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OPTIMAL STOPPING FOR THE LAST EXIT TIME

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Abstract

Given a one-dimensional downwards transient diffusion process X, we consider a random time ρ, the last exit time when X exits a certain level ℓ, and detect the optimal stopping time for it. In particular, for this random time ρ, we solve the optimisation problem $\inf_\tau \mathbb{E}[\lambda(\tau-\rho)_+ + (1-\lambda)(\rho-\tau)_+]$ over all stopping times τ. We show that the process should stop optimally when it runs below some fixed level κ_ℓ for the first time, where κ_ℓ is the unique solution in the interval $(0, \lambda\ell)$ of an explicitly defined equation.

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1. Introduction

We sometimes need to consider a stochastic process X and find a particular (stopping) time τ as close as possible to a random time ρ of interest, in the sense that the stopping time can be decided based on the current information. This is called an *optimal stopping* problem and is encountered very often in mathematical finance, economics and engineering. There are usually two ways to formulate this problem: one is to optimise in the time domain, that is, τ is as close as possible to ρ; the other is to optimise in the spatial domain, that is, X_τ is as close as possible to X_ρ. For some particular random times or stochastic processes, those two formulations are equivalent (see [5, 7]).

We detect the corresponding optimal stopping time of the *last exit time*, which is a random time, in the time domain, for a one-dimensional transient diffusion process. Some similar work has been done before. For example, [1, 2, 6] find the optimal stopping time for the last zero for a Brownian motion with drift, for a transient diffusion and for a spectrally negative Lévy process, respectively. Further to previous work, which seeks to minimise the expected value of the prediction error $\mathbb{E}[|\rho-\tau|]$, we aim to find a stopping time τ which minimises the expectation

$$\mathbb{E}[\lambda(\tau-\rho)_+ + (1-\lambda)(\rho-\tau)_+]. \tag{OS}$$

Here, ρ is the *last exit time* from some interval, λ is a weight parameter strictly between 0 and 1 and $(\cdot)_+$ is the positive function defined as $(\cdot)_+ := \max(\cdot, 0)$. That is, we aim to minimise the difference between τ and ρ, while putting different weights on the parts where the stopping time τ is before or after ρ. The classical expectation $\mathbb{E}[|\rho - \tau|]$ is recovered by taking $\lambda = 1/2$.

We solve this optimal stopping problem explicitly for the last exit time. Briefly, the process should stop when it moves down too low compared to some specific boundary, which can be found from a simple equation. Although the equation can only be solved explicitly for some special cases, it can readily be solved numerically.

The paper is organised as follows. In Section 2 we formulate the optimal stopping problem for general random times. To simplify the problem, we reduce the initial arbitrary transient process to a process without drift by the correlated scale function, so that we only need to concentrate on cases of driftless processes. Section 3 discusses the optimal stopping problem particularly for the last exit time. Explicit solutions for the cases when the diffusion coefficients are of power types are provided. Section 4 discusses some possible directions of future work.

2. Problem formulation

2.1. Diffusion set-up. Fix $a \in [-\infty, \infty)$ and $b \in \mathbb{R}$ with $a < b$. Let Ω be the canonical space of continuous functions from $\mathbb{R}_+$ to $[a, b)$ that remain constant after they reach the value a. (Note that the possibility $a = -\infty$ is allowed.) Let $Y = \{Y_t \mid t \in \mathbb{R}_+\}$ denote the coordinate process defined via $Y_t(\omega) = \omega(t)$ for all $\omega \in \Omega$; note that for each $\omega \in \Omega$, the function $t \mapsto Y_t(\omega)$ is continuous from $\mathbb{R}_+$ to $[a, b)$, and $Y_s(\omega) = a$ for all $s \geq t$ if $Y_t(\omega) = a$. Let $\mathcal{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ be the right-continuous augmentation of the natural filtration of Y, defined by

$$\mathcal{F}_t := \bigcap_{s>t} \sigma(Y_r \mid 0 \leq r \leq s), \tag{2.1}$$

and set the σ-field $\mathcal{F}$ equal to $\mathcal{F}_\infty$. Obviously, the process Y is adapted with respect to the filtration $\{\mathcal{F}_t\}$. Indeed, by [3, Proposition 1.1.13], Y is also progressively measurable to $\{\mathcal{F}_t\}$. For more information on the discussion below regarding one-dimensional diffusions, see [3, Section 5.5].

Consider functions $\alpha : [a, b) \to \mathbb{R}$ and $\beta : [a, b) \to \mathbb{R}_+$ such that β^2 is strictly positive and $\beta^{-2}(1 + |\alpha|)$ is locally integrable on (a, b). Let $\mathfrak{s} : (a, b) \to \mathbb{R}$ be an increasing twice-differentiable function satisfying

$$\alpha(y)\mathfrak{s}'(y) + (1/2)\beta^2(y)\mathfrak{s}''(y) = 0$$

for all $y \in (a, b)$. Then $\mathfrak{s}$ has a continuously differentiable inverse $\mathfrak{u} : (\mathfrak{s}(a), \mathfrak{s}(b)) \to \mathbb{R}$. Note that the solution $\mathfrak{s}$ is unique up to affine transformation. We assume that $-\infty < \mathfrak{s}(a+) < \mathfrak{s}(b-) = \infty$ holds for some (and then for any) function $\mathfrak{s}$. In view of this, one may consider a function $\mathfrak{s}$ with $\mathfrak{s}(a+) = 0$. In other words, for some $c \in (a, b)$,

$$\mathfrak{s}(y) = \int_a^y \exp\Big(-2 \int_c^v \frac{\alpha(w)}{\beta^2(w)}\, dw\Big)\, dv \quad \text{for } y \in [a, b). \tag{2.2}$$

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Now fix $y_0 \in (a, b)$. By [3, Theorem 5.5.15], under all the above assumptions, there exists a probability $\mathbb{P}$ on $\mathcal{F}$ (which coincides with the Borel σ-field on Ω) such that the coordinate process Y satisfies $\mathbb{P}[Y_0 = y_0] = 1$ and has dynamics

$$dY_t = \alpha(Y_t)\,dt + \beta(Y_t)\,dW_t \quad \text{for } t \in [0, \tau_a),$$

where $\tau_a = \inf\{t \in \mathbb{R}_+ \mid Y_t = a\}$ and W is a standard Brownian motion under $\mathbb{P}$. (Note that W is in general defined only up to time τ_a.) Then the $\mathfrak{s}$ function defined in (2.2) is a *scale function* of process Y. Moreover, by [3, Proposition 5.5.22],

$$\mathbb{P}\Big[\lim_{t\to\infty} Y_t = a\Big] = 1. \tag{2.3}$$

In words, Y is transient and drifts downwards (or explodes in finite time) to a under $\mathbb{P}$. Moreover, Y is a diffusion up to the 'explosion time' τ_a; after time τ_a, Y remains at a.

2.2. Optimal stopping. In the previous set-up, suppose that $\rho : \Omega \mapsto [0, \infty]$ is an $\mathcal{F}$-measurable random variable; we simply refer to ρ as a *random time*. We shall be concerned with finding a stopping time τ that is in a sense 'as close as possible' to a particular random time ρ_r^Y: the last exit time of Y from the interval (r, b). More precisely, the random time is defined as

$$\rho_r^Y = \sup\{t \in \mathbb{R}_+ \mid Y_t > r\}.$$

This random time obviously fails to be a stopping time, which makes the problem nontrivial. Though we focus on a particular random time—the last exit time ρ_r^Y—the discussion in this section is general and works for any arbitrary random time ρ.

Let $\mathcal{T}$ denote the set of all stopping times with respect to the filtration $\{\mathcal{F}_t\}$ defined in (2.1). A preliminary version of our objective is the problem

$$\begin{aligned}&\text{Find } \tau_* \in \mathcal{T} \text{ such that}\\ &\mathbb{E}[\lambda(\tau_* - \rho)_+ + (1-\lambda)(\rho - \tau_*)_+] = \inf_{\tau\in\mathcal{T}} \mathbb{E}[\lambda(\tau - \rho)_+ + (1-\lambda)(\rho - \tau)_+],\end{aligned} \tag{OS$'$}$$

where $\lambda \in (0, 1)$. In words, we wish to minimise the difference between τ and ρ, putting different weights on the parts where the stopping time τ is before or after ρ.

In the particular case $\lambda = 1/2$, this is equivalent to minimising $\mathbb{E}[|\rho - \tau|]$. On the other hand, for the constrained problems of minimising $\mathbb{E}[(\tau - \rho)_+]$ subject to $\mathbb{E}[(\rho - \tau)_+] \le \gamma$ or minimising $\mathbb{E}[(\rho - \tau)_+]$ subject to $\mathbb{E}[(\tau - \rho)_+] \le \gamma$, where $\gamma \in (0, \infty)$ is a tuning parameter, the corresponding Lagrangian formulation falls exactly within the scope of (OS$'$).

The formulation (OS$'$) is indeed satisfactory as long as $\mathbb{E}[\rho] < \infty$. However, when $\mathbb{E}[\rho] = \infty$, it may happen that the problem (OS$'$) returns an infinite value, which implies that all stopping times are trivially optimisers. (In this regard, see Remark 3.3 later on.) However, even in the case $\mathbb{E}[\rho] = \infty$, it may still be possible to formulate the problem in an alternative way and get well posedness and existence of unique optimisers. We explain how this can be accomplished.

Note that for all $\tau \in \mathcal{T}$, the following equality holds:

$$(1-\lambda)\rho - [\lambda(\tau-\rho)_+ + (1-\lambda)(\rho-\tau)_+] = \rho \wedge \tau - \lambda\tau,$$

where '$\wedge$' is used throughout to denote the minimum operation. In particular, if $\mathbb{E}[\rho] < \infty$, an optimal stopping time for the problem (OS′) is also an optimal stopping time for the problem of maximising $\mathbb{E}[\rho \wedge \tau - \lambda\tau]$ over all $\tau \in \mathcal{T}$. The last problem makes sense independently of whether $\mathbb{E}[\rho] < \infty$ holds or not.

A bit of care has to be exercised in ensuring that the expectation of $\rho \wedge \tau - \lambda\tau$ is well defined. Note that the negative part of the previous random variable is $(\lambda\tau - \rho)_+$; therefore, since this is a maximisation problem, a minimal condition to ask from a stopping time $\tau \in \mathcal{T}$ is that $\mathbb{E}[(\lambda\tau - \rho)_+] < \infty$. If this is the case, $\mathbb{E}[\rho \wedge \tau - \lambda\tau]$ is well defined and $(-\infty, \infty]$-valued. We then define

$$\mathcal{T}_\rho^\lambda := \{\tau \in \mathcal{T} \mid \mathbb{E}[(\lambda\tau - \rho)_+] < \infty\}$$

and consider the following problem:

$$\text{Find } \tau_* \in \mathcal{T}_\rho^\lambda \text{ such that } \mathbb{E}[\rho \wedge \tau_* - \lambda\tau_*] = \sup_{\tau \in \mathcal{T}_\rho^\lambda} \mathbb{E}[\rho \wedge \tau - \lambda\tau]. \tag{OS}$$

REMARK 2.1. When $\mathbb{E}[\rho] < \infty$, it is easily seen that $\mathcal{T}_\rho^\lambda$ comprises all stopping times τ such that $\mathbb{E}[\tau] < \infty$. It is clear also from the formulation (OS′) that, if $\mathbb{E}[\rho] < \infty$, a *necessary* condition for optimality of a stopping time τ is that $\mathbb{E}[\tau] < \infty$.

2.3. Reduction of the diffusion to a driftless diffusion. Recall the increasing and twice-differentiable scale function $\mathfrak{s}$ defined in (2.2). Define $X := \mathfrak{s}(Y)$ and $x_0 := \mathfrak{s}(y_0) \in (0, \infty)$. Then $\mathfrak{s}(y) \geq \mathfrak{s}(a+) = 0$ for all $y \in (a, b)$ implies that $X \geq 0$. Now apply Itô's formula to X:

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dW_s, \tag{2.4}$$

where the function $\sigma : [0, \infty) \mapsto \mathbb{R}$ is defined via $\sigma(x) = \mathfrak{s}'(\mathfrak{s}^{-1}(x))\beta(\mathfrak{s}^{-1}(x))$ for $x \in (0, \infty)$. Together with the assumptions of Section 2.1,

$$\sigma(0) = 0 \text{ and } \sigma^{-2} \text{ is locally integrable on } (0, \infty). \tag{2.5}$$

Therefore, X is a continuous-path nonnegative local martingale on $(\Omega, \mathbf{F}, \mathbb{P})$. Furthermore, by (2.3),

$$\mathbb{P}\Big[\lim_{t\to\infty} X_t = 0\Big] = 1.$$

Let $\ell = \mathfrak{s}(r) \in \mathbb{R}_+$. Then

$$\rho_r^Y = \rho_\ell^X = \sup\{t \in \mathbb{R}_+ \mid X_t > \ell\},$$

which is the last exit time of X from (ℓ, ∞) with $\ell \in (0, \infty)$. In particular, $\rho_\ell^X = 0$ if the set $\{t \in \mathbb{R}_+ \mid X_t > \ell\}$ is empty. Therefore, without loss of generality, in the rest of this paper, we shall be working with X satisfying (2.4), where the function $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}$ is such that (2.5) is valid. Moreover, by general results on the martingale problem of Stroock and Varadhan [3, Section 5.4], it follows that X possesses the strong Markov property on $(\Omega, \mathbf{F}, \mathbb{P})$.

3. The last exit time

In this section, we concentrate on solving the optimisation problem (OS) for ρ_ℓ^X, the last exit time of X from (ℓ, ∞). The main finding is Theorem 3.2. We prepare the ground with the following result.

LEMMA 3.1. *Assume that* $\int_1^\infty \sigma^{-2}(u)\, du < \infty$. *Define* $h : (0, \infty) \mapsto \mathbb{R}$ *via*

$$h(x) := 2 \int_x^\infty \frac{\lambda - 1 \wedge (u/\ell)}{\sigma^2(u)}\, du, \quad x \in (0, \infty). \tag{3.1}$$

Then the following statements are true:

(1) *h is decreasing on $(0, \lambda\ell)$ and increasing on $(\lambda\ell, \infty)$ and, in particular, $h(0) := \lim_{x \downarrow 0} h(x)$ is well defined and $(-\infty, \infty]$-valued;*

(2) *if $h(0) \in (0, \infty]$, there exists a unique $\kappa_\ell \in (0, \lambda\ell)$ such that $h(\kappa_\ell) = 0$.*

PROOF. Statement (1) follows on simple differentiation. Furthermore,

$$h(\lambda\ell) = -2(1 - \lambda) \int_{\lambda\ell}^\infty \frac{1}{\sigma^2(u)}\, du < 0;$$

therefore, statement (2) follows immediately from statement (1). □

THEOREM 3.2. *Assume that* $\int_1^\infty \sigma^{-2}(u)\, du < \infty$ *and define* $h : (0, \infty) \mapsto \mathbb{R}$ *via (3.1). If* $h(0) \in (0, \infty]$, *let* $\kappa = \kappa(\lambda, \ell) \in (0, \lambda\ell)$ *be the unique root of* $h(\kappa) = 0$; *otherwise, define* $\kappa = 0$. *Then the value of the problem (OS) is finite and the stopping time defined by* $\tau_\kappa = \inf\{t \in \mathbb{R}_+ \mid X_t \le \kappa\}$ *is optimal for (OS).*

The proof of Theorem 3.2 is quite involved and will be given in Section 3.1. We first give a remark and an example.

REMARK 3.3. When $\int_1^\infty \sigma^{-2}(u)\, du < \infty$ and $\int_0^1 u^2 \sigma^{-2}(u)\, du = \infty$, the problem (OS′) is not well posed, in the sense that it has infinite value and every stopping time is optimal. (This is shown in Section 3.1.4.) However, as Theorem 3.2 implies, the problem (OS) is well posed.

EXAMPLE 3.4. Let $\sigma^2(x) = \alpha x^p$ with $\alpha > 0, p > 1$, so that the assumption of Theorem 3.2, namely $\int_1^\infty \sigma^{-2}(u)\, du < \infty$, is satisfied. Then

$$h(x) = \frac{2}{\alpha} \int_x^\infty \frac{\lambda - 1 \wedge u/\ell}{u^p}\, du.$$

Since

$$\begin{aligned} h(0) &= \frac{2}{\alpha}\Big[\int_0^\ell \frac{\lambda - u/\ell}{\sigma^2(u)}\, du - (1 - \lambda) \int_\ell^\infty \frac{1}{\sigma^2(u)}\, du\Big] \\ &= \frac{2}{\alpha}\Big[\int_0^\ell \frac{\lambda - u/\ell}{\alpha u^p}\, du - (1 - \lambda) \int_\ell^\infty \frac{1}{\alpha u^p}\, du\Big] = +\infty > 0 \end{aligned}$$

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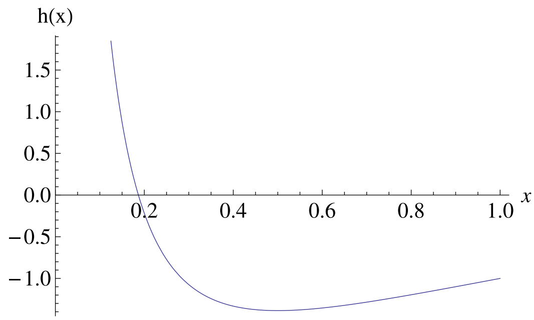

FIGURE 1. The function h for $\alpha = 1, \lambda = 1/2, p = 2, \ell = 1$.

for all $\lambda \in (0, 1)$, the equation

$$0 = \frac{\alpha h(x)}{2} = \begin{cases} -\dfrac{x^{-p+2}}{(p-2)\ell} + \dfrac{\lambda x^{-p+1}}{p-1} + \dfrac{\ell^{-p+1}}{(p-1)(p-2)} & \text{if } p \neq 2, \\ (1/\ell)\log(x/\ell) + \lambda/x - 1/\ell & \text{if } p = 2, \end{cases}$$

has a unique solution κ between 0 and $\lambda\ell$. Furthermore, the stopping time defined by $\tau_* = \tau_\kappa = \inf\{t \geq 0 : X_t \leq \kappa\}$ is optimal for the optimisation problem.

We specialise for a couple of values of p:

(i) $p = 2$. In this case X is a geometric Brownian motion process and

$$\alpha h(x)/2 = \frac{1}{\ell}\log(x/\ell) + \lambda/x - 1/\ell = 0$$

has a unique solution between 0 and $\lambda\ell$;

(ii) $p = 2(d-1)/(d-2)$ for some dimension $d > 2$. Then X is the transformed local martingale of a d-dimensional Bessel process via the scale function. In particular, for $d = 3$, that is, $p = 4$,

$$\alpha h(x)/2 = -1/(2\ell x^2) + \lambda/(3x^3) + 1/(6\ell^3) = 0$$

has a unique solution $\kappa = 2\ell\cos(\theta/3) \in (0, \lambda\ell)$, where $\cos\theta = -\lambda$ and $\sin\theta = -\sqrt{1-\lambda^2}$. For $d = 4$, that is, $p = 3$,

$$\alpha h(x)/2 = -1/(\ell x) + \lambda/(2x^2) + 1/(2\ell^2) = 0$$

has a unique solution $\kappa = \ell(1 - \sqrt{1-\lambda}) \in (0, \lambda\ell)$.

Figure 1 depicts the function h when $\alpha = 1, \lambda = 0.5, p = 2, \ell = 1$. The plot shows that h is strictly decreasing from 0 to $\lambda\ell = 0.5$ and intersects the x-axis at a point around 0.2, which is the κ we are seeking.

For the same parameter values, $\alpha = 1, \lambda = 0.5, p = 2, \ell = 1$, we have also simulated our results for X defined in (2.4) with initial value $x_0 = 1$. A few representative paths are given in Figure 2. The last exit time from the interval $(1, \infty)$ (labelled as ρ in each

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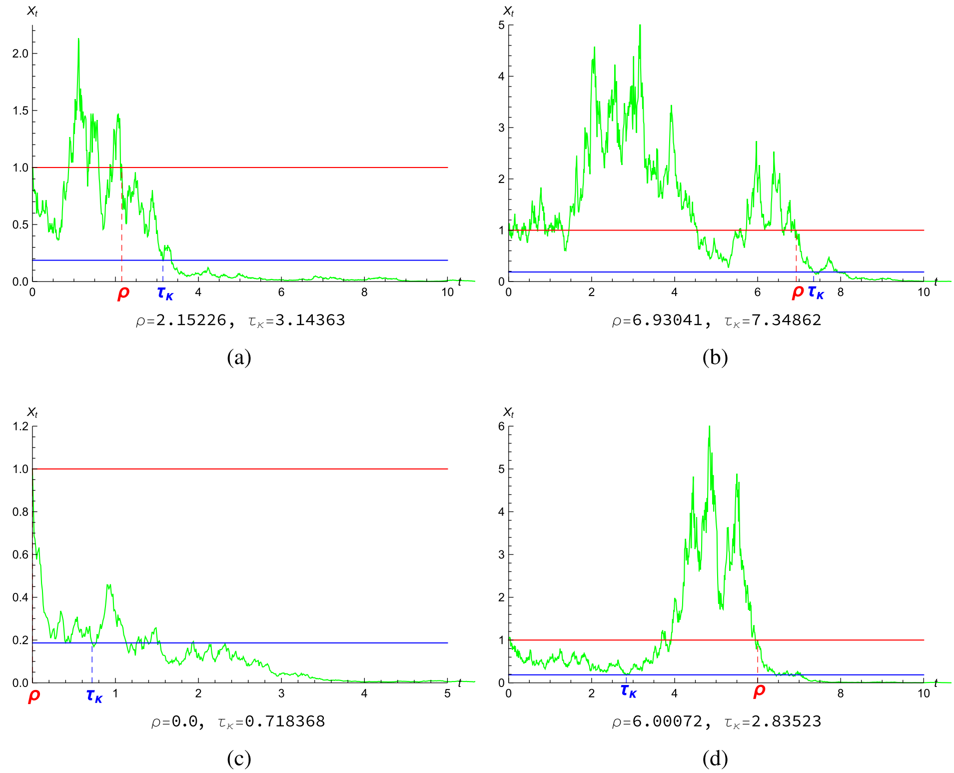

FIGURE 2. Simulated paths of X for $x_0 = 1, \alpha = 1, \lambda = 1/2, p = 2, \ell = 1$; ρ is the last exit time from the interval $(1, \infty)$ and τ_κ is the optimal stopping time.

figure) and the optimal stopping time (labelled as τ_κ in each figure) are both plotted. As expected, for many simulated paths, τ_κ is bigger than ρ, since once the drift-downward process X touches the low level $\kappa(< \lambda\ell)$, the chance that X will reach the high level ℓ again can be rare. However, theoretically and practically, X can still hit the level ℓ again (see Figure 2(d)).

3.1. Proof of Theorem 3.2. In the proof of Theorem 3.2, we keep $\ell \in (0, \infty)$ fixed and drop the dependence of quantities on ℓ for notational convenience. Instead of ρ_ℓ^X, we use ρ to denote the last exit time of X from the interval (ℓ, ∞).

The proof of Theorem 3.2 is split into steps. The main part is Section 3.1.3. At the end of the proof, in Section 3.1.4, the claim raised in Remark 3.3 is tackled.

Before we start with the proof, we introduce a notation. For $\gamma \in \mathbb{R}_+$, define

$$\tau_\gamma := \inf\{t \in \mathbb{R}_+ \mid X_t \leq \gamma\}.$$

It is clear that $\mathbb{P}[\tau_\ell \leq \rho \leq \tau_0] = 1$; therefore, optimal stopping times have to be greater than or equal to τ_ℓ and less than or equal to τ_0. Define $\mathcal{T}_0$ to be the set of all stopping

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times such that $\tau \le \tau_0$, and $\mathcal{T}_{0+}$ to be the set of all $\tau \in \mathcal{T}$ such that $\tau \le \tau_\gamma$ for some $\gamma \in (0,\infty)$. Clearly, $\mathcal{T}_{0+} \subseteq \mathcal{T}_0$ and, if an optimal stopping time exists, then it is an element of $\mathcal{T}_0$.

3.1.1. Reduction to an optimal control problem. The first step in solving (OS) is to find more 'explicit' forms for $\mathbb{E}[\rho \wedge \tau]$ and $\mathbb{E}[\tau]$.

Lemma 3.5. *Assume that $\int_1^\infty \sigma^{-2}(u)\,du < \infty$. Define a convex and decreasing function $G : (0,\infty) \mapsto \mathbb{R}$ via*

$$G(x) = -2\int_{x_0}^{x}\Big(\int_u^\infty \frac{1}{\sigma^2(v)}\,dv\Big)\,du, \quad x \in (0,\infty).$$

Similarly, define a convex and decreasing function $F : (0,\infty) \mapsto \mathbb{R}$ via

$$F(x) = -2\int_{x_0}^{x}\Big(\int_u^\infty \frac{1\wedge(v/\ell)}{\sigma^2(v)}\,dv\Big)\,du, \quad x \in (0,\infty).$$

Then $\mathbb{E}[\tau] = \mathbb{E}[G(X_\tau)] < \infty$ and $\mathbb{E}[\rho\wedge\tau] = \mathbb{E}[F(X_\tau)] < \infty$ for all $\tau \in \mathcal{T}_{0+}$. Furthermore, if $G(0) := \lim_{x\downarrow 0} G(x) < \infty$, the previous equalities hold for all $\tau \in \mathcal{T}_0$.

Proof. First, observe that both G and F are well defined in view of $\int_1^\infty \sigma^{-2}(u)\,du < \infty$. Note also that $F \le G$ for $x \le x_0$. Therefore, if $G(0) < \infty$, then also $F(0) < \infty$. The proof will now proceed in two steps.

Step (i). We shall first establish that $\mathbb{E}[\tau] = \mathbb{E}[G(X_\tau)]$ for $\tau \in \mathcal{T}_{0+}$ and in fact for all $\tau \in \mathcal{T}_0$ if $G(0) < \infty$.

Fix $\tau \in \mathcal{T}_0$ and let $\theta_n := \tau \wedge \tau_n \wedge \tau_{1/n} \wedge n$ for all $n \in \mathbb{N}$. Since $\tau \in \mathcal{T}_0$, the sequence $(\theta_n)_{n\in\mathbb{N}}$ monotonically converges to τ. Using Itô's formula and the fact that $G''(x) = 2/\sigma^2(x)$ for $x \in (0,\infty)$, we readily obtain

$$G(X_{\theta_n\wedge t}) - (\theta_n \wedge t) = -2\int_0^{\theta_n\wedge t} \sigma(X_s)\Big(\int_{X_s}^\infty \frac{1}{\sigma^2(u)}\,du\Big)\,dW_s.$$

By the definitions of θ_n and the function G, and the fact that G is decreasing,

$$-2\int_{x_0}^{1/n}\int_u^\infty \frac{1}{\sigma^2(v)}\,dv\,du \ge G(X_{\theta_n\wedge t}) - (\theta_n\wedge t) \ge -2\int_{x_0}^{n}\int_u^\infty \frac{1}{\sigma^2(v)}\,dv\,du - n.$$

Moreover, since $\int_1^\infty \sigma^{-2}(u)\,du < \infty$, the right-hand-side local martingale is indeed a martingale. Therefore, the process $(G(X_{\theta_n\wedge t}) - \theta_n \wedge t)_{t\in\mathbb{R}_+}$ is a martingale starting from zero for all $n \in \mathbb{N}$, which implies that

$$\mathbb{E}[\theta_n \wedge t] = \mathbb{E}[G(X_{\theta_n\wedge t})].$$

Let $t \to \infty$, apply the monotone convergence theorem to the left-hand side of the above equation and apply the dominated convergence theorem to the right-hand side. Since $X_{\theta_n\wedge t}$ is bounded between $1/n$ and n,

$$\mathbb{E}[\theta_n] = \mathbb{E}[G(X_{\theta_n})] \quad \text{for all } n \in \mathbb{N}.$$

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Upon sending $n \to \infty$, the left-hand side of this equation converges to $\mathbb{E}[\tau]$ in view of the monotone convergence theorem. The first claim follows if we can show that

$$\lim_{n\to\infty} \mathbb{E}[G(X_{\theta_n})] = \mathbb{E}[G(X_\tau)].$$

Note that X_{θ_n} converges to X_τ in probability. Thus, $G(X_{\theta_n})$ also converges to $G(X_\tau)$ in probability by the continuity of G. We shall show that the collection $\{G(X_{\theta_n}) \mid n \in \mathbb{N}\}$ of random variables is uniformly integrable when $\tau \in \mathcal{T}_{0+}$ and, if additionally $G(0) < \infty$, then the previous family is uniformly integrable for all $\tau \in \mathcal{T}_0$. Convergence in probability and uniform integrability will imply that $\lim_{n\to\infty} \mathbb{E}[G(X_{\theta_n})] = \mathbb{E}[G(X_\tau)]$.

Observe that G is a nonincreasing convex function. Since

$$\sup_{n\in\mathbb{N}} \mathbb{E}[X_{\theta_n}] \le n < \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{G(x)}{x} = \lim_{x\to\infty} G'(x) = 0$$

for any $\varepsilon > 0$, there exists a $C > 0$ such that $|G(x)|/|x| \le \varepsilon/n$ holds for all $G(x) \le -C$. Take any $G(X_{\theta_n})$. Then $|G(X_{\theta_n})| \le \varepsilon |X_{\theta_n}|/n$ whenever $G(X_{\theta_n}) \le -C$. Therefore,

$$\mathbb{E}[\mathbb{I}_{\{G(X_{\theta_n})\le -C\}}|G(X_{\theta_n})|] \le \frac{\varepsilon}{n}\mathbb{E}[\mathbb{I}_{\{G(X_{\theta_n})\le -C\}}|X_{\theta_n}|] \le \frac{\varepsilon}{n} n = \varepsilon,$$

which implies that the negative parts of $\{G(X_{\theta_n}) \mid n \in \mathbb{N}\}$ are uniformly integrable. On the other hand, if $\tau \in \mathcal{T}_{0+}$, there exists $\gamma \in (0, x_0)$ such that $G(X_{\theta_n}) \le G(\gamma) < \infty$ for all $n \in \mathbb{N}$. Furthermore, if $G(0) < \infty$, then $G(X_{\theta_n}) \le G(0) < \infty$ for all $n \in \mathbb{N}$. The uniform integrability of $G(X_{\theta_n})$ follows and hence the first claim.

Step (ii). We now discuss the validity of the equality $\mathbb{E}[\tau \wedge \rho] = \mathbb{E}[F(X_\tau)]$ for $\tau \in \mathcal{T}_{0+}$ and for all $\tau \in \mathcal{T}_0$ when $F(0) \le G(0) < \infty$. For $\tau \in \mathcal{T}_0$, we compute

$$\begin{aligned}\mathbb{E}[\tau \wedge \rho] &= \mathbb{E}\Big[\int_0^\infty \mathbb{I}_{\{t<\tau\wedge\rho\}}\,dt\Big] = \int_0^\infty \mathbb{E}[\mathbb{I}_{\{\tau>t\}}\mathbb{I}_{\{\rho>t\}}]\,dt \\ &= \int_0^\infty \mathbb{E}[\mathbb{I}_{\{\tau>t\}}\mathbb{P}[\rho > t \mid \mathcal{F}_t]]\,dt.\end{aligned}$$

Since X is a local martingale with $X_\infty = 0$ for $\mathbb{P}$-almost surely, Doob's maximal identity [4] shows that $\mathbb{P}[\rho > t \mid \mathcal{F}_t] = 1 \wedge (X_t/\ell)$ holds for $t \in \mathbb{R}_+$. In other words,

$$\mathbb{E}[\tau \wedge \rho] = \int_0^\infty \mathbb{E}[\mathbb{I}_{\{\tau>t\}}(1 \wedge (X_t/\ell))]\,dt = \mathbb{E}\Big[\int_0^\tau (1 \wedge (X_t/\ell))\,dt\Big].$$

Now define $A := \int_0^\cdot (1 \wedge (X_t/\ell))\,dt$. As in the previous paragraph, fix $\tau \in \mathcal{T}_0$ and let $\theta_n := \tau \wedge \tau_n \wedge \tau_{1/n} \wedge n$ for all $n \in \mathbb{N}$. Since $\tau \in \mathcal{T}_0$, $(\theta_n)_{n\in\mathbb{N}}$ converges monotonically to τ. By Itô's formula and the fact that $F''(x) = 2\sigma^{-2}(x)(1 \wedge (x/\ell))$ holds for $x \in (0, \infty)$, it follows that the process $(F(X_{\theta_n\wedge t}) - A_{\theta_n\wedge t})_{t\in\mathbb{R}_+}$ is a local martingale starting from zero. Thus, $\mathbb{E}[F(X_{\theta_n})] = \mathbb{E}[A_{\theta_n}]$ for all $n \in \mathbb{N}$. Send $n \to \infty$ and note that the right-hand side of this equation converges to $\mathbb{E}[A_\tau] = \mathbb{E}[\tau \wedge \rho]$ in view of the monotone convergence theorem. In order to show that the left-hand side of the equation converges to $\mathbb{E}[F(X_\tau)]$, we need to show that the family $\{F(X_{\theta_n}) \mid n \in \mathbb{N}\}$ is uniformly integrable for any $\tau \in \mathcal{T}_{0+}$ and in fact for all $\tau \in \mathcal{T}_0$ under the additional assumption that $F(0) < \infty$. This is done following *mutatis mutandis* the reasoning of the previous paragraph and is, therefore, omitted. □

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By Lemma 3.5, $\mathbb{E}[\tau] < \infty$ for all $\tau \in \mathcal{T}_{0+}$, which implies in particular that $\mathcal{T}_{0+} \subseteq \mathcal{T}_\rho^\lambda$. Furthermore, if $G(0) < \infty$, then $\mathbb{E}[\tau] < \infty$ for all $\tau \in \mathcal{T}_0$, which implies in particular that $\mathcal{T}_0 \subseteq \mathcal{T}_\rho^\lambda$.

3.1.2. Properties of H. In the notation of Lemma 3.5, define $H := F - \lambda G$. Then $\mathbb{E}[\rho \wedge \tau - \lambda\tau] = \mathbb{E}[H(X_\tau)]$ for $\tau \in \mathcal{T}_{0+}$ and actually for all $\tau \in \mathcal{T}_0$ if $G(0) < \infty$. Note that $H' = h$ for the function h defined in (3.1). Furthermore,

$$H''(x) = h'(x) = 2\frac{1 \wedge (x/\ell) - \lambda}{\sigma^2(x)}, \quad x \in (0, \infty).$$

Since H'' is negative on $(0, \lambda\ell)$ and positive on $(\lambda\ell, \infty)$, it follows that H is concave on $(0, \lambda\ell)$ and convex on $(\lambda\ell, \infty)$. Furthermore, since $H' = h$ is negative on $(\lambda\ell, \infty)$, it follows that H is decreasing in $(\lambda\ell, \infty)$.

Since H'' is negative on $(0, \lambda\ell)$, either $H'(0) = h(0) \in (0, \infty]$, in which case there exists a unique $\kappa \in (0, \lambda\ell)$ where H has a global maximum and $h(\kappa) = 0$; or $H'(0) = h(0) \in (-\infty, 0]$, in which case we set $\kappa = 0$ and $H(\kappa) = H(0) \in \mathbb{R}$ is again the global maximum of H.

3.1.3. Verification. We now verify that τ_κ is the optimal stopping time for (OS) and that the value of (OS) is finite. We consider two cases.

Case (A). Suppose that $H'(0) = h(0) \in (-\infty, 0]$. Then

$$\sup_{x \in (0,\infty)} H(x) = H(0) := \lim_{x \downarrow 0} H(x) > -\infty.$$

We shall use the following result.

LEMMA 3.6. *If $H(0) > -\infty$, then $G(0) < \infty$.*

PROOF. Straightforward manipulation of the functions F and G using Fubini's theorem shows that, for any $x \in (0, x_0)$,

$$F(x) = 2\int_0^\infty \frac{1 \wedge (v/\ell)}{\sigma^2(v)}(x_0 \wedge v - x)_+ \, dv,$$
$$G(x) = 2\int_0^\infty \frac{1}{\sigma^2(v)}(x_0 \wedge v - x)_+ \, dv.$$

In particular, by the monotone convergence theorem, $G(0) = \infty$ is equivalent to $\int_0^\epsilon v\sigma^{-2}(v)\, dv = \infty$ for all $\epsilon \in (0, \infty)$. For $x \in (0, x_0)$,

$$\begin{aligned} H(x) &= F(x) - \lambda G(x) \\ &= 2\int_0^\infty \frac{1 \wedge v/\ell - \lambda}{\sigma^2(v)}(x_0 \wedge v - x)_+ \, dv \\ &= -2\int_0^{\lambda\ell \wedge x_0} \frac{\lambda - v/\ell}{\sigma^2(v)}(v - x)_+ \, dv + 2\int_{\lambda\ell \wedge x_0}^\infty \frac{1 \wedge v/\ell - \lambda}{\sigma^2(v)}(x_0 \wedge v - x)_+ \, dv. \end{aligned}$$

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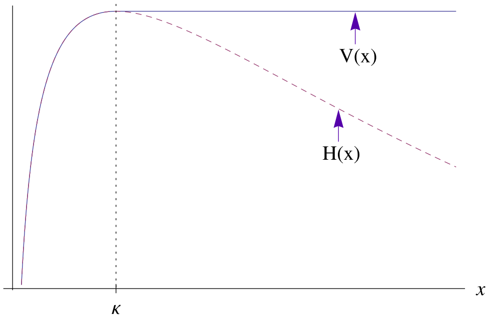

FIGURE 3. The functions H and V.

Recall the assumption that $\int_1^\infty \sigma^{-1}(v)\,dv < \infty$, implying that

$$\int_{\lambda\ell\wedge x_0}^{\infty} \frac{1\wedge v/\ell-\lambda}{\sigma^2(v)}(x_0\wedge v-x)_+\,dv<\infty.$$

On the other hand, $G(0)=\infty$ implies that

$$\lim_{x\downarrow 0}\int_0^{\lambda\ell\wedge x_0}\frac{(\lambda-v/\ell)}{\sigma^2(v)}(v-x)_+\,dv=\infty.$$

Therefore, by the monotone convergence theorem, $G(0)=\infty$ implies that $H(0)=-\infty$, which completes the proof. □

In view of Lemmas 3.5 and 3.6, $\mathbb{E}[\rho\wedge\tau-\lambda\tau]=\mathbb{E}[H(X_\tau)]$ for all $\tau\in\mathcal{T}_0$. Then

$$\mathbb{E}[\rho\wedge\tau-\lambda\tau]=\mathbb{E}[H(X_\tau)]\le H(0)=\mathbb{E}[H(X_{\tau_0})]=\mathbb{E}[\rho\wedge\tau_0-\lambda\tau_0],$$

which shows that $\tau_\kappa=\tau_0$ is optimal for (OS) and that the value of (OS) is $H(0)<\infty$.

Case (B). Suppose that $H'(0)=h(0)\in(0,\infty]$. Define a function $V:(0,\infty)\mapsto\mathbb{R}$ via $V(x)=H(x)$ for $x\in(0,\kappa)$ and $V(x)=H(\kappa)$ for $x\in[\kappa,\infty)$ (see Figure 3). It follows that $H\le V$, that V is continuous and concave and that $H(\kappa)=V(\kappa)$.

For any $\tau\in\mathcal{T}_{0+}$, there exists $\gamma\in(0,\kappa)$ such that $X_\tau\ge\gamma$. The concavity of V implies that $(V(X_{\tau\wedge t}))_{t\in\mathbb{R}_+}$ is a local supermartingale; since $V(X_\tau)\ge V(\gamma)>\infty$, $(V(X_{\tau\wedge t}))_{t\in\mathbb{R}_+}$ is a supermartingale. Therefore, $\mathbb{E}[V(X_\tau)]\le V(x_0)$.

On the other hand, consider $(V(X_{\tau_\kappa}))$. If $x_0\le\kappa$, then $\tau_\kappa=0$ and $\mathbb{E}[V(X_{\tau_\kappa})]=V(x_0)$; otherwise, if $x_0>\kappa$, then $\mathbb{E}[V(X_{\tau_\kappa})]=V(\kappa)=V(x_0)$. In both cases, $\mathbb{E}[V(X_{\tau_\kappa})]=V(x_0)$. (Indeed, $(V(X_{\tau_\kappa\wedge t}))_{t\in\mathbb{R}_+}$ is a true martingale.) Therefore, $\mathbb{E}[H(X_{\tau_\kappa})]=\mathbb{E}[V(X_{\tau_\kappa})]=V(x_0)$. It follows that

$$\mathbb{E}[\rho\wedge\tau-\lambda\tau]=\mathbb{E}[H(X_\tau)]\le\mathbb{E}[V(X_\tau)]\le V(x_0)=\mathbb{E}[H(X_{\tau_\kappa})]=\mathbb{E}[\rho\wedge\tau_\kappa-\lambda\tau_\kappa],$$

which shows the optimality of τ_κ in $\mathcal{T}_{0+}$.

Now pick any $\tau\in\mathcal{T}_\rho^\lambda$ and note that $\tau\wedge\tau_{1/n}\in\mathcal{T}_{0+}$ for all $n\in\mathbb{N}$. Noting that

$$\rho\wedge\tau\wedge\tau_{1/n}-\lambda(\tau\wedge\tau_{1/n})\ge-(\lambda\tau-\rho)_+$$

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for all $n \in \mathbb{N}$ and $\mathbb{E}[(\lambda\tau - \rho)_+] < \infty$ from the definition of $\mathcal{T}_\rho^\lambda$, Fatou's lemma implies that

$$\mathbb{E}[\rho \wedge \tau - \lambda\tau] \le \liminf_{n\to\infty} \mathbb{E}[\rho \wedge \tau \wedge \tau_{1/n} - \lambda(\tau \wedge \tau_{1/n})] \le V(x_0) = \mathbb{E}[\rho \wedge \tau_\kappa - \lambda\tau_\kappa],$$

which finally establishes the optimality of τ_κ in $\mathcal{T}_\rho^\lambda$ and the finiteness for the problem (OS).

3.1.4. Well posedness of the problem. To complete this section, we provide more details about Remark 3.3 on the well posedness of the optimal stopping problem for the last exit time.

Assume that $\int_1^\infty \sigma^{-2}(u)\,du < \infty$ and $\int_0^1 u^2\sigma^{-2}(u)\,du = \infty$. Let $\gamma \in (0, x_0 \wedge \ell)$. By Lemma 3.5 and since $\tau_\gamma \in \mathcal{T}_{0+}$,

$$\mathbb{E}[\rho \wedge \tau_\gamma] \ge \frac{1}{\ell}\int_\gamma^{x_0\wedge\ell}\Big(\int_u^{x_0\wedge\ell} \frac{v}{\sigma^2(v)}\,dv\Big)\,du = \frac{1}{\ell}\int_\gamma^{x_0\wedge\ell} \frac{v(v-\gamma)}{\sigma^2(v)}\,dv.$$

By the monotone convergence theorem, $\mathbb{E}[\rho] = \lim_{\gamma\downarrow 0} \mathbb{E}[\rho \wedge \tau_\gamma] = \infty$. Since this holds for all $x_0 \in (0,\infty)$, the strong Markov property gives $\mathbb{E}[(\rho - \tau)_+] = \infty$ whenever $\tau \in \mathcal{T}$ is such that $\mathbb{P}[\tau < \tau_0] > 0$. Since $\mathbb{P}[\rho \le \tau_0] = 1$, it easily follows that

$$\mathbb{E}[\lambda(\tau - \rho)_+ + (1-\lambda)(\rho - \tau)_+] = \infty$$

for all $\tau \in \mathcal{T}$. In other words, the optimisation problem (OS′) is not well posed, since every stopping time is trivially optimal. However, the optimisation problem (OS) is well posed by Theorem 3.2.

4. Conclusion

In this paper we have solved an optimal stopping problem for a one-dimensional downwards transient diffusion process. We have studied stopping as close as possible to the last exit time and have provided an explicit optimal stopping time: the diffusion process should stop optimally when it is too low compared to the exit level ℓ. This result makes intuitive sense, since the diffusion process is drifting downwards and it is highly possible that the process will not go up to level ℓ again.

Though our results are primarily focused on a one-dimensional transient diffusion process, we may extend the problem to some other one-dimensional processes or even for some multi-dimensional processes. We have mainly discussed an optimal stopping problem for a random time ρ with the *loss function* $[\lambda(\tau - \rho)_+ + (1-\lambda)(\rho - \tau)_+]$. This problem may be considered for more general loss functions. For example, let $u_1 : \mathbb{R}_+ \to \mathbb{R}_+$ and $u_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be two increasing functions such that $u_1(0) = u_2(0) = 0$. Then we may find the optimal stopping time τ (if it exists) for the random time ρ to minimise the expectation

$$\mathbb{E}[u_1((\tau - \rho)_+) + u_2((\rho - \tau)_+)].$$

Here, the functions u_1 and u_2 can be interpreted as the 'loss' caused by the estimated error $(\tau - \rho)_+$ or $(\rho - \tau)_+$, and the 'loss' is not necessarily linearly proportional to the estimated error. The optimal stopping problem for other random times can also be of interest.

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