2.7 we have

$$\{[3]\}F = F\{\alpha\}, \quad \{[2,1]\}F = F(\{\beta\} + \{\gamma\}), \quad \{[1^3]\}F = F\{\alpha\}$$

and

$$F\{\alpha\}F' = \{[3]\} + \{[1^3]\}, \quad F\{\beta\}F' = \{[2,1]\} = F\{\gamma\}\mathrm{F}',$$

as in 4.6.

5. Class algebra. We turn now to the algebra of classes and enquire to what extent the Frobenius Theorem can be dualized. Here we identify the idempotents first in 3.1b which becomes

$$(5.1\mathrm{a})\qquad \{T\} \downarrow (\{\hat{T}\}\varphi'), \quad \{\hat{T}\} \uparrow (\{T\}\varphi_1')$$

since by convention $\{T\}$ is a row vector. Again using 2.2 this becomes

$$(5.1\mathrm{b})\qquad \{C\} \downarrow (\{\hat{C}\}\Omega'), \quad \{\hat{C}\} \uparrow (\{C\}\Omega_1')$$

where $\Omega' = \hat{\Gamma}^{-1}\varphi'\Gamma$ and $\Omega_1' = \Gamma^{-1}\varphi_1'\hat{\Gamma}$. Analogously, we may identify the idempotents in 4.1 and 3.6 to yield

$$(5.2\mathrm{a})\qquad \{T\}\psi = \psi(\{\hat{T}\}\varphi'), \quad \psi\{\hat{T}\}\psi_1 = (\{T\}\varphi_1')$$

and again using 2.2(ii) we have

$$(5.2\mathrm{b})\qquad \{C\}\psi = \psi(\{\hat{C}\}\Omega'), \quad \psi\{\hat{C}\}\psi_1 = (\{C\}\Omega_1')$$

where $\psi = X^{-1}\varphi\hat{X}$ and $\psi_1 = \hat{X}^{-1}\varphi_1 X$.

Consider now the matrix $\Gamma' X_c$ which, in view of 2.1, we can write in the form

$$(5.3)\qquad U = \Gamma' X_c = X_c^{-1} D(g/f^\lambda) X_c = \sum_i a_i\{C_{i'}\}$$

Proof. If we set $a_i = \sum_\lambda \chi_i^\lambda$ then

$$g/f^\lambda = \sum_i a_i \gamma_{i'}^\lambda$$

for every λ, so $U = \sum_i a_i\{C_{i'}\}$ as claimed.

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We conclude from 5.3 that $U\{C_i\} = \{C_i\}U$ for all i, so that

$$\Omega = \Gamma'\varphi\hat{\Gamma}'^{-1} = \Gamma' X_c \cdot X_c^{-1}\varphi\hat{X}_c \cdot \hat{X}_c^{-1}\hat{\Gamma}'^{-1} = U\psi\hat{U}^{-1},$$
$$\Omega_1 = \hat{\Gamma}'\varphi_1\Gamma'^{-1} = \hat{\Gamma}'\hat{X}_c \cdot \hat{X}_c^{-1}\varphi_1 X_c \cdot X_c^{-1}\Gamma'^{-1} = \hat{U}\psi_1 U^{-1}.$$

It follows that we can write 5.2b as a

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5.5 *Example*. As before, $G = S_3$ with $\hat{G} = A_3$ so that

$$\{C_{2,1}\}\Omega = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 2 \\ 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 2 \\ 9 & 0 & 0 \\ -2 & 4 & 4 \end{bmatrix}$$
$$= \Omega(-\{I\} + 2\{(123)\} + 2\{(132)\})$$

and

$$\Omega\{(123)\}\Omega_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 2 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\frac{1}{6}\begin{bmatrix} 4 & 0 & 4 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 3 & 1 \\ 9 & 3 & 9 \\ 2 & 6 & 2 \end{bmatrix}$$
$$= (\{C\}\Omega_1') = \tfrac{1}{6}(\{C_{1^3}\} + 3\{C_{2,1}\} + \{C_3\})$$
$$= \Omega\{(132\})\Omega_1.$$

It is interesting to apply these ideas to relate the 'whole group' G to the subgroup $\hat{G}$. In terms of representations, this implies relating the regular representation of G to that of $\hat{G}$. If we denote the regular representation of G by $\{G\}_r$, it is not difficult to show that

$$\{G\}_r F = \frac{g}{\hat{g}} F\{\hat{G}\}_r, \qquad F\{\hat{G}\}_r F' = \{G\}_r. \tag{5.6}$$

In terms of classes, it is their sum over G which is in question. Denoting this by $\{G\}_c$ we have

$$\{G\}_c\Omega = \frac{g}{\hat{g}}\Omega\{\hat{G}\}_c, \qquad \Omega\{\hat{G}\}_c\Omega_1 = \{G\}_c, \tag{5.7}$$

utilizing the fact that the columns of Ω sum to $g/\hat{g}$ and those of Ω, to g. We illustrate the situation for classes with reference to Example 5.5:

$$\{C_{1^3} + C_{2,1} + C_3\}\Omega = 2\Omega\{I + (123) + (132)\} \tag{5.8}$$

and

$$\Omega\{I + (123) + (132)\}\Omega_1 = \{C_{1^3} + C_{2,1} + C_3\}, \tag{5.9}$$

as may easily be verified.

The author has tried to give an abstract description of Theorem 5.4 but with little success. We examine an important special case in the following section.

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6. A special case. The example which we have studied in such detail to illustrate our argument does not have the special feature $\hat{G} \simeq G/H$ which is of interest. In order to illustrate this case we take $G = S_4$ with $\hat{G} = S_3$,

$$H = I,\ (12)(34),\ (13)(24),\ (14)(23)$$

and

$$\begin{array}{r} \\ [4] \\ [3,1] \\ [2^2] \\ [2,1^2] \\ [1^4] \\ g_i \end{array}\begin{array}{ccccc} (1^4) & (2^2) & (2,1^2) & (4) & (3,1) \\ \hline \end{array}$$

$$\begin{array}{c|ccccc|} & (1^4) & (2^2) & (2,1^2) & (4) & (3,1) \\ \hline [4] & 1 & 1 & 1 & 1 & 1 \\ [3,1] & 3 & -1 & 1 & -1 & 0 \\ [2^2] & 2 & 2 & 0 & 0 & -1 \\ [2,1^2] & 3 & -1 & -1 & 1 & 0 \\ [1^4] & 1 & 1 & -1 & -1 & 1 \\ \hline g_i & 1 & 3 & 6 & 6 & 8 \end{array} : X, \quad \begin{bmatrix} 1 & 3 & 6 & 6 & 8 \\ 1 & -1 & 2 & -2 & 0 \\ 1 & 3 & 0 & 0 & -4 \\ 1 & -1 & -2 & 2 & 0 \\ 1 & 3 & -6 & -6 & 8 \end{bmatrix} : \Gamma,$$

so that

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \varphi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \psi = \tfrac{1}{4}\begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \tfrac{1}{4}\Omega,$$

$$\varphi_1 = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Omega_1 = \tfrac{1}{24}\begin{bmatrix} 13 & 13 & 3 & 3 & 1 \\ 9 & 9 & 15 & 15 & 9 \\ 2 & 2 & 6 & 6 & 14 \end{bmatrix}.$$

Thus

$$\{C_{1^4}\}\Omega = \Omega\{C_{1^3}\}, \quad \{C_{2^2}\}\Omega = 3\Omega\{C_{1^3}\}, \quad \{C_{2,1^2}\}\Omega = 2\Omega\{C_{2,1}\},$$
$$\{C_4\}\Omega = 2\Omega\{C_{2,1}\}, \quad \{C_{3,1}\}\Omega = 4\Omega\{C_3\}.$$

To explain the significance of these restricting relations we write the coset splitting of $G = S_4$ relative to H:

$$\begin{array}{lcccccc} S_4 = & H & +\ (12)H & +\ (13)H & +\ (23)H & +\ (123)H & +\ (132)H \\ & I & (12) & (13) & (23) & (123) & (132) \\ & (12)(34) & (34) & (1234) & (1342) & (134) & (234) \\ & (13)(24) & (1324) & (24) & (1243) & (243) & (124) \\ & (14)(23) & (1423) & (1432) & (14) & (142) & (143). \end{array}$$

Thus 5.4 describes the distribution of the classes in the cosets with the appropriate relative multiplicities,—*for the restricting process only*,—since the form of Ω_1 would clearly lead to quite different results.

In general when classes of G split in $\hat{G}$ the matrix φ has no left inverse. In this case, however, such an inverse exists and $\varphi'\varphi = I_3$ so that

$$\psi_0 = \hat{X}^{-1}\varphi' X = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = 4\Omega_0$$

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and $\psi_0\psi = \Omega_0\Omega = I_3$. Hence we have immediately from the restricting equation of 5.4

$$\Omega_0\{C\}\Omega = (\{\hat{C}\}\Omega') \tag{6.1}$$

in conformity with 5.1b. On the other hand, operating on the vector space with Ω_0' we have

$$(\{C\}\Omega_0')\Omega = \Omega(\{\hat{C}\}\Omega'\Omega_0') = \Omega\{\hat{C}\} \tag{6.2}$$

so that

$$\Omega\{\hat{C}\}\Omega_0 = (\{C\}\Omega_0')\Omega\Omega_0 = (\{C\}\Omega_0'), \tag{6.3}$$

since $\Omega\Omega_0 = \frac{1}{4}\{H\}$ and $\{\hat{C}H\}\frac{1}{4}\{H\} = \{\hat{C}H\}$. We have in particular:

$$4\Omega\{\hat{C}_{1^3}\}\Omega_0 = \{C_{1^4}\} + \{C_{2^2}\},$$
$$4\Omega\{\hat{C}_{2,1}\}\Omega_0 = \{C_{2,1^2}\} + \{C_4\},$$
$$4\Omega\{\hat{C}_3\}\Omega_0 = \{C_{3,1}\}.$$

Thus Ω_0 replaces Ω_1 in 5.1b and we can state another

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$$\Omega_0\{C\}\Omega = (\{\hat{C}\}\Omega'), \quad \Omega\{\hat{C}\}\Omega_0 = (\{C\}\Omega_0')$$

describe the coset structure of G relative to H.

We are utilizing here the *inverse* of the restricting process rather than inducing, since $\Omega_0\Omega\{\hat{C}\}\Omega_0\Omega = \{\hat{C}\}$.

REFERENCES

1. G. de. B. Robinson, *Group representations and geometry*, J. Mathematical Phys. *11* (1970), 3428–32.
2. ——— *The algebras of representations and classes of finite groups*, J. Mathematical Phys. *12* (1971), 2212–2215.
3. ——— *Tensor product representations*, J. Algebra *20* (1972), 118–123.
4. W. Burnside, *The theory of groups* (Cambridge Univ. Press, Cambridge, 1910).
5. A. Gamba, *Representations and classes in groups of finite order*, J. Mathematical Phys. *9* (1968), 186–192.

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