

TENSOR PRODUCTS OF POSITIVE DEFINITE QUADRATIC FORMS, V

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Our aim is to prove

THEOREM. *Let L be a positive lattice of E -type such that $[L: \tilde{L}] < \infty$ and \tilde{L} is indecomposable.*

(i) *If $L \cong L_1 \otimes L_2$ for positive lattices L_1, L_2 , then L_1, L_2 are of E -type and $[L_1: \tilde{L}_1], [L_2: \tilde{L}_2] < \infty$ and \tilde{L}_1, \tilde{L}_2 are indecomposable.*

(ii) *If L is indecomposable with respect to tensor product, then for each indecomposable positive lattice X we have*

(1) *$L \otimes X \cong L \otimes Y$ implies $X \cong Y$ for a positive lattice Y ,*

(2) *If $X = \otimes' L \otimes X'$ where X' is not divided by L , then $O(L \otimes X)$ is generated by $O(L), O(X')$ and interchanges of L 's, and*

(3) *$L \otimes X$ is indecomposable.*

We must explain notations and terminology. By a positive lattice we mean a lattice on positive definite quadratic space over the rational number field \mathbb{Q} . Let L be a positive lattice and put

$$m(L) = \min_{\substack{x \in L \\ x \neq 0}} Q(x)$$

where $Q(\)$ is a quadratic form associated with L . Set $\mathfrak{M}(L) = \{x \in L \mid Q(x) = m(L)\}$. If $\mathfrak{M}(L \otimes M) = \mathfrak{M}(L) \otimes \mathfrak{M}(M)$ ($= \{x \otimes y \mid x \in \mathfrak{M}(L), y \in \mathfrak{M}(M)\}$) for any positive lattice M , then L is called of E -type. \tilde{L} is a submodule of L spanned by $\mathfrak{M}(L)$. If $L \cong L_1 \otimes L_2$ implies $\text{rank } L_1$ or $\text{rank } L_2 = 1$, then we say that L is indecomposable with respect to tensor product. $O(L)$ denotes the orthogonal group of L . If a positive lattice X is isometric to $L \otimes K$ for a positive lattice K , then X is, by definition, divided by L . These notations and terminology will be used through this paper.

In § 1 we prove a theorem about weighted graphs. In § 2 we improve

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a result in [4] and in § 3 the above theorem is proved. In § 4 examples of a lattice L as in the theorem are given.

§ 1.

In this section we define a weighted graph and prove a fundamental theorem in this paper.

DEFINITION. Let A be a finite set and $[,]$ be a mapping from $A \times A$ into $\{t \mid 0 \leq t \leq 1\}$ such that

- (i) $[a, a'] = 1$ if and only if $a = a'$, and
- (ii) $[a, a'] = [a', a]$ for $a, a' \in A$.

Then we call $(A, [,])$ or simply A a weighted graph.

Let A be a weighted graph. A is called connected unless there exist subsets A_1, A_2 of A such that $A = A_1 \cup A_2, A_1 \cap A_2 = \phi$ and $[a_i, a_2] = 0$ for any $a_i \in A_i (i = 1, 2)$. If $A = \cup A_i$ (disjoint) satisfies

- (i) A_i is connected, and
- (ii) $[a, b] = 0$ if $a \in A_i, b \in A_j$ and $i \neq j$,

then each A_i is called a connected component of A . Let A, B be weighted graphs. For $(a, b), (a', b') \in A \times B$ we define $[(a, b), (a', b')]$ by $[a, a'] \cdot [b, b']$. Then $A \times B$ becomes a weighted graph. If there exists a bijection σ from A on B such that $[a, a'] = [\sigma(a), \sigma(a')]$ ($a, a' \in A$), then we say that A, B are isometric and write $\sigma: A \cong B$.

LEMMA 1. Let A, B, C be weighted graphs and assume that $A = \{e_i\}_{i=1}^n$ and $\sigma: A \times B \cong A \times C$. Take any element $b \in B$ and fix it. Define $f_i \in A, c_i \in C, g_{ij} \in A, b_{ij} \in B$ by

$$\sigma(e_i, b) = (f_i, c_i) \text{ and } \sigma(g_{ij}, b_{ij}) = (f_i, c_j).$$

Then we have $[e_i, e_j] = 0$ if $b_{ij} \neq b$.

Proof. Set $a_{ij} = [e_i, e_j]$. Then $a_{ij} = a_{ji}$ and

$$(0) \quad a_{ij} = [f_i, f_j][c_i, c_j].$$

Fix any integer $k (1 \leq k \leq n)$ and define $e'_s \in A, b_s \in B$ by $\sigma(e'_s, b_s) = (f_k, c_s) (1 \leq s \leq n)$. Put $S = \{s \mid b_s \neq b\}$. If $S \neq \phi$, then we take integers u, m such that

$$a_{um} = \max_{\substack{i: f_i = f_k \\ s \in S}} a_{is} \text{ and } f_u = f_k, m \in S.$$

If $a_{um} = 0$ can be shown, then the lemma will be proved. Assume $a_{um} \neq 0$ and put $b_m = b', e'_m = e_p$. Then we have

$$\sigma(e_p, b') = (f_k, c_m), \quad b' \neq b.$$

Since $\sigma(e_i, b) = (f_i, c_i)$, we have

- (1) $a_{ip}[b, b'] = [f_i, f_k][c_i, c_m],$
- (2) $a_{ip}[b, b'] = [c_i, c_m] \quad \text{if } f_i = f_k,$
- (3) $a_{mp}[b, b'] = [f_m, f_k].$

Hence $f_u = f_k$ implies

$$\begin{aligned} a_{um} &= [f_u, f_m][c_u, c_m] && \text{by (0)} \\ &= [f_k, f_m][c_u, c_m] \\ &= a_{mp}a_{up}[b, b']^2 && \text{by (2), (3)}. \end{aligned}$$

(4) $a_{um} = a_{mp}a_{up}[b, b']^2.$

Suppose $f_p = f_k$. Then $\sigma(e_p, b') = (f_k, c_m) = (f_p, c_m)$ implies $a_{um} \geq a_{pm}$. Hence we have

$$\begin{aligned} 0 < a_{um} &= a_{mp}a_{up}[b, b']^2 && \text{by (4)} \\ &\leq a_{um}a_{up}[b, b']^2 \\ &\leq a_{um}. \end{aligned}$$

This yields $a_{up}[b, b']^2 = 1$ and $[b, b'] = 1$. This contradicts $b \neq b'$. Therefore we get $f_p \neq f_k$. Suppose $p \in S$; then $a_{up} \leq a_{um}$ holds by definition. Hence we have

$$\begin{aligned} 0 < a_{um} &= a_{mp}a_{up}[b, b']^2 && \text{by (4)} \\ &\leq a_{mp}a_{um}[b, b']^2 \\ &\leq a_{um}. \end{aligned}$$

This implies $[b, b'] = 1$ and it contradicts $b \neq b'$. Hence we get $p \notin S$ and by definition of S there exists an integer t such that $\sigma(e_t, b) = (f_k, c_p)$. On the other hand $\sigma(e_t, b) = (f_t, c_t)$ holds. Hence we get $f_k = f_t, c_p = c_t$ and by (2)

$$a_{tp}[b, b'] = [c_t, c_m],$$

and by (1)

(5) $[b, b'] = [f_p, f_k][c_p, c_m].$

From these follows

$$\begin{aligned}
 [c_p, c_m] &= [c_t, c_m] \\
 &= a_{tp}[b, b'] \\
 &= a_{tp}[f_p, f_k][c_p, c_m].
 \end{aligned}$$

If $[c_p, c_m] \neq 0$, then $a_{tp}[f_p, f_k] = 1$ and this contradicts $f_p \neq f_k$. Hence we have $[c_p, c_m] = 0$ and $[b, b'] = 0$ by (5) and $[f_m, f_k] = 0$ by (3), and $a_{um} = [f_u, f_m][c_u, c_m] = [f_k, f_m][c_u, c_m] = 0$. This contradicts our assumption $a_{um} \neq 0$. Thus we have proved $a_{um} = 0$. Q.E.D.

THEOREM 1. *Let A, B, C be weighted graphs and assume that $A = \{e_i\}_{i=1}^n$ is connected and $\sigma: A \times B \cong A \times C$. Take any element $b \in B$ and put $\sigma(e_i, b) = (f_i, c_i)$. Then we have*

$$A \cong \{\sigma(e_i, b) \mid 1 \leq i \leq n\} = \{f_i \mid 1 \leq i \leq n\} \times \{c_i \mid 1 \leq i \leq n\}.$$

Proof. Put $C_i = \{c_k \mid k \text{ satisfies } f_k = f_i\}$ for $1 \leq i \leq n$, and denote by \tilde{C}_i a connected component of C_i which contains c_i . Suppose $[e_i, e_j] = [f_i, f_j][c_i, c_j] \neq 0$. We will show $\tilde{C}_i = \tilde{C}_j$. Since $[e_i, e_j] \neq 0$, Lemma 1 implies that there exists an element $e_t \in A$ such that $\sigma(e_t, b) = (f_i, c_j)$. Hence we have $f_t = f_i, c_t = c_j$ since $\sigma(e_t, b) = (f_t, c_t)$. By definition of C_i we have $c_j = c_t \in C_i$ and hence $c_j \in \tilde{C}_i$ since $c_i \in \tilde{C}_i$ and $[c_j, c_i] \neq 0$. Thus we have proved $\tilde{C}_i \cap \tilde{C}_j \neq \emptyset$. Take any element $x \in \tilde{C}_i \cap \tilde{C}_j$; then there exists u such that $x = c_u$ and $f_u = f_i$ since $x \in C_i$. Take any $y \in \tilde{C}_j$ such that $[y, x] \neq 0$. Then y can be written $y = c_k$ with $f_k = f_j$. $[e_u, e_k] = [f_u, f_k][c_u, c_k] = [f_i, f_j][x, y] \neq 0$ yields that $\sigma(e_s, b) = (f_u, c_k)$ for some s . From $f_s = f_u = f_i, [c_s, c_u] = [c_k, c_u] \neq 0$ follows that $y = c_k = c_s \in \tilde{C}_i$ since $c_u = x \in \tilde{C}_i, c_s \in C_i$ and \tilde{C}_i is a connected component of C_i . Thus we have shown that if $[x, y] \neq 0$ for $x \in \tilde{C}_i \cap \tilde{C}_j, y \in \tilde{C}_j$, then $y \in \tilde{C}_i$ holds. This implies $\tilde{C}_j \subset \tilde{C}_i$ and similarly $\tilde{C}_i \subset \tilde{C}_j$ and hence $\tilde{C}_i = \tilde{C}_j$ if $[e_i, e_j] \neq 0$. Since A is connected we get $\tilde{C}_1 = \dots = \tilde{C}_n$. Take any s, t ($1 \leq s, t \leq n$). From $c_t \in \tilde{C}_t = \tilde{C}_s \subset C_s$ follows that there exists i such that $c_t = c_i$ and $f_i = f_s$. Hence $(f_s, c_i) = (f_i, c_i) = \sigma(e_i, b)$ holds. Q.E.D.

§ 2.

Let L be an indecomposable positive lattice which satisfies the following condition (A').

(A') For any given positive lattices M, N and for any isometry σ from $L \otimes M$ on $L \otimes N$ which satisfies that $\sigma(L \otimes m) = L \otimes n$ ($m \in M, n \in N$) implies $m = 0, n = 0$, there exists a finite subset $\{v_1, \dots, v_m\}$ of L (depending on M, N, σ) such that

(1) each v_i is primitive in L and a submodule spanned by $\{v_1, \dots, v_m\}$ of L is of finite index in L ,

- (2) putting $M_{v_i} = \{m \in M \mid \sigma(L \otimes m) \subset v_i \otimes N\}$,
 $N_{v_i} = \{n \in N \mid \sigma^{-1}(L \otimes n) \subset v_i \otimes M\}$,

we have $\text{rank } M_{v_i} = \text{rank } N_{v_i} = \text{rank } M / \text{rank } L$, and

- (3) $\sigma(Q(v_i \otimes M_{v_i})) = Q(v_i \otimes N_{v_i})$.

Through this section the above L is fixed.

LEMMA 2. Let $M, N, \sigma, v_i, M_{v_i}, N_{v_i}$ be those as in the condition (A'). Then M, N are isometric and they are divided by L , and $\sigma(L \otimes M_{v_i}) = v_i \otimes N$.

Proof. By definition M_{v_i}, N_{v_i} are direct summands (as modules) of M, N respectively, and $\sigma(L \otimes M_{v_i}) \subset v_i \otimes N$, $\sigma^{-1}(L \otimes N_{v_i}) \subset v_i \otimes M$ and $\text{rank } \sigma(L \otimes M_{v_i}) = \text{rank } (v_i \otimes N)$, $\text{rank } \sigma^{-1}(L \otimes N_{v_i}) = \text{rank } (v_i \otimes M)$ imply $\sigma(L \otimes M_{v_i}) = v_i \otimes N$ and $\sigma^{-1}(L \otimes N_{v_i}) = v_i \otimes M$ since they are direct summands in $L \otimes N, L \otimes M$ respectively. This implies that M, N are divided by L . From (3) follows $\sigma(v_i \otimes M_{v_i}) = v_i \otimes N_{v_i}$ since they are direct summands of $L \otimes N$. Hence we can define an isometry $\mu_i: M_{v_i} \cong N_{v_i}$ by $\sigma(v_i \otimes m) = v_i \otimes \mu_i(m)$ for $m \in M_{v_i}$. For $m_i \in M_i, m_j \in M_j$ we show $B(m_i, m_j) = B(\mu_i(m_i), \mu_j(m_j))$ where B stands for bilinear forms associated with quadratic modules.

$$\begin{aligned} B(v_i, v_j)B(m_i, m_j) &= B(v_i \otimes m_i, v_j \otimes m_j) \\ &= B(\sigma(v_i \otimes m_i), \sigma(v_j \otimes m_j)) \\ &= B(v_i \otimes \mu_i(m_i), v_j \otimes \mu_j(m_j)) \\ &= B(v_i, v_j)B(\mu_i(m_i), \mu_j(m_j)). \end{aligned}$$

Hence we have $B(m_i, m_j) = B(\mu_i(m_i), \mu_j(m_j))$ if $B(v_i, v_j) \neq 0$. Suppose $B(v_i, v_j) = 0$, then we have

$$\begin{aligned} B(L \otimes M_{v_i}, L \otimes M_{v_j}) &= B(\sigma(L \otimes M_{v_i}), \sigma(L \otimes M_{v_j})) \\ &= B(v_i \otimes N, v_j \otimes N) \\ &= 0, \\ B(L \otimes \mu_i(M_{v_i}), L \otimes \mu_j(M_{v_j})) &= B(L \otimes N_{v_i}, L \otimes N_{v_j}) \\ &= B(\sigma^{-1}(L \otimes N_{v_i}), \sigma^{-1}(L \otimes N_{v_j})) \\ &= B(v_i \otimes M, v_j \otimes M) \\ &= 0. \end{aligned}$$

Hence $B(M_{v_i}, M_{v_j}) = B(\mu_i(M_{v_i}), \mu_j(M_{v_j})) = 0$ follows. Thus we have proved $B(m_i, m_j) = B(\mu_i(m_i), \mu_j(m_j))$ for $m_i \in M_{v_i}, m_j \in M_{v_j}$. By (1) we can choose

a subset of $\{v_1, \dots, v_m\}$, say $\{v_1, \dots, v_n\}$, so that it is a basis of QL . Then $\sigma(L \otimes M_{v_i}) = v_i \otimes N$ implies that $\sum_{i=1}^n M_{v_i}$ is a direct sum and $[M: \sum_{i=1}^n M_{v_i}] < \infty$. Hence a linear mapping μ from QM to QN defined by $\mu(\sum_{i=1}^n m_i) = \sum_{i=1}^n \mu_i(m_i)$ ($m_i \in M_{v_i}$) becomes an isometry from QM on QN . We have only to show $\mu(M) = N$. Take a basis $\{e_j\}$ of L and put

$$e_i = \sum_{j=1}^n a_{ij} v_j, \quad v_i = \sum_{j=1}^n b_{ij} e_j \quad (a_{ij}, b_{ij} \in \mathbb{Q}).$$

$\sum_{k=1}^n b_{ik} a_{kj} = \delta_{ij}$ (Kronecker's delta) is obvious. Take any element $m = \sum_{i=1}^n m_i$ ($m_i \in M_{v_i}$) of M and put $\sigma(v_j \otimes m_i) = v_i \otimes n_{ji}$ ($n_{ji} \in N$); then $n_{ii} = \mu(m_i)$ follows and

$$\begin{aligned} \sigma(e_k \otimes m) &= \sigma\left(\sum_{i,j} a_{kj} v_j \otimes m_i\right) \\ &= \sum_{i,j} a_{kj} v_i \otimes n_{ji} \\ &= \sum_i e_i \otimes \left(\sum_{i,j} a_{kj} b_{ii} n_{ji}\right). \end{aligned}$$

Since $\sigma(e_k \otimes m) \in L \otimes N$, we get $\sum_{i,j} a_{kj} b_{ii} n_{ji} \in N$. Summing up with respect to k , we have

$$\mu(m) = \sum \mu(m_i) = \sum n_{ii} \in N.$$

Thus $\mu(M) \subset N$ is proved. Since discriminants of M, N are equal, we have $\mu(M) = N$. Q.E.D.

LEMMA 3. *Let K, X, Y be positive lattices and assume that K is indecomposable and $\sigma: K \otimes X \cong K \otimes Y$. Then there exist submodules M_0, M of X and N_0, N of Y such that*

(i) M_0, M, N_0, N are direct summands of X, Y respectively and $[X: M_0 \perp M], [Y: N_0 \perp N] < \infty$, and

$$\sigma(K \otimes M_0) = K \otimes N_0, \quad \sigma(K \otimes M) = K \otimes N,$$

(ii) if $\sigma(K \otimes m) = K \otimes n$ ($m \in M, n \in N$), then $m = 0$ and $n = 0$, and

(iii) there exist orthogonal decompositions

$$M_0 = \bigoplus_{i=1}^t M_{0,i}, \quad N_0 = \bigoplus_{i=1}^t N_{0,i}$$

such that $\sigma(K \otimes M_{0,i}) = K \otimes N_{0,i}$ ($1 \leq i \leq t$) and

$$\sigma|_{K \otimes M_{0,i}} = \alpha_i \otimes \beta_i$$

where $\alpha_i \in O(K)$, $\beta_i: M_{0,i} \cong N_{0,i}$.

Proof. Suppose that m_1, \dots, m_r are linearly independent elements of X so that there exist elements $n_i \in Y$ such that $\sigma(K \otimes m_i) = K \otimes n_i$. We may assume that r is maximal. Then we put $M = Z[m_1, \dots, m_r]^\perp$, $N = Z[n_1, \dots, n_r]^\perp$ and $M_0 = M^\perp$, $N_0 = N^\perp$. Clearly (i), (ii) are satisfied. (iii) follows from Lemma 1 in § 3 in [2]. Q.E.D.

LEMMA 4. $L \otimes L$ is indecomposable and $O(L \otimes L)$ is generated by $O(L)$ and an interchange of L 's.

Proof. Take an isometry σ of $L \otimes L$. Suppose that there exist $x, y \in L$, $x \neq 0$ such that $\sigma(L \otimes x) = L \otimes y$. Supposing $K = X = Y = L$ in Lemma 3, a submodule of L corresponding to M of X in Lemma 3 is $\{0\}$ since its rank ($< \text{rank } L$) is divided by $\text{rank } L$ by Lemma 2. Hence we have $\sigma \in O(L) \otimes O(L)$ by Lemma 3. Suppose that there are no such elements x, y in L . Then, by Lemma 2, there is an element $v \in L$ such that

$$\sigma(L \otimes L_v) = v \otimes L,$$

where $L_v = \{x \in L \mid \sigma(L \otimes x) \subset v \otimes L\}$. Since $\text{rank } L_v = 1$, there is an element u such that $L_v = Z[u]$. Then $\mu\sigma(L \otimes u) = L \otimes v$ holds where $\mu \in O(L \otimes L)$ is defined by $\mu(x \otimes y) = y \otimes x (x, y \in L)$. Hence $\mu\sigma \in O(L) \otimes O(L)$ follows as above. The indecomposability of $L \otimes L$ is proved quite similarly as in the proof of Lemma 4 in [4]. Q.E.D.

LEMMA 5. $\otimes^m L$ is indecomposable provided that the orthogonal group $O(\otimes^m L)$ is generated by $O(L)$ and interchanges of L 's and that $\otimes^{m-1} L$ is indecomposable.

Proof. The proof is identical with that of Lemma 5 in [4].

THEOREM 2. Let X be an indecomposable positive lattice. Then we have

- (i) for any positive lattice Y , $L \otimes X \cong L \otimes Y$ implies $X \cong Y$,
- (ii) if $X = \otimes^t L \otimes X'$ where X' is a positive lattice which is not divided by L , then $O(L \otimes X)$ is generated by $O(L)$, $O(X')$ and interchanges of L 's,
- (iii) $L \otimes X$ is indecomposable.

Proof. We induct on $\text{rank } X$. In case of $\text{rank } X = 1$ our assertion is obvious. Suppose $\text{rank } X = k + 1$. Let Y be a positive lattice and $\sigma: L \otimes X \cong L \otimes Y$. Let M_0, M (resp. N_0, N) be submodules of X (resp. Y)

as in Lemma 3 for $K = L$. If $M = X$ (resp. $M_0 = X$), then $X \cong Y$ follows from Lemma 2 (resp. Lemma 3). Hence we may assume $M_0 \neq \{0\}$, $M \neq \{0\}$. Lemma 2 implies $M \cong N$. Hence we may assume $M = N = \perp K_i$ where K_i is indecomposable and suppose $K_i \cong \otimes^{r_i} L \otimes K'_i$ where K'_i is not divided by L . Since $\text{rank } K_i \leq \text{rank } M \leq k$, the inductive assumption implies that $L \otimes K_i$ is indecomposable and $O(L \otimes K_i)$ is generated by $O(L)$, $O(K'_i)$ and interchanges of L 's, identifying K_i and $\otimes^{r_i} L \otimes K'_i$. Hence, noting $L \otimes M \cong L \otimes N \cong \perp L \otimes K_i$, as in 2 in [4] for any basis $\{u_1, \dots, u_n\}$ of L we have

$$\begin{aligned}\sigma(L \otimes M_{u_i}) &= u_i \otimes N, & \sigma^{-1}(L \otimes N_{u_i}) &= u_i \otimes M, \\ \sigma(u_i \otimes M_{u_i}) &= u_i \otimes N_{u_i},\end{aligned}$$

where

$$\begin{aligned}M_{u_i} &= \{m \in M \mid \sigma(L \otimes m) \subset u_i \otimes N\}, \\ N_{u_i} &= \{n \in N \mid \sigma^{-1}(L \otimes n) \subset u_i \otimes M\}.\end{aligned}$$

Now $X = M_0 \perp M$, $Y = N_0 \perp N$ are proved quite similarly as in the proof of Theorem in § 1 in [3]. This is a contradiction since X is indecomposable. Thus (i) is proved. Let X be a positive lattice as in (ii). Assume that there exists an isometry $\sigma \in O(L \otimes X)$ which is not contained in a subgroup of $O(L \otimes X)$ generated by $O(L)$, $O(X')$ and interchanges of L 's. Suppose that there exist $x, y \in X$ such that $\sigma(L \otimes x) = L \otimes y$. We define M_0, M, N_0, N as in Lemma 3 for $K = L$, $Y = X$. Then $X = M_0 \perp M$ holds as above. Since $M_0 \neq \{0\}$ and X is indecomposable, we have $X = M_0$ and Lemma 3 implies $\sigma \in O(L) \otimes O(X)$. If X is divided by L , that is, $t \geq 1$, then $O(X) = O(L \otimes (\otimes^{t-1} L \otimes X'))$ is generated by $O(L)$ and $O(X')$ and interchanges of L 's since $\otimes^{t-1} L \otimes X'$ is indecomposable and $\text{rank } \otimes^{t-1} L \otimes X' \leq k$. Thus σ is contained in a subgroup generated by $O(L)$, $O(X')$ and interchanges of L 's in $O(L \otimes X)$. This is a contradiction. Therefore there exist no such elements x, y . Hence from Lemma 2 follows that $t \geq 1$ and there exists non-zero $v \in L$ such that $\sigma(L \otimes X_v) = v \otimes X$ where $X_v = \{x \in X \mid \sigma(L \otimes x) \subset v \otimes X\}$ by the assumption on L . Define $\mu_2 \in O(L \otimes X)$ by $\mu_2(x \otimes y \otimes z) = y \otimes x \otimes z$ ($x, y \in L, z \in \otimes^{t-1} L \otimes X'$); then $\mu_2 \sigma(L \otimes X_v) = L \otimes v \otimes \otimes^{t-1} L \otimes X'$. If there exist $x \in X_v, y \in v \otimes \otimes^{t-1} L \otimes X'$ such that $x \neq 0$ and $\mu_2 \sigma(L \otimes x) = L \otimes y$, then $\mu_2 \sigma \in O(L \otimes X)$ must be contained in a subgroup generated by $O(L)$, $O(X')$ and interchanges of L 's as above. This is also a contradiction. Repeating this operation we get, as in 1.6 in [4],

$$\mu_{t+1} \cdots \mu_2 \sigma(L \otimes X_{v, \dots, v' \dots}) = L \otimes v \otimes \cdots \otimes v' \dots \otimes X',$$

where $\mu_j \in O(L \otimes X)$ is defined by

$$\begin{aligned} \mu_j(x_1 \otimes \cdots \otimes x_j \otimes \cdots \otimes x_{t+1} \otimes y) \\ = x_j \otimes \cdots \otimes x_1 \otimes \cdots \otimes x_{t+1} \otimes y \quad (x_i \in L, y \in X'). \end{aligned}$$

If there exist $x \in X_{v, \dots, v' \dots}, y \in v \otimes \cdots \otimes X'$ such that $x \neq 0$ and $\mu_{t+1} \cdots \mu_2 \sigma(L \otimes x) = L \otimes y$, then $\mu_{t+1} \cdots \mu_2 \sigma$ is contained in a subgroup generated by $O(L), O(X')$ and interchanges of L 's. This is a contradiction. Hence Lemma 2 yields that $v \otimes \cdots \otimes v' \dots \otimes X'$ is divided by L . This contradicts the assumption on X' . Thus the proof of (ii) is completed. Let X be a positive lattice as in (ii). Then $O(L \otimes X) = O(\otimes^{t+1} L) \otimes O(X')$ has been proved as above. To complete the proof of (iii) we have only to show that $\otimes^{t+1} L$ is indecomposable by virtue of Lemma 3 in [4]. Since X is indecomposable, $\otimes^t L$ is also indecomposable. By virtue of (ii) $O(\otimes^{t+1} L)$ is generated by $O(L)$ and interchanges by L 's. Hence Lemma 5 implies that $\otimes^{t+1} L$ is indecomposable. Q.E.D.

Remark. By (i), (iii) and Theorem in 105:1 in [5] $L \otimes X \cong L \otimes Y$ implies $X \cong Y$ for any (not necessarily indecomposable) positive lattices X, Y .

§ 3.

Through this section we fix any positive lattice L of E -type such that $[L: \tilde{L}] < \infty$ and \tilde{L} is indecomposable.

LEMMA 6. *Let M, N be positive lattices and assume $\sigma: L \otimes M \cong L \otimes N$. Then for each $m \in \mathfrak{M}(M)$ we have $\sigma(L \otimes m) = F \otimes G$, where F, G are submodules of L, N respectively and $m(F) = m(L)$, and $m(G) = m(N)$.*

Proof. Let X be a positive lattice. For $x, y \in \mathfrak{M}(X)/\pm$, we put $[x, y] = |B(x, y)|/m(X)$. Then $\mathfrak{M}(X)/\pm$ becomes a weighted graph and \tilde{X} is indecomposable if and only if $\mathfrak{M}(X)/\pm$ is connected. Put $A = \mathfrak{M}(L)/\pm, B = \mathfrak{M}(M)/\pm, C = \mathfrak{M}(N)/\pm$. Since L is of E -type, we have $\mathfrak{M}(L \otimes M) = \mathfrak{M}(L) \otimes \mathfrak{M}(M), \mathfrak{M}(L \otimes N) = \mathfrak{M}(L) \otimes \mathfrak{M}(N)$ and σ induces an isometry from $A \times B$ on $A \times C$. By Theorem 1 there exist subsets $F' \subset A, G' \subset C$ such that $\sigma(A, m) = (F', G')$. Denoting by F_0, G_0 submodules of L, N spanned by $F' \subset \mathfrak{M}(L)/\pm, G' \subset \mathfrak{M}(N)/\pm$ respectively, we have $\sigma(\tilde{L} \otimes m) = F_0 \otimes G_0$ and $m(F_0) = m(L), m(G_0) = m(N)$. Put $F = \mathbf{Q}F_0 \cap L, G = \mathbf{Q}G_0 \cap N$; then

$[F: F_0], [G: G_0] < \infty, m(F) = m(L), m(G) = m(N)$ and $\sigma(L \otimes m), F \otimes G$ are direct summands of $L \otimes N$. Hence $\sigma(L \otimes m) = F \otimes G$ follows. Q.E.D.

THEOREM 3. *If $L \cong L_1 \otimes L_2$ for positive lattices L_1, L_2 , then L_1, L_2 are of E -type, $[L_1: \tilde{L}_1], [L_2: \tilde{L}_2] < \infty$ and \tilde{L}_1, \tilde{L}_2 are indecomposable.*

Proof. Define $\sigma \in O(L_1 \otimes L_2 \otimes L_2)$ by $\sigma(x \otimes y \otimes z) = x \otimes z \otimes y$ ($x \in L_1, y, z \in L_2$). For each $m \in \mathfrak{M}(L_2)$ $\sigma((L_1 \otimes L_2) \otimes m) = (L_1 \otimes m) \otimes L_2$ holds. Applying Lemma 6 in case of $M = N = L_2$, we have $m(L_1 \otimes m) = m(L)$. From Proposition 2 in [1] follows that $L_1 \otimes m$ is of E -type. Hence L_1 is of E -type. Similarly L_2 is of E -type. $\mathfrak{M}(L) = \mathfrak{M}(L_1) \otimes \mathfrak{M}(L_2)$ implies $[L_1: \tilde{L}_1], [L_2: \tilde{L}_2] < \infty$ and \tilde{L}_1, \tilde{L}_2 are indecomposable since $[L: \tilde{L}] < \infty$ and \tilde{L} is indecomposable. Q.E.D.

THEOREM 4. *Assume that L is indecomposable with respect to tensor product. Then L satisfies the condition (A') in § 2.*

Proof. Suppose that L is decomposable and $L = L_1 \perp L_2$ ($L_1, L_2 \neq 0$). Each $x \in \mathfrak{M}(L)$ is contained in L_1 or L_2 . If $\mathfrak{M}(L) \cap L_1 = \phi$, then $\mathfrak{M}(L) \subset L_2$ and hence $\tilde{L} \subset L_2$ and $\text{rank } L \leq \text{rank } L_2$. This is a contradiction. Hence we have $\mathfrak{M}(L) \cap L_i \neq \phi$ ($i = 1, 2$) and then \tilde{L} spanned by $\mathfrak{M}(L)$ is decomposable. This contradicts our assumption on L . Thus L is indecomposable. Set $\mathfrak{M}(L) = \{\pm v_1, \dots, \pm v_m\}$. We show that the condition (A') is satisfied for the subset $\{v_1, \dots, v_m\}$ of L by induction with respect to rank M . The first condition of (A') follows from our assumption on L . Let M, N be positive lattices and suppose that for $\sigma: L \otimes M \cong L \otimes N, \sigma(L \otimes m) = L \otimes n$ ($m \in M, n \in N$) implies $m = 0, n = 0$. Since L is of E -type, we have $\sigma(\mathfrak{M}(L) \otimes \mathfrak{M}(M)) = \mathfrak{M}(L) \otimes \mathfrak{M}(N)$ and hence $\sigma(\tilde{L} \otimes \tilde{M}) = \tilde{L} \otimes \tilde{N}$. Put $\tilde{M}^\perp = M', \tilde{N}^\perp = N', M'' = M'^\perp (\neq \{0\}), N'' = N'^\perp (\neq \{0\})$; then we have $[M: M' \perp M''], [N: N' \perp N''] < \infty, \sigma(L \otimes M') = L \otimes N'$ and $\sigma(L \otimes M'') = L \otimes N''$ by virtue of $[L: \tilde{L}] < \infty$. Assume $M' \neq 0$; then the inductive assumption implies $\text{rank } M'_{v_i} = \text{rank } N'_{v_i} = \text{rank } M'/\text{rank } L$ and $\text{rank } M''_{v_i} = \text{rank } N''_{v_i} = \text{rank } M''/\text{rank } L$, where

$$M'_{v_i} = \{m \in M' \mid \sigma(L \otimes m) \subset v_i \otimes N'\},$$

$$N'_{v_i} = \{n \in N' \mid \sigma^{-1}(L \otimes n) \subset v_i \otimes M'\}, \quad \text{and}$$

M''_{v_i}, N''_{v_i} are defined similarly for M'', N'' . Moreover M_{v_i}, N_{v_i} are defined similarly for M, N ; then $M_{v_i} \supset M'_{v_i} \perp M''_{v_i}$ and $N_{v_i} \supset N'_{v_i} \perp N''_{v_i}$ are obvious. Hence $\text{rank } M_{v_i} \geq \text{rank } M/\text{rank } L$ holds. Take any i ($1 \leq i \leq m$) and a subset S of $\{v_1, \dots, v_m\}$ such that S contains v_i and S is a basis of QL .

We may assume $i = 1$, $S = \{v_1, \dots, v_n\}$ ($n = \text{rank } L$). Then $\sigma(L \otimes M_{v_i}) \subset v_i \otimes N$ and $\sigma(L \otimes \sum_{i=1}^n M_{v_i}) \subset \sum_{i=1}^n v_i \otimes N$ imply that $\sum_{i=1}^n M_{v_i}$ is a direct sum. Thus we have $\text{rank } M \geq \sum_{i=1}^n \text{rank } M_{v_i} \geq \sum_{i=1}^n \text{rank } M / \text{rank } L = \text{rank } M$ and hence $\text{rank } M_{v_i} = \text{rank } M / \text{rank } L$. Hence $\text{rank } M_{v_i} = \text{rank } M / \text{rank } L$ for each i and similarly $\text{rank } N_{v_i} = \text{rank } N / \text{rank } L$ hold. From this follows that $QM_{v_i} = QM'_{v_i} \perp QM''_{v_i}$ and $QN_{v_i} = QN'_{v_i} \perp QN''_{v_i}$ and hence $\sigma(Q(v_i \otimes M_{v_i})) = \sigma(Q(v_i \otimes M'_{v_i}) \perp Q(v_i \otimes M''_{v_i})) = Q(v_i \otimes N'_{v_i}) \perp Q(v_i \otimes N''_{v_i}) = Q(v_i \otimes N_{v_i})$. Thus the condition (2), (3) are shown if $M' \neq 0$. Suppose $M' = 0$; then $[M: \tilde{M}]$, $[N: \tilde{N}] < \infty$ hold. For each $m \in \mathfrak{M}(M)$ Lemma 6 implies $\sigma(L \otimes m) = F \otimes G$ where F, G are submodules of L, N respectively and $m(F) = m(L)$. By the assumption on L we get $\text{rank } F$ or $\text{rank } G = 1$. $\text{rank } G = 1$ implies $\sigma(L \otimes m) = L \otimes n$ for some $n \in N$ and it contradicts our assumption on σ . Hence we have $F = Z[v]$ for $v \in \mathfrak{M}(L)$.

Thus for each $m \in \mathfrak{M}(M)$ there exists $v \in \mathfrak{M}(L)$ such that $\sigma(L \otimes m) \subset v \otimes N$.

Take any $v_i \in \mathfrak{M}(L)$ and fix it. For $n \in \mathfrak{M}(N)$ suppose $\sigma(v \otimes m) = v_i \otimes n$ for $v \in \mathfrak{M}(L)$, $m \in \mathfrak{M}(M)$. Since $\sigma(L \otimes m) \subset v_j \otimes N$ for $v_j \in \mathfrak{M}(L)$ as above, v_j must be equal to v_i and hence $\sigma(L \otimes m) \subset v_i \otimes N$, $m \in M_{v_i}$ and $m(M_{v_i}) = m(M)$. Therefore $v_i \otimes n = \sigma(v \otimes m) \in \sigma(L \otimes M_{v_i}) \subset v_i \otimes N$ holds for each $n \in \mathfrak{M}(N)$. Thus we get $v_i \otimes \tilde{N} \subset \sigma(L \otimes M_{v_i}) \subset v_i \otimes N$. From $[N: \tilde{N}] < \infty$ follows $\text{rank } M_{v_i} = \text{rank } N / \text{rank } L$, $m(M_{v_i}) = m(M)$ and $[M_{v_i}: \tilde{M}_{v_i}] < \infty$. Similarly $\text{rank } N_{v_i} = \text{rank } M / \text{rank } L$ follows.

For each $m \in \mathfrak{M}(M) \cap M_{v_i} = \mathfrak{M}(M_{v_i})$ we put $\sigma(v_i \otimes m) = v_i \otimes n$ ($n \in \mathfrak{M}(N)$). Then we have $\sigma^{-1}(L \otimes n) \subset v_i \otimes M$ by $\sigma^{-1}(v_i \otimes n) = v_i \otimes m$. Hence $n \in N_{v_i}$ follows. Conversely $n \in \mathfrak{M}(N) \cap N_{v_i} = \mathfrak{M}(N_{v_i})$ implies $\sigma^{-1}(v_i \otimes n) = v_i \otimes m$ for $m \in \mathfrak{M}(M)$ and $\sigma(L \otimes m) \subset v_i \otimes N$ by $\sigma(v_i \otimes m) = v_i \otimes n$. Hence we have $m \in M_{v_i}$ and $\sigma(v_i \otimes \mathfrak{M}(M_{v_i})) = v_i \otimes \mathfrak{M}(N_{v_i})$. $[M_{v_i}: \tilde{M}_{v_i}]$, $[N_{v_i}: \tilde{N}_{v_i}] < \infty$ yield $\sigma(Q(v_i \otimes M_{v_i})) = Q(v_i \otimes N_{v_i})$. This completes the proof of Theorem 4. Q.E.D.

Theorem 2, 3, 4 yield Theorem at the beginning of this paper.

§ 4.

In this section we give examples of positive lattices in Theorem.

PROPOSITION. Let $L = Z[e_1, \dots, e_n]$ be a quadratic lattice and put $ca_{ij} = B(e_i, e_j)$. Assume that

- (0) $c, a_{ij} \in \mathbf{Q}$, $c > 0$,
- (1) $a_{ii} = 1$ and $1 - \sum_{j \neq i} |a_{ij}| \geq 0$ for $i = 1, \dots, n$,

(2) for any non-empty subset S of $\{1, 2, \dots, n\}$

$$\#|S| - 1 \geq \sum_{\substack{i,j \in S \\ i \neq j}} |a_{ij}|.$$

Then L is a positive lattice of E -type and $L = \tilde{L}$.

Proof. By scaling we may suppose $c = 1$ without loss of generality. Let M be a positive lattice with $m(M) = 1$. Take any non-zero element $x = \sum_{i=1}^n e_i \otimes u_i \in L \otimes M$. Put $b_{ij} = a_{ij}/|a_{ij}|$ if $a_{ij} \neq 0$, $= 0$ if $a_{ij} = 0$, and $S = \{i | u_i \neq 0\}$ ($\neq \phi$). Then we have

$$\begin{aligned} Q(x) &= \sum a_{ij} B(u_i, u_j) \\ &= \sum Q(u_i) + \frac{1}{2} \sum_{i \neq j} a_{ij} (2B(u_i, u_j)) \\ &= \sum Q(u_i) + \frac{1}{2} \sum_{i \neq j} |a_{ij}| (Q(b_{ij}u_i + u_j) - Q(u_i) - Q(u_j)) \\ &= \sum_{i \in S} \left(1 - \sum_{\substack{j \in S \\ j \neq i}} |a_{ij}|\right) Q(u_i) + \frac{1}{2} \sum_{\substack{i,j \in S \\ i \neq j}} |a_{ij}| Q(b_{ij}u_i + u_j) \\ &\geq \#|S| - \sum_{\substack{i,j \in S \\ i \neq j}} |a_{ij}| \\ &\geq 1. \end{aligned}$$

Hence L is positive and $m(L)m(M) \geq m(L \otimes M) \geq 1$. $m(L) \leq 1, m(M) = 1$ imply $m(L \otimes M) = 1$ and $m(L) = 1$. If $Q(x) = 1$ and hence $x \in \mathfrak{M}(L \otimes M)$, then $b_{ij}u_i + u_j = 0$ and hence $u_i = \pm u_j$ for $i, j \in S$ with $i \neq j, a_{ij} \neq 0$. Suppose $S = S_1 \cup S_2$ and $a_{ij} = 0$ if $i \in S_1, j \in S_2$; then $x = (\sum_{i \in S_1} e_i \otimes u_i) + (\sum_{j \in S_2} e_j \otimes u_j)$ is an orthogonal sum and $x \in \mathfrak{M}(L \otimes M)$ implies that one of them must vanish. Thus we have S_1 or $S_2 = \phi$ and then $u_i = \pm u_j$ for $i, j \in S$. Therefore x should be $e \otimes u_i$ for $e \in L, i \in S$. By definition L becomes a lattice of E -type and $m(L) = 1$ implies $\mathfrak{M}(L) \supset \{e_i\}$ and hence $L = \tilde{L}$. Thus we complete the proof. Q.E.D.

Remark. If $a_{ii} = 1, |a_{ij}| < 1/n$ ($i \neq j$), then the conditions (1), (2) are satisfied and $\mathfrak{M}(L) = \{\pm e_i | 1 \leq i \leq n\}$. In this case it is easy to see whether L is indecomposable or not. Suppose that L is indecomposable and $L = L_1 \otimes L_2$. Then from our theorem follows that L_1, L_2 are of E -type and $\mathfrak{M}(L) = \mathfrak{M}(L_1) \otimes \mathfrak{M}(L_2), L_i = \tilde{L}_i$ and $|\mathfrak{M}(L_i)| = 2rkL_i$. Hence we can take minimal vectors as a basis of L_i , and then the matrix $(B(f_i, f_j))$ corresponding to L , where $\{\pm f_i\} = \{\pm e_i\}$, is a tensor product of matrices corresponding to L_i by their minimal vectors. Thus it is also easy to see whether L is indecomposable with respect to tensor product or not.

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